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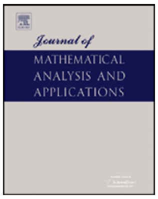

Existence results for quasilinear elliptic exterior problems involving convection term and nonlinear Robin boundary conditions [☆]

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ABSTRACT

In this paper, the authors establish the existence of solutions for a class of elliptic exterior problems involving convection terms and nonlinear Robin boundary conditions. The proof of the result is made by combining Galerkin method with *a priori* estimates for this kind of problem.

1. Introduction

In this work, we study the existence of solutions for the following problem

$$\begin{cases} -div\big(\xi(x)\nabla u\big) + u = \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ \xi(x)\partial_\nu u + \alpha(x,u)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a smooth exterior domain in $\mathbb{R}^N$, $N \geqslant 3$, that is, Ω is the complement of a bounded domain Ω_0 with smooth boundary (in this case, $\partial\Omega$ will be indicated by $\partial\Omega_0$), λ is a real parameter, ν is the unit vector of the outward normal on $\partial\Omega$, ξ is a positive continuous function, $f(x,u,\nabla u) = h(x,u) + g(x,\nabla u)$ is such that h is a sublinear function and g is bounded from above by a gradient term (or convection term) of the type $|\nabla u|^\beta$ with $0 < \beta < 1$, and α is a non-negative continuous function with subcritical growth at infinity, more exactly, α is a continuous function such that $0 \leqslant \alpha(x,\mu) \leqslant b(x)|\mu|^{p-2}$, $\forall(x,\mu) \in \Omega \times \mathbb{R}^*$ and $\alpha(x,0) = 0$, where b is a non-negative function and $p \in (1, \frac{2(N-1)}{N-2})$.

Elliptic problems involving convection terms are a challenge for many researchers, not only because in this situation they are not variational, but also they model several phenomena which may be viewed as prototypes of pattern formation in biology which is related to steady-state problems for a chemotactic aggregation model (see Keller and Segel [21]), and to study of activator–inhibitor systems modeling biological pattern formation (Gierer and Meinhardt [16]). See also, the Kardar–Parisi–Zhang model [20], the flame propagation model [9] and the papers [1,17,18] for more information about physical motivation of this kind of problem.

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In our work, together with the convection term and the unboundedness of the domain, we will also study the effect of concave and convex conditions on the nonlinearity and the boundary condition, respectively. This is done following a paper due to Ambrosetti, Brézis and Cerami [7], which treats an elliptic problem with Dirichlet boundary condition without convection term.

On the other hand, several authors worked on the problem (P_λ) (with gradient term) in a bounded domain, but with Dirichlet boundary condition; although we can be omitting some important papers, we would like to cite [12,13,15,27] with $0 < \beta < 1$, and [1,2,8,10,18,19,24,26,29] for $1 < \beta \leqslant 2$. We recall that $\beta = 2$ is considered in the literature as critical growth for the gradient, maybe because a classical existence result [23, Theorem 8.3] as well as *a priori* estimates for this class of problem hold for $0 < \beta \leqslant 2$.

Recently, Filippucci, Pucci and Rădulescu in [14] studied a problem involving p-Laplacian operator, namely,

$$\begin{cases} -div\big(\xi(x)|\nabla u|^{p-2}\nabla u\big) + |u|^{q-2}u = \lambda k(x)|u|^{r-2}u & \text{in } \Omega, \\ \xi(x)|\nabla u|^{p-2}\partial_\nu u + b(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega \end{cases} \tag{FPR}$$

without convection term, but in a smooth exterior domain Ω. Here, λ is a real parameter, ξ is a positive and Hölder continuous function in $L^\infty(\Omega)$, b is a continuous positive function on $\Gamma = \partial\Omega$ and k is a bounded function in $L^{p_o}(\Omega)$, with $p_o = \frac{p^*}{p^*-r}$ $(p < r < q < p^*)$, which is positive on a non-empty open subset of Ω. In that paper the authors showed, via variational methods, the existence of a real number $\lambda^* > 0$ such that for $\lambda < \lambda^*$, problem (*FPR*) has no solution, and for $\lambda \geqslant \lambda^*$, the problem has at least one solution.

See also Chabrowski and Ruf [11], Mihăilescu and Rădulescu [25] and Yu [28], for related problems involving Neumann condition.

Our concern in problem (P_λ) is mainly motivated by results proved in [14], by considering only the case $p = q = 2$, but adding a term involving the gradient, and also with a nonlinear Robin boundary condition. We recall that, in this case, $p = q = 2$, the boundary condition in (*FPR*) is reduced to the linear case.

