

On positive solutions for a class of singular quasilinear elliptic systems

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Abstract

We study through the lower and upper-solution method, the existence of positive weak solution to the quasilinear elliptic system with weights

$$\begin{cases} -\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) = \lambda |x|^{-(a+1)p+c_1} u^\alpha v^\gamma & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-bq} |\nabla v|^{q-2} \nabla v) = \lambda |x|^{-(b+1)q+c_2} u^\delta v^\beta & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $0 \leq \alpha < p-1$, $0 \leq \beta < q-1$, $\delta, \gamma, c_1, c_2 > 0$ and $\theta := (p-1-\alpha)(q-1-\beta) - \gamma\delta > 0$, for each $\lambda > 0$.
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1. Introduction

In this paper, we will study through the lower and upper-solution method, the existence of positive weak solution to the quasilinear elliptic system with weights

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda|x|^{-(a+1)p+c_1}u^\alpha v^\gamma & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v) = \lambda|x|^{-(b+1)q+c_2}u^\delta v^\beta & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of \mathbb{R}^N , satisfying the interior sphere condition; that is, for each $x_0 \in \partial\Omega$ there exists $B(y_0, r) \subset \Omega$ such that $x_0 \in \partial B(y_0, r)$; $0 \in \Omega$, $1 < p, q < N$, $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $\delta, \gamma, c_1, c_2 > 0$ and $\theta := (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$, for each $\lambda > 0$. Here, $B(y_0, r)$ denotes an open ball centered at y_0 with radius r and $\partial B(y_0, r)$ stands for the boundary of $B(y_0, r)$.

The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [16]). On the other hand, the qualitative theory for elliptic problems has been extensively studied because of various applications in mathematical physics and rich mathematical structure. In particular, the quasilinear elliptic systems are used in the study of the population dynamics. In this case, a weak solution (u, v) , where each component is nontrivial, nonnegative and continuous, it is called a ‘‘coexistence state,’’ for instance see [11,12] and [13].

Even in the regular case, that is, when $a = b = 0$ and $c_1 = p = c_2 = q$, the quasilinear elliptic equations involving p -laplacian operator have attracted much attention in late years and we would like to mention the following papers [1,4,5,18] and [28]. In those works the nonlinearities have subcritical and critical growth at infinity and they behave like a function s^d ($d \geq p - 1$) at the origin. Roughly speaking, in this case we say that the nonlinearities are convex or that is ‘‘fast diffusion’’ case (see [11]).

Still in the regular situation, when the nonlinearities are concave or in the case ‘‘slow diffusion,’’ Díaz and Saa in [19] gave a necessary and sufficient condition to the existence of positive solution in the scalar case, that is, for a class of quasilinear elliptic equations involving p -laplacian operator. By using the lower and upper solution method, Hai and Shivaji in [23], treated a class of elliptic systems involving the p -laplacian operator with concave nonlinearities. Recently, Chen in [14], by using the same approach, got some existence and nonexistence results for a class of systems, namely, system (1.1) with $a = b = 0$, $c_1 = p$ and $c_2 = q$.

The aim of this work is to extend or complement some of the above results for the quasilinear elliptic systems involving singularity. For the quasilinear elliptic equations involving singularity, that is, in the scalar case see, e.g., [3,6–10,15–17,20–22,29,31,32] and references therein. Because of the singularity in the weights, in addition to work in a Banach space framework instead of in a Hilbert space it requires a careful analysis, for example, to get regularity results as well as the strong maximum principle and comparison principle result (see also [6,25,27] and [30]). Mainly to obtain the behavior of the eigenfunction associated to the first eigenvalue related to our operator. Finally, we recall that the system studied here, also in [14,23], are neither variational nor hamiltonian type, bringing an additional difficult.

Our main result is as follows:

Theorem 1.1. *Suppose that Ω is a bounded smooth domain of \mathbb{R}^N satisfying the interior sphere condition, $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $c_1, c_2 > 0$, $0 \leq \alpha < p - 1$,*

$0 \leq \beta < q - 1$, $\delta, \gamma > 0$ and $\theta := (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$. Then system (1.1) possesses a weak solution $(\psi_1, \psi_2) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$, where each component is positive and belongs to $C^{0,\rho}(\overline{\Omega}) \cap C^{1,\mu}(\Omega \setminus \{0\})$, for some $\rho \in (0, 1]$, $\mu > 0$, and each $\lambda > 0$.

Next we will establish a nonexistence result for our system.

Theorem 1.2. *Suppose that Ω is a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, $c_1, c_2, \delta, \gamma > 0$, $\theta = 0$, $p\gamma = q(p - 1 - \alpha)$ and $(a + 1)p - c_1 = (b + 1)q - c_2$. Then there exists $\lambda_0 > 0$ such that system (1.1) does not possess any weak solution, for all $0 < \lambda < \lambda_0$, where each component is nontrivial and nonnegative.*

Notations. In the rest of this paper Ω is a bounded smooth domain of \mathbb{R}^N , containing the origin and we will make use of the following notations: $\int_{\Omega} f(x) dx$ and $\int_{\mathbb{R}^N} f(x) dx$ will be denoted by $\int_{\Omega} f$ and $\int_{\mathbb{R}^N} f$, respectively. The constant C denotes (possibly different) positive constants. We say that a pair (u_1, u_2) is positive (respectively nontrivial, bounded, nonnegative) if and only if u_i is positive (respectively nontrivial, bounded, nonnegative), for $i = 1, 2$.

Remark 1.1. It is well known that all bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega \in C^k$ ($k \geq 2$) satisfies the interior sphere condition, see [2, Lemma 2.2].

2. Lower and upper-solution theorem

Our main tool will be a general method of lower and upper-solution. This method, in the scalar situation, has been used by many authors; for instance, [6, 11, 14, 24] and [26]. The proof for the system case follows arguing as in [11] when $a = b = 0$, and $p = q = c_1 = c_2$.

Consider the system:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c_1}h(x, u, v) & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v) = |x|^{-(b+1)q+c_2}k(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p, q < N$, $-\infty < a < \frac{N-p}{p}$, $-\infty < b < \frac{N-q}{q}$, $c_1, c_2 > 0$ and $h, k : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

(HK1) $h(x, s, t)$, $k(x, s, t)$ are Carathéodory functions and they are bounded if s, t belong to bounded sets.

