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Non-autonomous perturbations for a class of quasilinear elliptic equations on $\mathbb{R}$

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Abstract

This paper is concerned with the existence of two positive solutions for a class of quasilinear elliptic equations on $\mathbb{R}$ involving the p-Laplacian, with a non-autonomous perturbation. The first solution is obtained as a local minimum in a neighborhood of 0 and the second one by a mountain-pass argument. The special features of the problem here is the "complex" structure of the linear part which, in particular, oblige to work into the space $W^{1,p}(\mathbb{R})$. Then one faces problems in the convergence of the sequences of derivatives $u_n' \to u$.

Keywords: Non-autonomous perturbations; Schrödinger equation; p-Laplacian; Variational method

1. Introduction

We study the following class of quasilinear elliptic problems on $\mathbb{R}$ involving the p-Laplacian:

$$\begin{cases} Lu + V(x)|u|^{p-2}u = |u|^{q-2}u + g(x) & \text{in } \mathbb{R}, \\ u \in W^{1,p}(\mathbb{R}), \quad u \geqslant 0 \text{ in } \mathbb{R}, \end{cases} \tag{1.1}$$

where the operator L is defined by

$$Lu \equiv -\left[|u'|^{p-2}u'\right]' - K_0\left\{\left[\left(|u|^\beta\right)'\right]^{p-2}\left(|u|^\beta\right)'\right\}'|u|^{\beta-2}u,$$

$K_0 > 0$, $\beta > 1$, $p > 1$, $q \geqslant p\ \beta$ $(q > p)$, $g \in L^s(\mathbb{R})$ for some $s \geqslant 1$ and $V : \mathbb{R} \to \mathbb{R}$ is a given bounded potential function verifying the basic condition

$$\text{there exists } \alpha_0 > 0 \text{ such that } \inf_{\mathbb{R}} V(x) \geqslant \alpha_0 > 0. \tag{V$_0$}$$

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The quasilinear equations of the type (1.1) have been accepted as a model for several physical phenomena. In the case $p = \beta = 2$, these equations are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + V(x)z - h(|z|^2)z - K_0\Delta f(|z|^2)f'(|z|^2)z \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where V is a given potential, K_0 is a real constant, h and f are real functions. For instance, the case $f(s) = s$ was used as a model for the superfluid film equations in plasma physics by Kurihura in [25]. Besides Eq. (1.2) with $f(s) = (1+s)^{\frac{1}{2}}$ models the self-channeling of a high-power ultra short laser in matter, see Borovskii and Galkin [13] and De Bouard, Hayashi and Saut [14]. Eq. (1.2) also appears in the plasma physics and the fluid mechanics (see [7,35]), in mechanics and in condensed matter theory (see [19,30], respectively).

The case $K_0 = 0$, has been intensively studied in recent years, considering various types of g, see for example [8,18,22], as well as in their references. More exactly, Rabinowitz in [36] (see also [6,16]) treated the case when V satisfies some coercivity conditions such as

$$\lim_{|x|\to\infty} V(x) = +\infty. \tag{V$_1$}$$

The situation in which V is bounded and satisfies the periodicity condition

$$V(x+p) = V(x), \quad x \in \mathbb{R},\ p \in \mathbb{Z}, \tag{V$_2$}$$

was studied, e.g., by Coti-Zelati and Rabinowitz [17], Kryszewski and Szulkin [24] and Montecchiari [31].

On the other hand, when V is asymptotic to a constant $\overline{V} \equiv \sup_{x\in\mathbb{R}} V(x)$, that is,

$$\lim_{|x|\to\infty} V(x) = \overline{V} \quad \text{and} \quad V \leqslant \overline{V}, \tag{V$_{3'}$}$$

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^n$, problem (1.1) in $\mathbb{R}^n$ with $1 < p < n$ and $g = 0$ was considered by Zhu and Yang [42]. Alves, Carrião and Miyagaki in [1] treated problem (1.1) when $g = 0$ and V is asymptotic to a periodic function V_p, namely,

$$\lim_{|x|\to\infty} V(x) = V_p \quad \text{and} \quad V \leqslant V_p, \tag{V$_3$}$$

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^n$.

Still in the case $K_0 = 0$, Rădulescu and Smets in [37] obtained multiplicity result for problem (1.1) when $g \neq 0$ and $\beta = p = 2$.