In our work, due to the presence of the gradient term, as well as a nonlinear boundary condition on exterior domain, it was necessary to impose more restricting on β, $0 < \beta < 1$, and adapt some arguments used in Alves and de Figueiredo in [6], see also [5].

In order to establish our result, we set the basic hypotheses on the functions ξ, h, g and α as follows:

(H_1) The functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are locally Hölder continuous, $\xi : \Omega \to \mathbb{R}$ is a C^1 function, and there exists a constant $k_0 > 0$ such that $k_0 \leqslant \xi(x)$, $\forall x \in \Omega$.

(H_2) There are a constant $0 < r_1 < 1$ and continuous functions a_i $(i = 0, 1, 2)$, $a_1 \in L^2(\mathbb{R}^N)$ and $a_2 \in L^{\frac{2}{1-r_1}}(\mathbb{R}^N)$, such that

$$0 < a_0(x) \leqslant h(x,\mu) \leqslant a_1(x) + a_2(x)|\mu|^{r_1}, \quad \forall (x,\mu) \in \Omega \times \mathbb{R}.$$

(H_3) There are a constant $0 < r_2 < 1$, and continuous functions $a_3 \in L^2(\mathbb{R}^N)$ and $a_4 \in L^{\frac{2}{1-r_2}}(\mathbb{R}^N)$ such that

$$0 \leqslant g(x,\eta) \leqslant a_3(x) + a_4(x)|\eta|^{r_2}, \quad \forall (x,\eta) \in \Omega \times \mathbb{R}^N.$$

(H_4) The function $\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous if $\mu \neq 0$ and satisfies

$$0 \leqslant \alpha(x,\mu) \leqslant b(x)|\mu|^{p-2}, \quad \forall (x,\mu) \in \Omega \times \mathbb{R} - \{0\} \quad \text{and} \quad \alpha(x,0) = 0,$$

where $1 < p < \frac{2(N-1)}{N-2}$ and $b(x)$ is a non-negative continuous function.

Our approach consists in considering a class of auxiliary problems $(P_R)_\lambda$ defined in a bounded smooth domain $\Omega_R \subset \mathbb{R}^N$. Namely, we consider the problem

$$\begin{cases} -div\big(\xi(x)\nabla u\big) + u = \lambda\big(h(x,u) + g(x,\nabla u)\big) & \text{in } \Omega_R, \\ \xi(x)\partial_\nu u + \alpha(x,u)u = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{$(P_R)_\lambda$}$$

where $\Omega_R = B_R(0) \cap \Omega$ is such that $\overline{\Omega}_0 \subset B_R(0)$. Notice that $\partial\Omega_R = \partial\Omega \cup \partial B_R(0)$. Fixing $\lambda \neq 0$, and using Galerkin's method, we show the existence of a weak solution to problem $(P_R)_\lambda$. Taking $R = n$, we obtain a family of solutions $\{u_n\}$ to problem $(P_n)_\lambda$. Combining an *a priori* estimate with a diagonal argument, and passing to the limit in $(P_n)_\lambda$ as $n \to \infty$, we obtain a solution of (P_λ).

By a solution of problem (P_λ) we mean a function $u \in C^2(\Omega) \cap H^1(\Omega)$ verifying the equation weakly in Ω.

Our main result is the following.

Theorem 1.1. *Assume the conditions* (H_1)–(H_4).

(i) *If* $\lambda < 0$, *then problem* (P_λ) *has at least one solution. In this case, all the solutions should be either negative or sign changing solutions*;

(ii) *If* $\lambda = 0$, *then problem* (P_λ) *has only one solution* $u \equiv 0$;

(iii) *If* $\lambda > 0$, *then problem* (P_λ) *has at least one solution. In this case, all the solutions should be either positive or sign changing solutions.*

Example 1.1. Taking $\xi(x) = (\mathrm{arctg}(x) + \pi)$, $h(x,u) = (1 + |\mathrm{sen}(u)||u|^{r_1})\exp(-|x|)$, $g(x,\nabla u) = \frac{|\cos(|\nabla u|)||\nabla u|^{r_2}}{1+|x|^2}$ and $\alpha(x,u)u = |\mathrm{sen}(u)||u|^{p-2}u\exp(|x|)$, with $r_1, r_2 \in (0,1)$, then Theorem 1.1 holds.

2. Auxiliary results

In this section, we will present some notations and results which will be necessary in the next section.