(HK2) There exists a function $g : \mathbb{R} \rightarrow \mathbb{R}$ continuous, nondecreasing, with $g(0) = 0$, $0 \leq g(s) \leq C(1 + |s|^{r-1})$, $\forall s \in \mathbb{R}$, where $r = \min\{p, q\}$ for some $C > 0$, and the applications $s \mapsto h(x, s, t) + g(s)$, $t \mapsto k(x, s, t) + g(t)$ are nondecreasing, for a.e. $x \in \Omega$.

Now, we are setting some spaces and their norms. If $\alpha \in \mathbb{R}$ and $l \geq 1$, we define $L^l(\Omega, |x|^\alpha)$ as being the subspace of $L^l(\Omega)$, of the Lebesgue measurable functions $u : \Omega \rightarrow \mathbb{R}$, satisfying

$$\|u\|_{L^l(\Omega, |x|^\alpha)} := \left(\int_{\Omega} |x|^\alpha |u|^l \right)^{\frac{1}{l}} < \infty.$$

If $1 < p < N$ and $-\infty < a < \frac{N-p}{p}$, we define $W^{1,p}(\Omega, |x|^{-ap})$ (respectively $W_0^{1,p}(\Omega, |x|^{-ap})$) as being the closure of $C^\infty(\Omega)$ (respectively $C_0^\infty(\Omega)$), with respect to the norm $\|\cdot\|$ defined by

$$\|u\| := \left(\int_{\Omega} |x|^{-ap} |\nabla u|^p \right)^{\frac{1}{p}}.$$

The following Sobolev–Hardy inequality with weights was proved by Caffarelli, Kohn and Nirenberg in [7], which is called the Caffarelli–Kohn–Nirenberg inequality. Supposing that $1 < p < N$, there exists $C_{a,e} > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |x|^{-ep^*} |u|^{p^*} \right)^{\frac{p}{p^*}} \leq C_{a,e} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p,$$

where $-\infty < a < \frac{N-p}{p}$, $a \leq e \leq a + 1$, $d = 1 + a - e$ and $p^* := p^*(a, e) = \frac{Np}{N-dp}$.

The following version of the strong maximum principle is obtained by applying Pucci and Serrin result, more exactly, [27, Theorem 8.1].

Theorem 2.1 (Strong Maximum Principle). Consider Ω a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$ and $\Psi : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ a nonnegative Carathéodory function. If $u \in W_0^{1,p}(\Omega, |x|^{-ap}) \cap C^0(\Omega) \cap C^1(\Omega \setminus \{0\})$, and $u \geq 0$ satisfies

$$\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) + \Psi(x, u, \nabla u) \leq 0, \quad \forall x \in \Omega,$$

then either $u \equiv 0$ or $u > 0$ in Ω .

Proof. Suppose that $u \neq 0$. Since $\Psi(x, s, \eta) \geq 0$, we have

$$\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) \leq 0, \quad \forall x \in \Omega.$$

Consider $R > 0$ such that $B(0, R) \subset \Omega$ and $u \neq 0$ in $\overline{\Omega} = \Omega \cup \overline{B(0, R/2)}$. Applying the Pucci–Serrin strong maximum principle theorem [27, Theorem 8.1] for $\overline{\Omega}$, we obtain $u > 0$ in $\Omega \setminus \overline{B(0, R/2)}$.

Notice that, there exists $\delta > 0$ with $\delta \leq u(x)$ for all $x \in \partial B(0, R)$, because u is continuous and positive in $\Omega \setminus \overline{B(0, R/2)}$. Moreover, if $\phi \in C_0^\infty(B(0, R))$, $\phi \geq 0$ and defining $\bar{u} = u|_{B(0, R)}$, $v \equiv \delta$ in $B(0, R)$, we get

$$\int_{B(0, R)} |x|^{-ap} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \phi \geq \int_{B(0, R)} |x|^{-ap} |\nabla v|^{p-2} \nabla v \nabla \phi$$

and by the comparison principle theorem (see [6, Lemma 3.2]), we obtain $0 < \delta = v(x) \leq u(x)$ for a.e. $x \in B(0, R)$. Since $u \in C^0(\Omega)$, it follows that $u \geq \delta$ in $B(0, R)$ and $u > 0$ in Ω . \square

Xuan in [32] proved that, if Ω is a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p < N$, $0 \leq a < \frac{N-p}{p}$ and $c_1 > 0$, then there exists the first eigenvalue $\lambda_1 > 0$ of problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) = \lambda |x|^{-(a+1)p+c_1} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

which is associated to an eigenfunction $\phi_1 \in C^{1,\alpha_1}(\Omega \setminus \{0\})$, with $\phi_1 > 0$ in $\Omega \setminus \{0\}$, for some $\alpha_1 > 0$.

The following result is crucial in the study the behavior of the first eigenfunction.

Theorem 2.2. *Suppose that Ω is a bounded smooth domain of \mathbb{R}^N satisfying the interior sphere condition, $0 \in \Omega$, $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $c > 0$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ a nonnegative function. If $u \in W_0^{1,p}(\Omega, |x|^{-ap}) \cap C^{1,\mu}(\overline{\Omega} \setminus \{0\})$, with $u > 0$ in Ω and $\mu > 0$, is a weak solution of problem*

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c} f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$|\nabla u(x)| \geq \sigma$$

for some $\sigma > 0$ and every $x \in \partial\Omega$.

Proof. Consider $x_0 \in \partial\Omega$. Since $\partial\Omega$ satisfies the interior sphere condition, there exists $B(y_0, r) \subset \Omega$ such that $x_0 \in \partial B(y_0, r)$. Also, we can suppose $B(y_0, r) \subset \Omega \setminus \overline{B(0, R)}$, for some $R > 0$, with $B(0, R) \subset \Omega$.

Define the function $b : \mathbb{R}^N \rightarrow \mathbb{R}$ given by $b(x) = k(e^{-\alpha|x-y_0|^2} - e^{-\alpha r^2})$, where α, k are positive constants that will be fixed later.

Firstly, we will prove that

$$-\operatorname{div}(|x|^{-ap}|\nabla b|^{p-2}\nabla b) \leq 0, \quad \forall x \in B(y_0, r) \setminus \overline{B(y_0, r/3)}, \tag{2.3}$$

if $\alpha > 0$ is sufficiently large and independent of $k > 0$.