The case in which $K_0 \neq 0$, $4 \leqslant q+1 \leqslant \frac{4n}{n-2}$ if $n \geqslant 3$, and $q \geqslant 3$ if $n = 1,2$, problem (1.1) was studied in [4,27,28,34]. More especially Liu, Wang and Wang in [29] established the existence of solution both one-sign and nodal ground states of soliton type, by using the Nehari method. Poppenberg, Schmitt and Wang in [34] and Liu and Wang in [27], by using a constrained minimization argument, obtained a solution for problem (1.2) with a Lagrange multiplier λ associated to the non-linear term. Afterwards, Liu, Wang and Wang in [28], by using a change of variables, reduced the quasilinear problem (1.1) in $\mathbb{R}^n$ with $p = \beta = 2$ and $g = 0$ to a semilinear one and their result does not involve the constant λ any more. This argument was also used by Colin and Jeanjean in [15] when $p = 2$, and Severo [38] for $n \geqslant p > 1$, $\beta = 2$, $p \neq 2$. These authors proved the existence of solutions by applying the classical results by Berestycki and Lions [12] when $n = 1$ or $n \geqslant 3$, and by Berestycki, Gallouët and Kavian [9] when $n = 2$. For $n = 1$, by using a variational perturbative approach, Ambrosetti and Wang in [4] were also able to drop the above Lagrange multiplier. In [2], Alves, Carrião and Miyagaki by employing some techniques developed in [4] and by combining the concentration–compactness principle due to Lions in [26] with a minimization approach, proved the existence of non-negative solutions for problem (1.1) when $g = 0$. Finally, Eq. (1.2) in $n = 2$, involving a critical exponential non-linearities was treated, for instance, in [22,32,38].

A main difficulty in our work is because for the case $p \neq 2$, we need to prove the convergence of the sequence of the derivatives, that is,

$$u_n' \to u' \quad \text{a.e. in } \mathbb{R}, \text{ as } n \to \infty,$$

for a bounded sequence $\{u_n\}$ in a Sobolev spaces. This convergence can be obtained adapting some arguments used by Boccardo and Murat in [10]. (See also [5], for $p > n$.)

Another difficulty is the case $K_0 \neq 0$ and $n = 1$. In these cases the ideas of [15,28] also [38] do not in a straightforward way.

This paper was motivated by [5,21,37,40] and by applying the variational method we will prove the existence of two positive solutions for problem (1.1). Our main results are the following:

Theorem 1.1. *Let V be a bounded potential satisfying conditions* (V_0) *and either* (V_2) *or* (V_3). *Suppose that $K_0 = 1$, $\beta > 1$, $p > 1$ and $q \geqslant p\,\beta$ ($q > p$), then problem* (1.1) *has at least one solution if $g \neq 0$ and $|g|_s$ is sufficiently small. Moreover, if $g \in C_+$, then this solution is positive.*

Theorem 1.2. *Let V be a bounded potential satisfying conditions* (V_0) *and either* (V_2) *or* (V_3). *Suppose that $K_0 = 1$, $\beta > 1$, $p > 1$ and $q \geqslant p\,\beta$ ($q > p$), then for each $f \in C_+$ problem* (1.1) *with $g = \epsilon f$ has at least two positive solutions, for all $\epsilon > 0$ sufficiently small.*

Remark 1.3. We recall that

$$|g|_p :\equiv \left[\int_{\mathbb{R}} \left|g(x)\right|^p dx\right]^{\frac{1}{p}} \text{ denotes the usual norm in } L^p(\mathbb{R});$$

C_+ is the positive cone of the dual space $(L^{s'}(\mathbb{R}))^* := L^s(\mathbb{R})$, $s \geqslant 1$, specifically: $f \in C_+$ if and only if $f \neq 0$ and $\int_{\mathbb{R}} f(x)u(x)\,dx \geqslant 0$ for all $u \in W^{1,p}(\mathbb{R})$ such that $u(x) \geqslant 0$ a.e. on $\mathbb{R}$.

Our result can be proved with the assumption (V_1), because in this case we can use compactness.

To prove Theorems 1.1 and 1.2 we will need some preliminary results that are described by several lemmata established below.