If $D \subset \mathbb{R}^N$, $N \geqslant 3$ and $s \geqslant 1$, we consider $H^1(D)$ (the Sobolev space $W^{1,2}(D)$) and $L^s(D)$ equipped with the respective usual norms

$$\|\cdot\|_{H^1(D)} = \left(\int_D |\nabla u|^2\,dx + \int_D |u|^2\,dx\right)^{\frac{1}{2}} \quad \text{and}$$

$$\|\cdot\|_{L^s(D)} = \left(\int_D |u|^s\,dx\right)^{\frac{1}{s}}.$$

The result below is a consequence of Brouwer's Fixed Point Theorem and its proof can be found in Kesavan [22].

Lemma 2.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function with $\langle F(x), x\rangle \geqslant 0$, for all x verifying $|x| = R > 0$, where $\langle x, y\rangle$ is the usual inner product of $\mathbb{R}^N$. Then there exists $z_0 \in \overline{B}_R(0)$ such that $F(z_0) = 0$.*

The next result has an important role in our proof.

Lemma 2.2 *(Regularity). If conditions (H_1)–(H_3) hold and assuming that $u \in H^1(\Omega_n)$ is a weak solution of problem $(P_n)_\lambda$, then u belongs to $C^2(\Omega_n)$.*

Proof. Define the function

$$\Phi(x) = \lambda\big(h(x,u) + g(x,\nabla u)\big).$$

Since $u \in H^1(\Omega_n)$, by (H_2)–(H_3) we have

$$\Phi \in L^{\frac{2}{r}}(\Omega_n),$$

where $r = \max\{r_i,\ i = 1,2\}$. Therefore, by Agmon regularity result [3, Theorem 7.1′], we infer that all weak solutions $u \in H^1(\Omega_n)$ of the problem

$$-div\big(\xi(x)\nabla u\big) + u = \Phi(x) \quad \text{in } \Omega_n$$

belong to $W^{2,s_1}_{\mathrm{loc}}(\Omega_n)$, where $s_1 = 2/r$. By applying the same argument, we conclude that

$$\Phi \in L^{\frac{s_1}{r}}(\Omega_n),$$
$$u \in W^{2,s_2}_{\mathrm{loc}}(\Omega_n)$$

with $s_2 = 2/r^2$. This argument is known as *bootstrap*.

Since $r \in (0,1)$, we can repeat this argument k times, until we obtain

$$u \in W^{2,s_k}_{\mathrm{loc}}(\Omega_n) \quad \text{and} \quad s_k = 2/r^k > N.$$

Therefore, by the Sobolev–Morrey embedding, we have that u belongs to $C^2(\Omega_n)$. □

In the next theorem, under the assumptions (H_1)–(H_4), we obtain a solution of the problem $(P_n)_\lambda$ by using Galerkin's method. This result is a key point on the proof of Theorem 1.1.

Theorem 2.1. *Let $\lambda \neq 0$ be a fixed parameter and assume (H_1)–(H_4). Then, there exists $n_0 = n_0(\Omega_0) \in \mathbb{N}$ such that for all $n \geqslant n_0$, problem $(P_n)_\lambda$ possesses at least one weak solution $v_n \in H^1(\Omega_n)$.*

Proof. Let n_0 be the smallest natural number such that $\Omega_0 \subset B_{n_0}(0)$, fix $n \geqslant n_0$. Let $\Sigma = \{e_1, \ldots, e_m, \ldots\}$ be an orthonormal base of the Hilbert space $H^1(\Omega_n)$. For each $m \in \mathbb{N}$, define the following subspace

$$V_m = [e_1, \ldots, e_m],$$

that is, V_m is an m-dimensional space generated by the orthonormal set $\{e_1,\dots,e_m\}$. It is well known that $(V_m, \|\cdot\|_{H^1(\Omega_n)})$ and $(\mathbb{R}^m, |\cdot|)$ are isomorphic by the natural linear transformation $T: V_m \to \mathbb{R}^m$ given by

$$v=\sum_{i=1}^{m}\gamma_i e_i \to T(v)=\gamma=(\gamma_1,\dots,\gamma_m)$$

which also satisfies

$$\|v\|_{H^1(\Omega_n)}=\bigl|T(v)\bigr|=|\gamma|$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}^m$. We will use the identification

$$\gamma\mapsto\sum_{i=1}^{m}\gamma_i e_i=v.$$

Now, consider the compact linear operator given by

$$\mathfrak{T}: W^{1,q}(\Omega)\to L^p(\partial\Omega),$$

with $1<p<q^*=\frac{q(N-1)}{N-q}$ if $q<N$ (see [4, Theorems 7.53–7.57]) and define the function $F=(F_1,\dots,F_m):\mathbb{R}^m\to\mathbb{R}^m$ by

$$F_i(\gamma)=\int\limits_{\Omega_n}\xi(x)\nabla v\nabla e_i\,dx+\int\limits_{\Omega_n}ve_i\,dx+\int\limits_{\Gamma_n}\alpha(x,\widetilde{v})\widetilde{v}e_i\,ds-\int\limits_{\Omega_n}\lambda\bigl(h(x,v)+g(x,\nabla v)\bigr)e_i\,dx,$$

where $\Gamma_n=:\partial\Omega_n$ and $\widetilde{v}\equiv\mathfrak{T}(v)$.