In fact, it is not difficult to check that there exist positive constants γ_0 and γ_1 verifying the following inequalities

$$\sum_{i,j=1}^N \left| \frac{\partial}{\partial \eta_i} (|\eta|^{p-2} \eta_j) \right| \leq \gamma_0 |\eta|^{p-2}, \quad \forall \eta \in \mathbb{R}^N \setminus \{0\}, \tag{2.4}$$

and

$$\sum_{i,j=1}^N \frac{\partial}{\partial \eta_i} (|\eta|^{p-2} \eta_j) \xi_i \xi_j \geq \gamma_1 |\eta|^{p-2} |\xi|^2, \quad \forall \eta, \xi \in \mathbb{R}^N, \text{ with } \eta \neq 0. \tag{2.5}$$

Notice that

$$b(x) = 0, \quad \forall x \in \partial B(y_0, r),$$

moreover, we obtain

$$\frac{\partial b}{\partial x_i}(x) = -2\alpha k(x_i - y_{0i})e^{-\alpha|x-y_0|^2} \tag{2.6}$$

and

$$\frac{\partial^2 b}{\partial x_j \partial x_i}(x) = -2\alpha k e^{-\alpha|x-y_0|^2} \delta_{ij} + 4\alpha^2 k(x_i - y_{0i})(x_j - y_{0j})e^{-\alpha|x-y_0|^2} \tag{2.7}$$

for all $x \in \mathbb{R}^N$.

We observe that, there exist $K_1, K_2 > 0$, such that

$$K_1 \leq |x|, \quad |\nabla b|^{p-2}, \quad e^{-\alpha|x-y_0|^2} \leq K_2, \quad \forall x \in B(y_0, r) \setminus \overline{B(y_0, r/3)}. \quad (2.8)$$

Then, by using (2.4)–(2.8), we infer that

$$\operatorname{div}(|x|^{-ap}|\nabla b|^{p-2}\nabla b) \geq 0, \quad \forall x \in B(y_0, r) \setminus \overline{B(y_0, r/3)},$$

for $\alpha > 0$ sufficiently large and independent of $k > 0$. This proves (2.3).

Hence, from (2.3) if $\phi \in C_0^\infty(B(y_0, r) \setminus \overline{B(y_0, r/2)})$, with $\phi \geq 0$, we have

$$\int_{B(y_0, r) \setminus \overline{B(y_0, r/2)}} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla \phi \geq \int_{B(y_0, r) \setminus \overline{B(y_0, r/2)}} |x|^{-ap}|\nabla b|^{p-2}\nabla b \nabla \phi.$$

On the other hand, since $u \in C^{1,\mu}(\overline{\Omega} \setminus \{0\})$ and $u > 0$ in Ω , there exists $\delta > 0$ such that

$$\delta \leq u(x), \quad \forall x \in \partial B(y_0, r/2),$$

and choosing $k > 0$ sufficiently small, we get

$$b(x) \leq \delta \leq u(x), \quad \forall x \in \partial B(y_0, r/2).$$

Consequently

$$b(x) \leq u(x)$$

for all $x \in \partial(B(y_0, r) \setminus \overline{B(y_0, r/2)}) = \partial B(y_0, r) \cup \partial B(y_0, r/2)$.

Applying the comparison principle theorem (see [6, Lemma 3.2]), we obtain $b(x) - u(x) \leq 0$ for each $x \in B(y_0, r) \setminus \overline{B(y_0, r/2)}$. Moreover $b(x_0) - u(x_0) = 0$, then

$$\frac{\partial b(x_0)}{\partial \nu} - \frac{\partial u(x_0)}{\partial \nu} \geq 0,$$

where $\nu : \partial(B(y_0, r)) \rightarrow \mathbb{R}^N$, given by $\nu(x) = \frac{x-y_0}{|x-y_0|}$, is the outer unit normal vector to $\partial B(y_0, r)$.

Hence,

$$\nabla u(x_0)\nu(x_0) = \frac{\partial u}{\partial \nu}(x_0) \leq \frac{\partial b}{\partial \nu}(x_0) = \nabla b(x) \frac{x_0 - y_0}{|x_0 - y_0|} = -2\alpha k r e^{-\alpha r^2} < 0.$$

Since $u \in C^{1,\mu}(\overline{\Omega} \setminus \{0\})$, we have $\nabla u \in C^{0,\mu}(\overline{\Omega} \setminus \{0\})$. Then, there exists $\sigma > 0$ such that

$$|\nabla u(x_0)| \geq \sigma, \quad \forall x_0 \in \partial \Omega. \quad \square$$

In the next result, we will study some properties of the first eigenfunction of problem (2.2).

Theorem 2.3. *Suppose that Ω is a bounded smooth domain of \mathbb{R}^N , satisfying the interior sphere condition, $0 \in \Omega$, $1 < p < N$, $0 \leq a < \frac{N-p}{p}$ and $c_1 > 0$. If λ_1 and ϕ_1 are the eigenvalue and eigenfunction, respectively, of problem (2.2), then ϕ_1 belongs to $C^{0,\rho_1}(\overline{\Omega}) \cap C^{1,\mu_1}(\overline{\Omega} \setminus \{0\})$, $\phi_1 > 0$ in Ω and $|\nabla \phi_1| \geq \sigma$ on $\partial \Omega$, for some $\rho_1 \in (0, 1]$, $\mu_1 > 0$ and $\sigma > 0$.*

Proof. Let

$$p - 1 < \bar{q} < \min \left\{ \frac{Np}{N-p} - 1; p - 1 + \frac{c_1}{N-p(a+1)} \right\}$$

and $g(x, s) := \lambda_1 s^{p-1}$, it follows that

$$|g(x, s)| \leq C(1 + |s|^q), \quad \forall s \in \mathbb{R}, \forall x \in \Omega.$$

Therefore, by the regularity result (see [6, Theorem 2.1]) we conclude that $\phi_1 \in C^{0,\rho_1}(\overline{\Omega})$, for some $\rho_1 \in (0, 1]$. By the regularity result (see [25, Theorem 1]) we have $\phi_1 \in C^{1,\mu_1}(\overline{\Omega} \setminus \{0\})$, for some $\mu_1 > 0$. Applying Theorem 2.1 it follows that $\phi_1 > 0$ in Ω , also, by Theorem 2.2, we conclude that $|\nabla\phi_1(x)| \geq \sigma > 0$, for every $x \in \partial\Omega$. \square

We also have the following lemma.