2. Preliminary results for first solution

First of all, define the energy functional $I : W^{1,p}(\mathbb{R}) \to \mathbb{R}$ associated to problem (1.1) by

$$I(u) \equiv \int_{\mathbb{R}} \left\{ \frac{1}{p}\left[|u'|^p + V(x)|u|^p\right] + \frac{\beta^{p-1}}{p} K_0 |u|^{p(\beta-1)}|u'|^p - \frac{1}{q}|u|^q - gu \right\} dx \tag{2.1}$$

and its Fréchet derivative

$$\begin{aligned} I'(u)\cdot z = {} & \int_{\mathbb{R}} \left(\left[|u'|^{p-2}u'\right]z' + V(x)\left[|u|^{p-2}u\right]z\right) dx + \int_{\mathbb{R}} \beta^{p-1}(\beta-1)\left[|u|^{p(\beta-1)-1}|u'|^p\right]z\,dx \\ & + \int_{\mathbb{R}} \beta^{p-1}\left[|u|^{p(\beta-1)}|u'|^{p-2}u'\right]z'\,dx - \int_{\mathbb{R}} \left[|u|^{q-2}u\right]z\,dx - \int_{\mathbb{R}} g(x)z(x)\,dx. \end{aligned} \tag{2.2}$$

We cannot apply directly such method because I and I' are not well defined in $W^{1,p}(\mathbb{R})$. Moreover, the second integral in the expression of I, as well as the second and the third integral, in the expression of I' are very hard to analyze. Mainly because the functions are living in the space $W^{1,p}(\mathbb{R})$ and because the Brezis–Lieb identity type (see [12]) does not hold in our case, even in the case $p = 2$, as was mentioned in [34] and [4]. To overcome these difficulties we will use an argument developed by Liu, Wang and Wang in [28] (see also [15] for $p = 2$ and [38] for $p \neq 2$, $\beta = 2$ and $n \geqslant p > 1$).

We make the change of variable $u = f(v)$ ($v = f^{-1}(u)$), where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(0) = 0, \qquad f(-t) = -f(t) \quad \text{on } (-\infty, 0],$$

and

$$f'(t) = \frac{1}{[1 + \beta^{p-1}|f(t)|^{p(\beta-1)}]^{\frac{1}{p}}} \quad \text{on } [0, \infty).$$

In addition, f satisfies the following properties, which can be proved arguing as Colin and Jeanjean in [15, Lemma 2.1].

Lemma 2.1.

(a) *f is uniquely defined, $C^2(\mathbb{R})$ and invertible.*
(b) $|f'(t)| \leqslant 1$, *for all $t \in \mathbb{R}$.*
(c) $|f(t)| \leqslant |t|$, *for all $t \in \mathbb{R}$.*
(d) $\frac{f(t)}{t} \to 1$, *as $t \to 0$.*
(e) *For $t \geqslant 0$ we have $\frac{1}{\beta} f(t) \leqslant t f'(t) \leqslant f(t)$.*
(f) $\frac{f(t)}{t^{1/\beta}} \to C > 0$, *as $t \to \infty$.*

Thus, if u is a classical solution for problem (1.1), making this change of variables we obtain the following equation:

$$-\Delta_p v \equiv -\left(|v'|^{p-2}v'\right)' = \frac{1}{[1+\beta^{p-1}|f(t)|^{p(\beta-1)}]^{\frac{1}{p}}} H(x,v), \quad x \in \mathbb{R}, \tag{2.3}$$

where

$$H(x,v) = -V(x)\left|f(v)\right|^{p-2} f(v) + \left|f(v)\right|^{q-2} f(v) + g(x).$$

The energy functional associated to problem (1.1) is given by

$$J(v) \equiv \int_{\mathbb{R}} \left\{ \frac{1}{p}\left[|v'|^p + V(x)\left|f(v)\right|^p\right] - \frac{1}{q}\left|f(v)\right|^q - g(x) f(v) \right\} dx \tag{2.4}$$

and its Fréchet derivative is given by

$$\begin{aligned} J'(v) \cdot w = {} & \int_{\mathbb{R}} \left[|v'|^{p-2}v'\right] w' \, dx + \int_{\mathbb{R}} V(x)\left[\left|f(v)\right|^{p-2} f(v) f'(v)\right] w \, dx \\ & - \int_{\mathbb{R}} \left[\left|f(v)\right|^{q-2} f(v) f'(v)\right] w \, dx - \int_{\mathbb{R}} g(x) f'(v) w \, dx. \end{aligned} \tag{2.5}$$

They are well defined on the space $W^{1,p}(\mathbb{R})$ under suitable assumptions on the potential V and the proprieties of the function f.