Taking $K=\min\{k_0,1\}$, we have that,

$$\begin{aligned}\bigl\langle F(\gamma),\gamma\bigr\rangle&=\int\limits_{\Omega_n}\xi(x)|\nabla v|^2\,dx+\int\limits_{\Omega_n}|v|^2\,dx+\int\limits_{\Gamma_n}\alpha(x,\widetilde{v})|\widetilde{v}|^2\,ds-\int\limits_{\Omega_n}\lambda\bigl(h(x,v)v+g(x,\nabla v)v\bigr)\,dx\\&\geqslant K\|v\|^2_{H^1(\Omega_n)}-|\lambda||a_1|_{L^2(\Omega_n)}|v|_{L^2(\Omega_n)}-|\lambda||a_2|_{L^{\frac{2}{1-r_1}}(\Omega_n)}|v|^{r_1+1}_{L^2(\Omega_n)}\\&\quad-|\lambda||a_3|_{L^2(\Omega_n)}|v|_{L^2(\Omega_n)}-|\lambda|\biggl(\int\limits_{\Omega_n}a_4^2|\nabla v|^{2r_2}\,dx\biggr)^{\frac12}|v|_{L^2(\Omega_n)}\\&\geqslant K\|v\|^2_{H^1(\Omega_n)}-c_1|a_1|_{L^2(\Omega_n)}\|v\|_{H^1(\Omega_n)}-c_2|a_2|_{L^{\frac{2}{1-r_1}}(\Omega_n)}\|v\|^{r_1+1}_{H^1(\Omega_n)}\\&\quad-c_3|a_3|_{L^2(\Omega_n)}\|v\|_{H^1(\Omega_n)}-c_4|a_4|_{L^{\frac{2}{1-r_2}}(\Omega_n)}\|v\|^{r_2+1}_{H^1(\Omega_n)},\end{aligned}$$

where $c_i\in\mathbb{R}_+^*$, $i=1,\dots,4$, are independent of m. Therefore,

$$\bigl\langle F(\gamma),\gamma\bigr\rangle\geqslant K|\gamma|^2-c_1|a_1|_{L^2(\Omega_n)}|\gamma|-c_2|a_2|_{L^{\frac{2}{1-r_1}}(\Omega_n)}|\gamma|^{r_1+1}-c_3|a_3|_{L^2(\Omega_n)}|\gamma|-c_4|a_4|_{L^{\frac{2}{1-r_2}}(\Omega_n)}|\gamma|^{r_2+1}.$$

Since $r_j+1<2$, $j=1,2$, it follows that we may choose $\rho, r>0$ independently of m, such that

$$\bigl\langle F(\gamma),\gamma\bigr\rangle\geqslant r>0\quad\text{on } |\gamma|=\rho.$$

By Lemma 2.1, since F is a continuous function, for each $m\in\mathbb{N}$ there exists $\gamma_{m,n}\in\mathbb{R}^m$ (not identically null number, due to hypothesis (H_2)) verifying

$$F(\gamma_{m,n})=0,\quad|\gamma_{m,n}|\leqslant\rho.\tag{2.1}$$

In the following pages, we use

$$v_{m,n}\in V_m\subset H^1(\Omega_n)\quad\text{such that } T(v_{m,n})=\gamma_{m,n}.$$

Hence, $\|v_{m,n}\|_{H^1(\Omega_n)}\leqslant\rho$, $\forall m,n\in\mathbb{N}$. Thus

$$\int\limits_{\Omega_n}\xi(x)\nabla v_{m,n}\nabla\omega\,dx+\int\limits_{\Omega_n}v_{m,n}\omega\,dx+\int\limits_{\Gamma_n}\alpha(x,\widetilde{v}_{m,n})\widetilde{v}_{m,n}\omega\,ds=\int\limits_{\Omega_n}\lambda\bigl(h(x,v_{m,n})\omega+g(x,\nabla v_{m,n})\omega\bigr)\,dx,\tag{2.2}$$

$\forall\omega\in V_k$, and $m\geqslant k$.