Lemma 2.1. Consider Ω a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$ and $c > 0$. Assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a nonnegative function. Suppose that u is a weak solution of problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c} f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then u is nonnegative for a.e. in Ω .

Proof. Since u is a weak solution, taking $u^- := \max\{0, -u\} \in W_0^{1,p}(\Omega, |x|^{-ap})$, we get

$$-\int_{\Omega} |x|^{-ap}|\nabla u^-|^{p-2}|\nabla u^-|^2 = \int_{\Omega} |x|^{-(a+1)p+c} f(x, u)u^-.$$

From this equality, we obtain

$$\int_{\Omega} |x|^{-ap}|\nabla u^-|^p = 0$$

and therefore $u^- = 0$ a.e. in Ω . Hence $u \geq 0$, for a.e. in Ω . \square

In order to establish a version of the abstract lower and upper-solution method for our class of the operators, we will introduce some definitions.

Definition 2.1. We say that the pair $(\underline{u}, \underline{v})$, where $\underline{u} \in W^{1,p}(\Omega, |x|^{-ap}) \cap L^\infty(\Omega)$ and $\underline{v} \in W^{1,q}(\Omega, |x|^{-bq}) \cap L^\infty(\Omega)$, is a weak lower-solution of system (2.1), if

$$\begin{cases} \int_{\Omega} |x|^{-ap}|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla \phi \leq \int_{\Omega} |x|^{-(a+1)p+c_1} h(x, \underline{u}, \underline{v})\phi, \\ \int_{\Omega} |x|^{-bq}|\nabla \underline{v}|^{q-2}\nabla \underline{v}\nabla \psi \leq \int_{\Omega} |x|^{-(b+1)q+c_2} k(x, \underline{u}, \underline{v})\psi, \\ \underline{u}, \underline{v} \leq 0, \quad \text{on } \partial\Omega, \end{cases}$$

for $\phi \in W_0^{1,p}(\Omega, |x|^{-ap})$, $\psi \in W_0^{1,q}(\Omega, |x|^{-bq})$, with $\phi, \psi \geq 0$. (A function $u \in W^{1,p}(\Omega, |x|^{-ap})$ is said to be less than or equal to $w \in W^{1,p}(\Omega, |x|^{-ap})$ on $\partial\Omega$ when $\max\{0, u - w\} \in W_0^{1,p}(\Omega, |x|^{-ap})$.)

Similarly one defines a weak upper-solution (\bar{u}, \bar{v}) of system (2.1), by considering the reversed inequalities in the above definition.

Notation. If $u, v \in L^\infty(\Omega)$, with $u(x) \leq v(x)$ for a.e. $x \in \Omega$, we denote by $[u, v]$ the set $\{w \in L^\infty(\Omega) : u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega\}$.

Now, we establish a version of the abstract lower and upper-solution method for our class of the operators.

Theorem 2.4 (Lower- and Upper-Solution). Consider system (2.1), under the hypotheses (HK1) and (HK2). Suppose that $(\underline{u}, \underline{v}), (\bar{u}, \bar{v})$ are respectively, a weak lower-solution and a weak upper-solution of system (2.1), with $\underline{u}(x) \leq \bar{u}(x), \underline{v}(x) \leq \bar{v}(x)$, for a.e. $x \in \Omega$. Then there exists a minimal (u_*, v_*) (and, respectively, a maximal (u^*, v^*)) weak solution for system (2.1) in the set $[\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$. In particular, every weak solution $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$ of system (2.1) satisfies:

$$u_*(x) \leq u(x) \leq u^*(x) \quad \text{and} \quad v_*(x) \leq v(x) \leq v^*(x)$$

for a.e. $x \in \Omega$.

Proof. As in [11] (also see [6] in the scalar case), let us consider the set $[\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}] \subset L^\infty(\Omega) \times L^\infty(\Omega)$ endowed with the topology given by the convergence a.e. in Ω . Let $p' > 1$ and $q' > 1$ be the conjugate exponents to p and q , respectively.

We define the operator

$$S : [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}] \rightarrow L^{p'}(\Omega, |x|^{-(a+1)p+c_1}) \times L^{q'}(\Omega, |x|^{-(b+1)q+c_2}) \equiv L_{p'q'},$$

$$S(u, v) := (h(\cdot, u(\cdot), v(\cdot)) + g(u(\cdot)), k(\cdot, u(\cdot), v(\cdot)) + g(v(\cdot))),$$

which by (HK1) and (HK2) is well defined, it is bounded and nondecreasing in each component. Let $\{(u_m, v_m)\} \subset [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$ and $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$, with $(u_m(x), v_m(x)) \rightarrow (u(x), v(x))$, for a.e. $x \in \Omega$, then, by the Lebesgue dominated convergence theorem, we obtain

$$\int_{\Omega} |x|^{-(a+1)p+c_1} [h(x, u_m, v_m) + g(u_m) - h(x, u, v) - g(u)]^{p'} \rightarrow 0,$$

$$\int_{\Omega} |x|^{-(b+1)q+c_2} [k(x, u_m, v_m) + g(v_m) - k(x, u, v) - g(v)]^{q'} \rightarrow 0$$

as $m \rightarrow \infty$, that is, the operator S is continuous. From [6, Theorem 3.1] it follows that the operator

$$T : L_{p'q'} \rightarrow W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq}),$$

$$T(f_1, f_2) := (T_p(f_1), T_q(f_2))$$

is well defined, which is continuous and nondecreasing in each component, where $T_p(f_1)$ and $T_q(f_2)$ are operators defined in [6, Theorem 3.1].