We use the following notation:

We say that (u_n) is a Palais–Smale sequence ($(\mathrm{PS})_c$ in short) for the functional ψ at the level $c \in \mathbb{R}$, if

$$\psi(u_n) \to c \quad \text{and} \quad \psi'(u_n) \to 0, \quad \text{as } n \to \infty.$$

Lemma 2.2. *Let $(u_n) \subset W^{1,p}(\mathbb{R})$ be a bounded $(\mathrm{PS})_c$ sequence for the functional I. Then there exists a subsequence of (u_n) (which we also denote by (u_n)) such that*

(a) *$u_n \rightharpoonup u$, weakly in $W^{1,p}(\mathbb{R})$, as $n \to \infty$.*
(b) *$u_n \to u$ a.e. on $\mathbb{R}$, as $n \to \infty$.*
(c) *$u_n' \to u'$ a.e. on $\mathbb{R}$, as $n \to \infty$.*

Proof. The proof of items (a) and (b) are immediate. The proof of the item (c) follows by adapting some arguments used in Boccardo and Murat [10] (see also [2,5,33]). For the sake of completeness, we will give the sketch of the proof. Define the family of functions

$$\tau_\eta(s) = \begin{cases} s & \text{if } |s| \leqslant \eta, \\ \eta \frac{s}{|s|} & \text{if } |s| > \eta. \end{cases}$$

Fixing a compact set $K \subset \mathbb{R}$, we take a cut-off function $\phi_K : \mathbb{R} \to \mathbb{R}$, satisfying

$$\phi_K \in C_0^\infty(\mathbb{R}), \quad 0 \leqslant \phi_K \leqslant 1 \text{ in } \mathbb{R} \quad \text{and} \quad \phi_K = 1 \quad \text{on } K.$$

Consider the test function $\phi_K \tau_\eta(u_n - u) \in W^{1,p}(\mathbb{R})$. Since (u_n) is a $(\mathrm{PS})_c$ sequence for functional I, we have

$$\begin{aligned}
o(1) &= I'(u_n)\phi_K \tau_\eta(u_n - u) - I'(u)\phi_K \tau_\eta(u_n - u) \\
&= \int_{\mathbb{R}} \left[1 + \beta^{p-1}|u_n|^{p(\beta-1)}\right]\left[|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right]\left(\phi_K \tau_\eta(u_n - u)\right)' dx \\
&\quad + \int_{\mathbb{R}} V(x)\left[|u_n|^{p-2}u_n - |u|^{p-2}u\right]\phi_K \tau_\eta(u_n - u)\, dx \\
&\quad + \int_{\mathbb{R}} \beta^{p-1}(\beta - 1)\left[|u_n|^{p(\beta-1)-1}|u_n'|^p - |u|^{p(\beta-1)-1}|u'|^p\right]\phi_K \tau_\eta(u_n - u)\, dx \\
&\quad + \int_{\mathbb{R}} \beta^{p-1}\left[|u_n|^{p(\beta-1)} - |u|^{p(\beta-1)}\right]|u'|^{p-2}u'\left(\phi_K \tau_\eta(u_n - u)\right)' dx \\
&\quad - \int_{\mathbb{R}} \left[|u_n|^{q-2}u_n - |u|^{q-2}u\right]\phi_K \tau_\eta(u_n - u)\, dx - \int_{\mathbb{R}} g(x)\phi_K \tau_\eta(u_n - u)\, dx.
\end{aligned} \tag{2.6}$$

Affirmation 2.3. *We claim that*

(a) $$\left|\int_K V(x)\left[|u_n|^{p-2}u_n - |u|^{p-2}u\right]\phi_K \tau_\eta(u_n - u)\, dx\right| = o(1).$$

(b) $$\left|\int_K \left[|u_n|^{p(\beta-1)-1}|u_n'|^p - |u|^{p(\beta-1)-1}|u'|^p\right]\phi_K \tau_\eta(u_n - u)\, dx\right| = o(1).$$

(c) $$\left|\int_K \left[|u_n|^{p(\beta-1)} - |u|^{p(\beta-1)}\right]|u'|^{p-2}u'\left(\phi_K \tau_\eta(u_n - u)\right)' dx\right| = o(1).$$