Moreover, passing to a subsequence if necessary, we can assume that there exists $v_n \in H^1(\Omega_n)$ such that

$$v_{m,n} \rightharpoonup v_n \quad \text{in } H^1(\Omega_n),$$
$$v_{m,n}(x) \to v_n(x) \quad \text{a.e. in } \Omega_n, \text{ as } m \to \infty.$$

The claim below is a key point to conclude the proof of this theorem.

Claim 2.1. *The sequence* $\{v_{m,n}\}_m$ *is strongly convergent to* v_n *in* $H^1(\Omega_n)$.

Admitting for now that the claim is true, passing to the limit as $m \to \infty$ in equality (2.2), from the Sobolev trace embedding (see [4, Theorems 7.53–7.57]), it follows that

$$\int_{\Omega_n} \xi(x)\nabla v_n \nabla\omega\,dx + \int_{\Omega_n} v_n\omega\,dx + \int_{\Gamma_n} \alpha(x,\widetilde{v}_n)\widetilde{v}_n\omega\,ds = \int_{\Omega_n} \lambda\big(h(x,v_n)\omega + g(x,\nabla v_n)\omega\big)\,dx, \quad \forall\omega \in V_k. \tag{2.3}$$

For each $\phi \in H^1(\Omega_n)$, there exists $\{\gamma_i\} \subset \mathbb{R}$ verifying $\phi = \sum_{i=1}^{\infty} \gamma_i e_i$, and therefore, the sequence

$$\phi_k = \sum_{i=1}^{k} \gamma_i e_i \in V_k,$$

is strongly convergent to ϕ in $H^1(\Omega_n)$. Putting $w = \phi_k$ in (2.3) and taking the limit as $k \to \infty$ together with the compactness of the imbedding $H^1(\Omega) \hookrightarrow L^p(\Gamma)$, we obtain

$$\int_{\Omega_n} \xi(x)\nabla v_n \nabla\phi\,dx + \int_{\Omega_n} v_n\phi\,dx + \int_{\Gamma_n} \alpha(x,\widetilde{v}_n)\widetilde{v}_n\phi\,ds = \int_{\Omega_n} \lambda\big(h(x,v_n)\phi + g(x,\nabla v_n)\phi\big)\,dx. \tag{2.4}$$

From the above study, we prove the existence of a weak solution v_n of $(P_n)_\lambda$ and the proof of Theorem 2.1 is finished. □

Proof of Claim 2.1. Using the convergence $v_{m,n} \rightharpoonup v_n$ in $H^1(\Omega_n)$, hypothesis (H_2), and the Dominated Convergence Theorem, it follows that

$$\int_{\Omega_n} \xi(x)\nabla v_{m,n}\nabla\omega\,dx \to \int_{\Omega_n} \xi(x)\nabla v_n\nabla\omega\,dx, \tag{2.5}$$

$$\int_{\Omega_n} v_{m,n}\omega\,dx \to \int_{\Omega_n} v_n\omega\,dx, \tag{2.6}$$

$$\int_{\Omega_n} h(x,v_{m,n})\omega\,dx \to \int_{\Omega_n} h(x,v_n)\omega\,dx. \tag{2.7}$$

To verify the convergence

$$\int_{\Gamma_n} \alpha(x,\widetilde{v}_{m,n})\widetilde{v}_{m,n}\omega\,ds \to \int_{\Gamma_n} \alpha(x,\widetilde{v}_n)\widetilde{v}_n\omega\,ds, \tag{2.8}$$

note that Sobolev trace immersion $W^{1,2}(\Omega_n) \hookrightarrow L^p(\Gamma_n)$ for $1 < p < \frac{2(N-1)}{N-2}$ is compact. Hence, for almost every $x \in \Gamma_n$, we have that $\widetilde{v}_{m,n} \to \widetilde{v}_n$ and a function $\Psi \in L^p(\Gamma_n)$ exists such that $|\widetilde{v}_{m,n}| \leqslant \Psi$. By hypothesis (H_4) and by the observation below, we obtain that $|\alpha(x,\widetilde{v}_{m,n})\widetilde{v}_{m,n}\omega| \leqslant b(x)\Psi^{p-1}|\omega|$, a.e. $x \in \Gamma_n$. Since $\alpha(x,\mu)\mu$ is continuous, by the Dominated Convergence Theorem, (2.8) holds.