We defined the operator $F : [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}] \rightarrow W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$, by $F := T \circ S$. Thus for each $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$, $F(u, v) = (F_1(u, v), F_2(u, v))$ is the unique weak solution of system

$$\begin{cases} L(x, e) = |x|^{-(a+1)p+c_1} [h(x, u, v) + g(u)] & \text{in } \Omega, \\ L(x, w) = |x|^{-(b+1)q+c_2} [k(x, u, v) + g(v)] & \text{in } \Omega, \\ e = w = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$L(x, e) = -\operatorname{div}(|x|^{-ap}|\nabla e|^{p-2}\nabla e) + |x|^{-(a+1)p+c_1}g(e)$$

and

$$L(x, w) = -\operatorname{div}(|x|^{-bq}|\nabla w|^{q-2}\nabla w) + |x|^{-(b+1)q+c_2}g(w).$$

Writing $(u_1, v_1) := F(\underline{u}, \underline{v})$ and $(u^1, v^1) := F(\bar{u}, \bar{v})$, we get, for $\phi \in W_0^{1,p}(\Omega, |x|^{-ap})$, with $\phi \geq 0$,

$$\begin{aligned} & \int_{\Omega} |x|^{-ap}|\nabla u_1|^{p-2}\nabla u_1\nabla\phi + \int_{\Omega} |x|^{-(a+1)p+c_1}g(u_1)\phi \\ &= \int_{\Omega} |x|^{-(a+1)p+c_1}[h(x, \underline{u}, \underline{v}) + g(\underline{u})]\phi \\ &\geq \int_{\Omega} |x|^{-ap}|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\phi + \int_{\Omega} |x|^{-(a+1)p+c_1}g(\underline{u})\phi \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} |x|^{-ap}|\nabla u^1|^{p-2}\nabla u^1\nabla\phi + \int_{\Omega} |x|^{-(a+1)p+c_1}g(u^1)\phi \\ &\leq \int_{\Omega} |x|^{-ap}|\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla\phi + \int_{\Omega} |x|^{-(a+1)p+c_1}g(\bar{u})\phi. \end{aligned}$$

In addition, we have $u_1 = 0 \geq \underline{u}$ and $u^1 = 0 \leq \bar{u}$ on $\partial\Omega$, then, by the comparison principle theorem (see [6, Lemma 3.2]), it follows that $\underline{u}(x) \leq u_1(x)$ and $u^1(x) \leq \bar{u}(x)$, for a.e. $x \in \Omega$. Similarly, $\underline{v}(x) \leq v_1(x)$ and $v^1(x) \leq \bar{v}(x)$, for a.e. $x \in \Omega$.

Observing that F_i is nondecreasing, $i = 1, 2$, we obtain

$$\begin{cases} \underline{u}(x) \leq u_1(x) \leq F_1(u, v) \leq u^1(x) \leq \bar{u}(x), \\ \underline{v}(x) \leq v_1(x) \leq F_2(u, v) \leq v^1(x) \leq \bar{v}(x), \end{cases}$$

for all $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$ and a.e. $x \in \Omega$. By repeating the same reasoning, the sequences $\{(u_m, v_m)\}, \{(u^m, v^m)\} \subset [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$, given by

$$\begin{aligned} (u_0, v_0) &:= (\underline{u}, \underline{v}), & (u_{m+1}, v_{m+1}) &:= F(u_m v_m), \\ (u^0, v^0) &:= (\bar{u}, \bar{v}), & (u^{m+1}, v^{m+1}) &:= F(u^m v^m), \end{aligned}$$

satisfy

$$\begin{cases} u_0 \leq u_1 \leq \dots \leq u_m \leq F_1^m(u, v) \leq u^m \leq \dots \leq u^1 \leq u^0, \\ v_0 \leq v_1 \leq \dots \leq v_m \leq F_2^m(u, v) \leq v^m \leq \dots \leq v^1 \leq v^0, \end{cases}$$

for all $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$, for a.e. in Ω and every $m \in \mathbb{N}$. In particular, suppose that $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$ is a weak solution of system (2.1), and write $(\hat{u}, \hat{v}) = F(u, v)$, we get, for $\phi \in W_0^{1,p}(\Omega, |x|^{-ap})$, with $\phi \geq 0$,

$$\begin{aligned} & \int_{\Omega} |x|^{-ap} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla \phi + \int_{\Omega} |x|^{-(a+1)p+c_1} g(\hat{u}) \phi \\ &= \int_{\Omega} |x|^{-(a+1)p+c_1} [h(x, u, v) + g(u)] \phi \\ &= \int_{\Omega} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi + \int_{\Omega} |x|^{-(a+1)p+c_1} g(u) \phi. \end{aligned}$$

Furthermore, we have $\hat{u} = u = 0$ on $\partial\Omega$, then the comparison principle theorem (see [6, Lemma 3.2]) implies that $F_1(u, v) = \hat{u} = u$. Analogously, $F_2(u, v) = \hat{v} = v$. Thus

$$\begin{cases} u_0 \leq u_1 \leq \dots \leq u_m \leq u \leq u^m \leq \dots \leq u^1 \leq u^0, \\ v_0 \leq v_1 \leq \dots \leq v_m \leq v \leq v^m \leq \dots \leq v^1 \leq v^0, \end{cases}$$

for a.e. in Ω .

Therefore, we infer that $u_m(x) \rightarrow u_*(x)$, $v_m(x) \rightarrow v_*(x)$, $u^m(x) \rightarrow u^*(x)$, and $v^m(x) \rightarrow v^*(x)$, as $m \rightarrow \infty$, for a.e. $x \in \Omega$. Since the operator F is continuous, we have

$$F(u_*, v_*) = (u_*, v_*) \quad \text{and} \quad F(u^*, v^*) = (u^*, v^*). \quad \square$$

3. Existence of weak upper-solution

We will establish the existence of a positive weak upper-solution for system (1.1), where each component belongs to $C^{0,\rho}(\overline{\Omega})$, for some $\rho \in (0, 1]$.

Lemma 3.1. *Suppose that Ω is a bounded smooth domain of \mathbb{R}^N , with $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, $\delta, \gamma > 0$, $\theta := (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$ and $c_1, c_2 > 0$. Then system (1.1) possesses a positive weak upper-solution $(z_1, z_2) \in C^{0,\rho_1}(\overline{\Omega}) \times C^{0,\rho_2}(\overline{\Omega})$, for some $\rho_i \in (0, 1]$, $i = 1, 2$, and each $\lambda > 0$.*

Proof. By combining Lemma 2.1 with [6, Theorems 2.1 and 3.1], we can choose $e_i \in C^{0,\rho_i}(\overline{\Omega})$, for $i = 1, 2$, with (e_1, e_2) a nonnegative weak solution of system (1.1) with $\lambda = 1$ and $\alpha = \beta = \gamma = \delta = 0$. Evidently e_i is nontrivial, $i = 1, 2$. Applying a regularity result of [30], we obtain $e_i \in C^{1,\alpha_i}(\Omega \setminus \{0\})$, for some $\alpha_i > 0$ and $i = 1, 2$. Then, by the strong maximum principle Theorem 2.1, we get $e_i > 0$ in Ω , $i = 1, 2$.