(d) $$\left|\int_K \left[|u_n|^{q-2}u_n - |u|^{q-2}u\right]\phi_K \tau_\eta(u_n - u)\, dx\right| \leqslant C\eta.$$

(e) $$\left|\int_{\mathbb{R}} g(x)\phi_K \tau_\eta(u_n - u)\, dx\right| = o(1).$$

(f) $$\limsup_{n\to\infty} g_n \leqslant C\eta,$$

where

$$g_n \equiv \left|\int_K \left[1 + K_0\beta^{p-1}|u_n|^{p(\beta-1)}\right]\left[|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right]\left(\phi_K \tau_\eta(u_n - u)\right)' dx\right|.$$

By assuming the above affirmation, we conclude the proof of Lemma 2.2. In fact, define the functions e_n by

$$e_n(x) = \left[1 + K_0\beta^{p-1}|u_n|^{p(\beta-1)}\right]\left[|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right]\left(\tau_\eta(u_n - u)\right)'.$$

Since $[1 + K_0\beta^{p-1}|u_n|^{p(\beta-1)}] > 0$, from the inequality

$$\left[|x|^{p-2}x - |y|^{p-2}y\right](x - y) \geqslant \begin{cases} C\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2, \\ C|x - y|^p & \text{if } p \geqslant 2, \ \forall x, y \in \mathbb{R}^N \end{cases} \tag{2.7}$$

(see [33,39]), we infer that $e_n \geqslant 0$. We also have that

$$\int_{\mathbb{R}} e_n(x)\,dx < \infty.$$

Fix θ with $0 < \theta < 1$ and split the set K into

$$S_n^\eta = \{x \in K/|u_n - u| \leqslant \eta\} \quad \text{and} \quad G_n^\eta = \{x \in K/|u_n - u| > \eta\}.$$

Using Hölder's inequality we obtain

$$\int_K e_n^\theta(x)\,dx = \int_{S_n^\eta} e_n^\theta(x)\,dx + \int_{G_n^\eta} e_n^\theta(x)\,dx \leqslant \left\{\int_{S_n^\eta} e_n(x)\,dx\right\}^\theta |S_n^\eta|^{1-\theta} + \left\{\int_{G_n^\eta} e_n(x)\,dx\right\}^\theta |G_n^\eta|^{1-\theta}.$$

Now, fix η; then $|G_n^\eta| \to 0$, as $n \to \infty$, because $u_n \to u$ uniformly and a.e. in K, as $n \to \infty$. From the uniform boundedness of (e_n) in $L^1(\mathbb{R})$ we get

$$\limsup_{n\to\infty} \int_{I_n} e_n^\theta(x)\,dx \leqslant (C\eta)^\theta |S_n^\eta|^{1-\theta}.$$

Letting $\eta \to 0$, we get that $e_n^\theta \to 0$ in $L^1(K)$, as $n \to \infty$. Hence the term in the integral tends to zero a.e. in K, as $n \to \infty$. Since $[1 + \beta^{p-1} K_0 |u_n|^{p(\beta-1)}] > 0$, using a sequence of compact sets K and the inequality (2.7) we obtain

$$u_n' \to u' \quad \text{a.e. on } \mathbb{R},\ \text{as } n \to \infty. \qquad \square$$

Proof of Affirmation 2.3. The proofs of items (a) through (d) follow as in [2, Proposition 4.2]. The item (e) is held because $g \in L^s(\mathbb{R})$ for some $s \geqslant 1$ and (u_n) is bounded in $W^{1,p}(\mathbb{R})$. Item (f) follows by taking lim sup in Eq. (2.6) and by the previous items. $\square$

Lemma 2.4. *The sequence $(v_n) \subset W^{1,p}(\mathbb{R})$ is a bounded $(\mathrm{PS})_c$ sequence for the functional J defined in* (2.4) *if and only if $u_n = f(v_n) \in W^{1,p}(\mathbb{R})$ is also a bounded $(\mathrm{PS})_c$ sequence for functional I defined in* (2.1). *Moreover, $v_n' \to v'$ a.e on $\mathbb{R}$, as $n \to \infty$.*