From now on, for each $m \in \mathbb{N}$, we consider the function

$$G_m(x) := g\big(x, \nabla v_{m,n}(x)\big).$$

From (H_3),

$$|G_m|_{L^{\frac{2N}{(N+2)r_2}}(\Omega_n)} \leqslant |a_3|_{L^{\frac{2N}{(N+2)r_2}}(\Omega_n)} + \left(\int_{\Omega_n} \big|a_4(x)\big|^{\frac{2N}{(N+2)r_2}} |\nabla v_{m,n}|^{\frac{2N}{N+2}}\,dx\right)^{\frac{(N+2)r_2}{2N}}. \tag{2.9}$$

Using (2.1) and Hölder's inequality with exponents $\frac{N+2}{N}$ and $\frac{N+2}{2}$, we get the estimate

$$\begin{aligned}|G_m|_{L^{\frac{2N}{(N+2)r_2}}(\Omega_n)} &\leqslant |a_3|_{L^{\frac{2N}{(N+2)r_2}}(\Omega_n)} + |a_4|_{L^{\frac{N}{r_2}}(\Omega_n)} |\nabla v_{m,n}|^{r_2}_{L^2(\Omega_n)} \\ &\leqslant c_1 + c_2\rho^{r_2}.\end{aligned} \tag{2.10}$$

Since $L^{\frac{2N}{(N+2)r_2}}(\Omega_n)$ is reflexive, up to a subsequence, there exists $G \in L^{\frac{2N}{(N+2)r_2}}(\Omega_n)$ such that $G_m \rightharpoonup G$ in $L^{\frac{2N}{(N+2)r_2}}(\Omega_n)$, that is,

$$\int_{\Omega_n} G_m\varphi\,dx \to \int_{\Omega_n} G\varphi\,dx, \quad \forall \varphi \in L^{\theta}(\Omega_n), \tag{2.11}$$

where $\frac{1}{\theta} + \frac{(N+2)r_2}{2N} = 1$.

Since $\theta < 2^*$, by embedding $H^1(\Omega_n) \hookrightarrow L^{\theta}(\Omega_n)$, and by (2.5)–(2.8) it follows arguing as in (2.4), that

$$\int_{\Omega_n} \xi(x)\nabla v_n \nabla\phi\,dx + \int_{\Omega_n} v_n\phi\,dx + \int_{\Gamma_n} \alpha(x, \widetilde{v}_n)\widetilde{v}_n\phi\,ds - \int_{\Omega_n} \lambda h(x, v_n)\phi\,dx - \int_{\Omega_n} \lambda G(x)\phi\,dx = 0, \tag{2.12}$$

for all $\phi \in H^1(\Omega_n)$. Now, notice that

$$\int_{\Omega_n} \xi(x)|\nabla v_{m,n} - \nabla v_n|^2\,dx = \int_{\Omega_n} \xi(x)|\nabla v_{m,n}|^2\,dx - \int_{\Omega_n} \xi(x)\nabla v_{m,n}\nabla v_n\,dx - \int_{\Omega_n} \xi(x)\nabla v_n\nabla(v_{m,n} - v_n)\,dx.$$

Since $v_{m,n} \rightharpoonup v_n$ in $H^1(\Omega_n)$, we have

$$\int_{\Omega_n} \xi(x)\nabla v_n\nabla(v_{m,n} - v_n)\,dx = o_m(1).$$

By the equalities (2.2) and (2.12) and by the convergence (2.8), we have

$$K\|v_{m,n} - v_n\|^2_{H^1(\Omega_n)} \leqslant \int_{\Omega_n} \lambda\big(h(x, v_{m,n}) - h(x, v_n)\big)v_{m,n}\,dx - \int_{\Omega_n} \lambda\big(G_m(x) - G(x)\big)v_{m,n}\,dx + o_m(1).$$

Using the weak convergence $v_{m,n} \rightharpoonup v_n$ in $H^1(\Omega_n)$, we can prove that the limit, as $m \to +\infty$, of each of the right terms of the last inequality goes to zero, and, therefore,

$$\|v_{m,n} - v_n\|^2_{H^1(\Omega_n)} \to 0.$$

So, $v_{m,n} \to v_n$ in $H^1(\Omega_n)$, and this conclude the proof of Claim 2.1. □

3. Proof of Theorem 1.1

The case (ii). This proof is immediate.

The cases (i) and (iii). Under assumption (H_1)–(H_4), we will obtain a solution of the problem (P_λ) for a $\lambda \neq 0$, by combining the result obtained in Theorem 2.1 with an *a priori* estimate and diagonal argument.