We define

$$(z_1(x), z_2(x)) := (Ae_1(x), Be_2(x)),$$

where A, B are positive constants that will be fixed later. Let $f_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$, $f_2 \in W_0^{1,q}(\Omega, |x|^{-bq})$, with $f_1, f_2 \geq 0$.

Then, we obtain

$$\begin{aligned} \int_{\Omega} |x|^{-ap} |\nabla z_1|^{p-2} \nabla z_1 \nabla f_1 &= A^{p-1} \int_{\Omega} |x|^{-ap} |\nabla e_1|^{p-2} \nabla e_1 \nabla f_1 \\ &= A^{p-1} \int_{\Omega} |x|^{-(a+1)p+c_1} f_1 \end{aligned} \tag{3.1}$$

and similarly

$$\int_{\Omega} |x|^{-bq} |\nabla z_2|^{q-2} \nabla z_2 \nabla f_2 = B^{q-1} \int_{\Omega} |x|^{-(b+1)q+c_2} f_2. \tag{3.2}$$

If $l := \|e_1\|_{\infty}$, $L := \|e_2\|_{\infty}$, $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, $\lambda > 0$ and $\theta > 0$, it is easy to prove that there exist positive constants A, B , such that

$$A^{p-1-\alpha} = \lambda B^{\gamma} l^{\alpha} L^{\gamma} \quad \text{and} \quad B^{q-1-\beta} = \lambda A^{\delta} l^{\delta} L^{\beta}. \tag{3.3}$$

Thus, from (3.3), we obtain

$$\begin{aligned} \lambda z_1^{\alpha}(x) z_2^{\gamma}(x) &\leq \lambda A^{\alpha} B^{\gamma} l^{\alpha} L^{\gamma} \leq A^{p-1}, \quad \forall x \in \Omega, \\ \lambda z_1^{\delta}(x) z_2^{\beta}(x) &\leq \lambda A^{\delta} B^{\beta} l^{\delta} L^{\beta} \leq B^{q-1}, \quad \forall x \in \Omega. \end{aligned} \tag{3.4}$$

Therefore, by using (3.1), (3.2) and (3.4), we conclude that

$$\int_{\Omega} |x|^{-ap} |\nabla z_1|^{p-2} \nabla z_1 \nabla f_1 \geq \lambda \int_{\Omega} |x|^{-(a+1)p+c_1} z_1^{\alpha} z_2^{\gamma} f_1$$

and

$$\int_{\Omega} |x|^{-bq} |\nabla z_2|^{q-2} \nabla z_2 \nabla f_2 \geq \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} z_1^{\delta} z_2^{\beta} f_2.$$

Hence, $(z_1, z_2) \in C^{0,\rho_1}(\overline{\Omega}) \times C^{0,\rho_2}(\overline{\Omega})$ is a positive weak upper-solution of system (1.1). \square

4. Existence of weak lower-solution

We will prove the existence of a positive weak lower-solution for system (1.1), where each component belongs to $C^0(\overline{\Omega})$.

Lemma 4.1. *Suppose that Ω is a bounded smooth domain of \mathbb{R}^N , satisfying the interior sphere condition, with $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, $\delta, \gamma, c_1, c_2 > 0$ and $\theta := (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$. Then system (1.1) possesses a positive weak lower-solution $(\psi_1, \psi_2) \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$, for each $\lambda > 0$.*

Proof. Applying Theorem 2.3, with $1 < p < N$, $0 \leq a < \frac{N-p}{p}$ and $c_1 > 0$, we have $\lambda_1 > 0$ and ϕ_1 the eigenvalue and eigenfunction, respectively, of problem (2.2), with ϕ_1 belongs to $C^{0,\rho_1}(\overline{\Omega}) \cap C^{1,\mu_1}(\overline{\Omega} \setminus \{0\})$, $\phi_1 > 0$ in Ω and $|\nabla \phi_1| \geq \sigma_1$ on $\partial\Omega$, for some positive constants σ_1, μ_1 and ρ_1 . Changing $1 < p < N$, $0 \leq a < \frac{N-p}{p}$ and $c_1 > 0$ by $1 < q < N$, $0 \leq b < \frac{N-q}{q}$ and $c_2 > 0$, respectively, we have $\lambda_2 > 0$ and ϕ_2 the eigenvalue and eigenfunction, respectively, of problem (2.2), satisfying $\phi_2 \in C^{0,\rho_2}(\overline{\Omega}) \cap C^{1,\mu_2}(\overline{\Omega} \setminus \{0\})$, $\phi_2 > 0$ in Ω and $|\nabla \phi_2| \geq \sigma_2$ on $\partial\Omega$, for some constants $\sigma_2, \mu_2 > 0$ and $\rho_2 > 0$.

We define

$$(\psi_{1c}(x), \psi_{2c}(x)) := (c\phi_1^m(x), c^k\phi_2^n(x)),$$

which belongs to $(C^0(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \{0\})) \times (C^0(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \{0\}))$, with $c > 0$ to be fixed later and

$$\frac{\delta}{q - 1 - \beta} < k < \frac{p - 1 - \alpha}{\gamma}, \quad m = \frac{p}{p - 1}, \quad n = \frac{q}{q - 1}, \tag{4.1}$$

because $\theta > 0$, $p - 1 - \alpha > 0$ and $q - 1 - \beta > 0$.

Then, for all $f_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$, $f_2 \in W_0^{1,q}(\Omega, |x|^{-bq})$, with $f_1, f_2 \geq 0$, we have

$$\begin{aligned} & \int_{\Omega} |x|^{-ap} |\nabla \psi_{1c}|^{p-2} \nabla \psi_{1c} \nabla f_1 \\ &= \int_{\Omega} |x|^{-ap} (cm)^{p-1} \phi_1^{(m-1)(p-2)+(m-1)} |\nabla \phi_1|^{p-2} \nabla \phi_1 \nabla f_1 \\ &= (cm)^{p-1} \int_{\Omega} |x|^{-ap} |\nabla \phi_1|^{p-2} \nabla \phi_1 [\nabla(\phi_1 f_1) - (\nabla \phi_1) f_1] \\ &= (cm)^{p-1} \int_{\Omega} [\lambda_1 |x|^{-(a+1)p+c_1} \phi_1^p - |x|^{-ap} |\nabla \phi_1|^p] f_1. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_{\Omega} |x|^{-bq} |\nabla \psi_{2c}|^{q-2} \nabla \psi_{2c} \nabla f_2 \\ &= (c^k n)^{q-1} \int_{\Omega} [\lambda_2 |x|^{-(b+1)q+c_2} \phi_2^q - |x|^{-bq} |\nabla \phi_2|^q] f_2. \end{aligned}$$

Since $\phi_i = 0$ and $|\nabla \phi_i| \geq \sigma_i$ on $\partial\Omega$, for $i = 1, 2$, there exists $\eta > 0$ such that for every $x \in \Omega_{\eta} := \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \eta\}$, we have

$$\lambda_1 |x|^{-(a+1)p+c_1} \phi_1^p - |x|^{-ap} |\nabla \phi_1|^p \leq 0, \quad \lambda_2 |x|^{-(b+1)q+c_2} \phi_2^q - |x|^{-bq} |\nabla \phi_2|^q \leq 0.$$

Then, for each $\lambda > 0$, we get

$$\int_{\Omega_{\eta}} |x|^{-ap} |\nabla \psi_{1c}|^{p-2} \nabla \psi_{1c} \nabla f_1 \leq 0 \leq \lambda \int_{\Omega_{\eta}} |x|^{-(a+1)p+c_1} \psi_{1c}^{\alpha} \psi_{2c}^{\gamma} f_1 \tag{4.2}$$

for all $f_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$, $f_1 \geq 0$, and

$$\int_{\Omega_{\eta}} |x|^{-bq} |\nabla \psi_{2c}|^{q-2} \nabla \psi_{2c} \nabla f_2 \leq 0 \leq \lambda \int_{\Omega_{\eta}} |x|^{-(b+1)q+c_2} \psi_{1c}^{\delta} \psi_{2c}^{\beta} f_2 \tag{4.3}$$

for all $f_2 \in W_0^{1,q}(\Omega, |x|^{-bq})$, $f_2 \geq 0$.

Now, as $\phi_i > 0$ in Ω and ϕ_i is continuous, $i = 1, 2$, then there exists $\mu > 0$ such that $\phi_i(x) \geq \mu > 0$ for all $x \in \Omega \setminus \Omega_{\eta}$ and $i = 1, 2$. Therefore from (4.1) we obtain $a_0 > 0$ such that the following inequalities hold

$$\lambda_2 n^{q-1} c^{k(q-1-\beta)-\delta} \phi_2^{q-n\beta}(x) \leq \lambda \mu^{m\delta} \leq \lambda \phi_1^{m\delta}(x), \quad \forall x \in \Omega \setminus \Omega_{\eta}, \tag{4.4}$$

and

$$\lambda_1 m^{p-1} c^{p-1-\alpha-k\gamma} \phi_1^{p-m\alpha}(x) \leq \lambda \mu^{n\gamma} \leq \lambda \phi_2^{n\gamma}(x), \quad \forall x \in \Omega \setminus \Omega_{\eta}, \tag{4.5}$$

for each $c \in (0, a_0)$.

Then,

$$\begin{aligned}
 & (cm)^{p-1} (\lambda_1 |x|^{-(a+1)p+c_1} \phi_1^p - |x|^{-ap} |\nabla \phi_1|^p) \\
 & \leq |x|^{-(a+1)p+c_1} \lambda_1 (cm)^{p-1} \phi_1^p \\
 & = |x|^{-(a+1)p+c_1} \lambda_1 m^{p-1} c^{p-1-\alpha-k\gamma} \phi_1^{p-m\alpha} [c^{k\gamma} c^\alpha \phi_1^{m\alpha}] \\
 & \stackrel{(4.5)}{\leq} \lambda |x|^{-(a+1)p+c_1} \phi_2^{n\gamma} c^{k\gamma} c^\alpha \phi_1^{m\alpha} \\
 & = \lambda |x|^{-(a+1)p+c_1} \psi_{1c}^\alpha \psi_{2c}^\gamma
 \end{aligned}$$

and similarly, from (4.4), we have

$$\begin{aligned}
 & (c^k n)^{q-1} (\lambda_2 |x|^{-(b+1)q+c_2} \phi_2^q - |x|^{-bq} |\nabla \phi_2|^q) \\
 & \leq \lambda |x|^{-(b+1)q+c_2} \psi_{1c}^\delta \psi_{2c}^\beta
 \end{aligned}$$

in $\Omega \setminus \Omega_\eta$ and each $c \in (0, a_0)$.

Therefore,

$$\int_{\Omega \setminus \Omega_\eta} |x|^{-ap} |\nabla \psi_{1c}|^{p-2} \nabla \psi_{1c} \nabla f_1 \leq \lambda \int_{\Omega \setminus \Omega_\eta} |x|^{-(a+1)p+c_1} \psi_{1c}^\alpha \psi_{2c}^\gamma \tag{4.6}$$

and

$$\int_{\Omega \setminus \Omega_\eta} |x|^{-bq} |\nabla \psi_{2c}|^{q-2} \nabla \psi_{2c} \nabla f_2 \leq \lambda \int_{\Omega \setminus \Omega_\eta} |x|^{-(b+1)q+c_2} \psi_{1c}^\delta \psi_{2c}^\beta. \tag{4.7}$$

Hence from (4.2), (4.3), (4.6) and (4.7), it follows that (ψ_{1c}, ψ_{2c}) is a positive weak lower-solution of system (1.1), for each $c \in (0, a_0)$. \square

5. Proof of Theorem 1.1

Let (z_1, z_2) and (ψ_{1c}, ψ_{2c}) be a weak upper and lower-solution, given by Lemmas 3.1 and 4.1, respectively, where $c \in (0, a_0)$. Observing that $\bar{\psi}_{ic_0}(x) \leq z_i(x)$ for every $x \in \Omega$, for some $c_0 \in (0, a_0)$ sufficiently small and $i = 1, 2$, we obtain by Theorem 2.4, a weak solution (u_0, v_0) with $\psi_{1c_0}(x) \leq u_0(x) \leq z_1(x)$ and $\psi_{2c_0}(x) \leq v_0(x) \leq z_2(x)$, for a.e. $x \in \Omega$.