In what follows, v_n is a weak solution of the problem $(P_n)_\lambda$ obtained in Theorem 2.1, and let Ω_n be as already defined. Notice that

$$\int_{\Omega_n} \xi(x)|\nabla v_n|^2\,dx + \int_{\Omega_n} |v_n|^2\,dx + \int_{\Gamma_n} \alpha(x, v_n)|v_n|^2\,ds = \int_{\Omega_n} \lambda h(x, v_n)v_n\,dx + \int_{\Omega_n} \lambda g(x, \nabla v_n)v_n\,dx. \tag{3.1}$$

Since $K = \min\{k_0, 1\}$, by hypotheses (H_2)–(H_3) and Sobolev embedding, we have

$$\begin{aligned}K\|v_n\|^2_{H^1(\Omega_n)} &\leqslant K\|v_n\|^2_{H^1(\Omega_n)} + \int_{\Gamma_n} \alpha(x, v_n)|v_n|^2\,ds \\ &\leqslant |\lambda||a_1|_{L^2(\Omega_n)}|v_n|_{L^2(\Omega_n)} + |\lambda||a_2|_{L^{\frac{2}{1-r_1}}(\Omega_n)}|v_n|^{r_1+1}_{L^2(\Omega_n)} \\ &\quad + |\lambda||a_3|_{L^2(\Omega_n)}|v_n|_{L^2(\Omega_n)} + |\lambda|\bigg(\int_{\Omega_n} a_4^2|\nabla v_n|^{2r_2}\,dx\bigg)^{\frac{1}{2}}|v_n|_{L^2(\Omega_n)} \\ &\leqslant \widetilde{C}\big(\|v_n\|_{H^1(\Omega_n)} + \|v_n\|^{r_1+1}_{H^1(\Omega_n)} + \|v_n\|^{r_2+1}_{H^1(\Omega_n)}\big).\end{aligned}$$

This, together with the fact that $r_1, r_2 \in (0,1)$ implies that there exists a constant $c > 0$, independent of n, such that

$$\|v_n\|_{H^1(\Omega_n)} < c. \tag{3.2}$$

If $n > m$, by (3.2) we have

$$\|v_n\|_{H^1(\Omega_m)} \leqslant \|v_n\|_{H^1(\Omega_n)} < c. \tag{3.3}$$

Now, let be $m \in \mathbb{N}$ fixed. If $n > m$, by (3.3) (up to a subsequence) there exists $v \in H^1(\Omega_m)$ verifying

$$v_n(x) \to v(x), \quad \text{a.e. } x \in \Omega_m,$$
$$v_n \rightharpoonup v \quad \text{in } H^1(\Omega_m)$$

and

$$v \neq 0 \quad \text{by hypothesis } (H_2).$$

Now, notice that

$$\begin{aligned} K\|v_n - v\|^2_{H^1(\Omega_m)} &\leqslant \int_{\Omega_m} \xi(x)|\nabla v_n|^2\,dx + \int_{\Omega_m} |v_n|^2\,dx - \int_{\partial\Omega_m} \xi(x)\partial_\nu v_n v_n\,ds \\ &\quad - \left(\int_{\Omega_m} \xi(x)\nabla v_n \nabla v\,dx + \int_{\Omega_m} v_n v\,dx - \int_{\partial\Omega_m} \xi(x)\partial_\nu v_n v\,ds\right) + \int_{\partial\Omega_m} \xi(x)\partial_\nu v_n(v_n - v)\,ds + o(1) \\ &= \int_{\Omega_m} f(x, v_n, \nabla v_n)(v_n - v)\,dx - \int_{\partial\Omega_m} \xi(x)\partial_\nu v_n(v_n - v)\,ds + o(1). \end{aligned}$$

Using Hölder inequality together with Schauder estimates and trace embedding theorems (see [4]) we have

$$\|v_n - v\|^2_{H^1(\Omega_m)} \to 0.$$

So we obtain that v is a weak solution of the problem

$$\begin{cases} -div\big(\xi(x)\nabla v\big) + v = \lambda f(x, v, \nabla v) & \text{in } \Omega_m, \\ \xi(x)\partial_\nu v + \alpha(x, v)v = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Now, by a diagonal argument, there exists a subsequence (still denoted by $\{v_n\}$) and a function v such that

$$v_n(x) \to v(x), \quad \text{a.e. } x \in \Omega,$$
$$v_n \to v \quad \text{in } H^1_{\text{loc}}(\Omega),$$

and again using (H_2), the last limits imply that v is a non-trivial solution to the problem (P_λ) with $\lambda \neq 0$.

Now, we will show that this solution belongs to $H^1(\Omega) \cap C^2(\Omega)$.

Let E be a bounded subset of Ω, and let $m_0 \in \mathbb{N}$ be such that if $n \geqslant m_0$ then $E \subset \Omega_n$. We have

$$\int_E h(x, v_n)v_n\,dx \leqslant \int_E |a_1||v_n|\,dx + \int_E |a_2||v_n|^{r_1+1}\,dx.$$

By Young's inequality, for each $\epsilon > 0$, $C_1(\epsilon), C_2(\epsilon) > 0$ exist such that

$$|a_1||v_n| \leqslant C_1(\epsilon)|a_1|^2 + \epsilon|v_n|^2$$

and

$$|a_2||v_n|^{r_1+1} \leqslant C_2(\epsilon)|a_2|^{\frac{2}{1-r_1}} + \epsilon|v_n|^2.$$

Fixing $\epsilon = \frac{K}{8|\lambda|}$, we get

$$\int_E h(x, v_n)v_n\,dx \leqslant \frac{1}{4}\int_E |v_n|^2\,dx + C_1\int_E |a_1|^2\,dx + C_2\int_E |a_2|^{\frac{2}{1-r_1}}\,dx. \tag{3.4}$$

Applying the same argument, we obtain

$$g(x, \nabla v_n)v_n \leqslant \frac{K}{2|\lambda|}|\nabla v_n|^2 + C_3|a_4|^{\frac{2}{1-r_2}} + \frac{K}{4|\lambda|}|v_n|^2 + C_4|a_3|^2,$$

and so

$$\int_E g(x,\nabla v_n)v_n\,dx \leqslant \frac{K}{2|\lambda|}\int_E |\nabla v_n|^2\,dx + C_3\int_E |a_4|^{\frac{2}{1-r_2}}\,dx + \frac{K}{4|\lambda|}\int_E |v_n|^2\,dx + C_4\int_E |a_3|^2\,dx. \tag{3.5}$$

Since v_n is a weak solution to the problem $(P_n)_\lambda$ and $E\subset\Omega_n$, by (3.4) and (3.5), it follows that

$$\int_E \left(|\nabla v_n|^2+|v_n|^2\right)dx \leqslant \int_{\Omega_n}\left(|\nabla v_n|^2+|v_n|^2\right)dx \leqslant C\left(\int_{\Omega_n}|a_1|^2\,dx+\int_{\Omega_n}|a_2|^{\frac{2}{1-r_1}}\,dx+\int_{\Omega_n}|a_4|^{\frac{2}{1-r_2}}\,dx+\int_{\Omega_n}|a_3|^2\,dx\right).$$

Using the hypotheses that $a_1,a_3\in L^2(\mathbb{R}^N)$, $a_2\in L^{\frac{2}{1-r_1}}(\mathbb{R}^N)$ and $a_4\in L^{\frac{2}{1-r_2}}(\mathbb{R}^N)$, the previous inequality implies that there exists $C>0$ such that

$$\int_E \left(|\nabla v_n|^2+|v_n|^2\right)dx\leqslant C \quad \text{for all } E \text{ subset of } \Omega,$$

from where it follows that

$$\|v\|_{H^1(E)}\leqslant C,$$

consequently $v\in H^1(\Omega)$ and therefore

$$\lim_{|x|\to\infty} v(x)=0.$$

By arguments similar to those used in Lemma 2.2, we can conclude that the solution v belongs to $H^1(\Omega)\cap C^2(\Omega)$.

To conclude the proof of Theorem 1.1, suppose by contradiction that the problem (P_λ) with $\lambda<0$ possesses a positive solution u. Then we have

$$\int_\Omega \xi(x)|\nabla u|^2\,dx+\int_\Omega u^2\,dx+\int_\Gamma \alpha(x,u)u^2\,ds=\lambda\int_\Omega f(x,u,\nabla u)u\,dx.$$

Since $f(x,u,\nabla u)>0$, we obtain

$$\|u\|_{H^1(\Omega)}+\int_\Gamma \alpha(x,u)u^2\,ds=0$$

and therefore $u=0$, which is a contradiction. Now, if $\lambda>0$ and u is a negative solution of the problem (P_λ), using the same ideas employed above, again we get a contradiction. This proves (i) and (iii), and consequently, the proof of Theorem 1.1 is finished.

4. Final comments

Our result still holds if the hypothesis (H_1) is replaced by:

$(H_1)'$ The functions $\xi:\Omega\to\mathbb{R}$, $h:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Omega\times\mathbb{R}^N\to\mathbb{R}$ are locally Hölder continuous where ξ is a positive function for all $x\in\Omega$.

In this case, we should look for a solution of the problem (P_λ), in a Sobolev space $X\cap C^2(\Omega)$, where X is the completion of $C_0^\infty(\mathbb{R}^N)$, restricted to Ω, with respect to the norm

$$\|u\|_\xi=\left(\int_\Omega\left(\xi(x)|\nabla u|^2+|u|^2\right)dx\right)^{\frac{1}{2}}.$$

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