Let

$$p - 1 < \bar{q} < \min \left\{ \frac{Np}{N - p} - 1; p - 1 + \frac{c_1}{N - p(a + 1)} \right\}$$

and $g(x, s) := \lambda v_0^\gamma(x) s^\alpha$. Since v_0 is bounded in $L^\infty(\Omega)$ and $0 \leq \alpha < p - 1$, we infer that

$$|g(x, s)| \leq \lambda \|v_0\|_{L^\infty(\Omega)}^\gamma |s|^\alpha \leq C(1 + |s|^{\bar{q}}), \quad \forall s \in \mathbb{R} \text{ and uniformly in } x \in \Omega.$$

Therefore, by the regularity result (see [6, Theorem 2.1]) we conclude that $u_0 \in C^{0,\rho_1}(\bar{\Omega})$, for some $\rho_1 \in (0, 1)$. Similarly, we have $v_0 \in C^{0,\rho_2}(\bar{\Omega})$, for some $\rho_2 \in (0, 1)$. By a regularity result of [30] (also in [25]) follows that $u_0 \in C^{1,\mu_1}(\Omega \setminus \{0\})$ and $v_0 \in C^{1,\mu_2}(\Omega \setminus \{0\})$, for some $\mu_1, \mu_2 > 0$. As $\psi_{ic_0} > 0$ in Ω , $i = 1, 2$, it follows that $u_0, v_0 > 0$ in Ω .

Remark 5.1. Suppose that $\Omega := B(0, R)$. By a result in [17, Proposition 3.1] problem (2.2) has a radial eigenfunction $\phi_1 \in C^2(B(0, R)) \cap C^{1,\mu_1}(\overline{B(0, R)})$, $\mu_1 > 0$, associated to the eigenvalue $\lambda_1 > 0$, if $1 < p < N$, $-\infty < a < \frac{N-p}{p}$ and $c_1 > p - 1$. Moreover, $\phi_1 > 0$ in $B(0, R)$ and $|\nabla\phi_1| > 0$ on $\partial\Omega$. Then, repeating the proof of Lemmas 3.1 and 4.1, we obtain that system (1.1) possesses weak upper and lower-solution. Therefore, by Theorem 2.4, we can recover Theorem 1.1 with $\Omega := B(0, R)$, $1 < p, q < N$, $-\infty < a < \frac{N-p}{p}$, $-\infty < b < \frac{N-q}{q}$, $c_1 > p - 1$ and $c_2 > q - 1$.

6. Proof of Theorem 1.2

We will prove this result by contradiction. Define

$$\mu_1 = p/(1 + \alpha), \quad \mu_2 = p/(p - 1 - \alpha), \quad \theta_1 = q/(q - 1 - \beta), \quad \theta_2 = q/(1 + \beta).$$

By hypotheses $(p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta = 0$ and $p\gamma = q(p - 1 - \alpha)$, then

$$\begin{aligned} \mu_1^{-1} + \mu_2^{-1} &= \theta_1^{-1} + \theta_2^{-1} = 1, & \mu_1(\alpha + 1) &= p, & \mu_2\gamma &= q, \\ \theta_1\delta &= p, & \theta_2(\beta + 1) &= q, & \mu_1, \mu_2, \theta_1, \theta_2 &> 1. \end{aligned}$$

Let λ_1 and λ_2 be the eigenvalues given as in the proof of Lemma 4.1. We recall that they can be characterized by

$$\begin{aligned} \lambda_1 &:= \inf \left\{ \frac{\int_{\Omega} |x|^{-ap} |\nabla w|^p}{\int_{\Omega} |x|^{-(a+1)p+c_1} |w|^p} : w \in W_0^{1p}(\Omega, |x|^{-ap}) \setminus \{0\} \right\} > 0, \\ \lambda_2 &:= \inf \left\{ \frac{\int_{\Omega} |x|^{-bq} |\nabla w|^q}{\int_{\Omega} |x|^{-(b+1)q+c_2} |w|^q} : w \in W_0^{1q}(\Omega, |x|^{-bq}) \setminus \{0\} \right\} > 0. \end{aligned}$$

Let $\lambda_0 := \frac{1}{2} \min\{\lambda_1, \lambda_2\}$. Suppose by contradiction that there exists a nontrivial and non-negative weak solution (u, v) of system (1.1), with $0 < \lambda < \lambda_0$. By Young inequality, we obtain

$$\begin{aligned} \lambda_1 \int_{\Omega} |x|^{-(a+1)p+c_1} u^p &\leq \int_{\Omega} |x|^{-ap} |\nabla u|^p = \lambda \int_{\Omega} |x|^{-(a+1)p+c_1} u^{\alpha+1} v^{\gamma} \\ &\leq \lambda \int_{\Omega} |x|^{-(a+1)p+c_1} \left(\frac{u^p}{\mu_1} + \frac{v^q}{\mu_2} \right) \end{aligned}$$

and similarly

$$\lambda_2 \int_{\Omega} |x|^{-(b+1)q+c_2} v^q \leq \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} \left(\frac{u^p}{\theta_1} + \frac{v^q}{\theta_2} \right).$$

Since $(a + 1)p - c_1 = (b + 1)q - c_2$, we get

$$\begin{aligned} \lambda_1 \int_{\Omega} |x|^{-(a+1)p+c_1} u^p + \lambda_2 \int_{\Omega} |x|^{-(b+1)q+c_2} v^q \\ \leq \lambda(\mu_1^{-1} + \theta_1^{-1}) \int_{\Omega} |x|^{-(a+1)p+c_1} u^p + \lambda(\mu_2^{-1} + \theta_2^{-1}) \int_{\Omega} |x|^{-(b+1)q+c_2} v^q \\ \leq 2\lambda \left[\int_{\Omega} |x|^{-(a+1)p+c_1} u^p + \int_{\Omega} |x|^{-(b+1)q+c_2} v^q \right], \end{aligned}$$

then

$$0 < (\lambda_1 - 2\lambda) \int_{\Omega} |x|^{-(a+1)p+c_1} u^p + (\lambda_2 - 2\lambda) \int_{\Omega} |x|^{-(b+1)q+c_2} v^q \leq 0,$$

which is a contradiction. \square

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