Proof. To conclude the equivalence that $(v_n) \subset W^{1,p}(\mathbb{R})$ is a bounded $(\mathrm{PS})_c$ sequence for the functional J defined in (2.4) if and only if $u_n = f(v_n) \in W^{1,p}(\mathbb{R})$ is also a bounded sequence for functional I, we will prove that

(a) $I(u_n) = J(v_n)$.
(b) $I'(u_n) \cdot z = J'(v_n) \cdot w$, if $z = f'(v_n) \cdot w$.
(c) If the sequence (v_n) is bounded in $W^{1,p}(\mathbb{R})$, then the sequence (u_n) is also bounded in $W^{1,p}(\mathbb{R})$.
(d) Every $(\mathrm{PS})_c$ sequence for the functional J is bounded in $W^{1,p}(\mathbb{R})$.

Using $u_n' = f'(v_n) \cdot v_n'$, item (a) follows because

$$|v_n'|^p = |u_n'|^p + \beta^{p-1} |u_n|^{p(\beta-1)} |u_n'|^p.$$

In fact,

$$|u_n'|^p + \beta^{p-1}|u_n|^{p(\beta-1)}|u_n'|^p = |f'(v_n) \cdot v_n'|^p + \beta^{p-1}|f(v_n)|^{p(\beta-1)}|f'(v_n) \cdot v_n'|^p,$$
$$|f'(v_n) \cdot v_n'|^p [1 + \beta^{p-1}|f(v_n)|^{p(\beta-1)}] = |f'(v_n) \cdot v_n'|^p |f'(v_n)|^{-p} = |v_n'|^p.$$

To prove item (b) we use again $u_n' = f'(v_n) \cdot v_n'$ and $z' = [f'(v_n)]' \cdot w + f'(v_n) \cdot w'$, where

$$f''(v_n) = -\beta^{p-1}(\beta - 1)|f'(v_n)|^{p+2}|f(v_n)|^{p(\beta-1)-2} f(v_n) v_n'.$$

We infer that

$$\begin{aligned}
&\left|u_n'\right|^{p-2}u_n'z' + \beta^{p-1}|u_n|^{p(\beta-1)}\left|u_n'\right|^{p-2}u_n'z' + \beta^{p-1}(\beta-1)|u_n|^{p(\beta-1)-1}\left|u_n'\right|^p z \\
&\quad = \left|f'(v_n)v_n'\right|^{p-2}f'(v_n)v_n'z'\left|f'(v_n)\right|^{-p} + \beta^{p-1}(\beta-1)\left|f(v_n)\right|^{p(\beta-1)-1}\left|f'(v_n)v_n'\right|^p z \\
&\quad = \left|v_n'\right|^{p-2}v_n'\left|f'(v_n)\right|^{-1}\left[-\beta^{p-1}(\beta-1)\left|f'(v_n)\right|^{p+2}\left|f(v_n)\right|^{p(\beta-1)-2}f(v_n)v_n'w + f'(v_n)w'\right] \\
&\qquad + \beta^{p-1}(\beta-1)\left|f(v_n)\right|^{p(\beta-1)-1}\left|f'(v_n)v_n'\right|^p f'(v_n)w \\
&\quad = \left|v_n'\right|^{p-2}v_n'w'.
\end{aligned}$$

Hence the item (b) follows.

Item (c) follows from Lemma 2.1(c). In fact,

$$\|u_n\| = \left\|f(v_n)\right\| \leqslant \|v_n\| \leqslant M.$$

Remark 2.5. We recall that the embedding $W^{1,p}(\mathbb{R})$ into $L^s(\mathbb{R})$ for $s=\infty$ or for $s \geqslant p$ is continuous.

Now we prove the item (d). Using Lemma 2.1(e), the Hölder's inequality, (V_0) and Remark 2.5 we obtain that

$$\begin{aligned}
\frac{1}{\beta}c_1 + \frac{1}{q}\left|J'(v_n)\right|_{(W^{1,p}(\mathbb{R}))^*}\cdot\|v_n\| + 1 &\geqslant \frac{1}{\beta}J(v_n) - \frac{1}{q}J'(v_n)\cdot v_n \\
&\geqslant C_1\left(\frac{1}{p\beta} - \frac{1}{q}\right)\|v_n\|^p - C\left(\frac{1}{\beta} - \frac{1}{q\beta}\right)|g|_s\|v_n\|.
\end{aligned}$$

Hence v_n is bounded in $W^{1,p}(\mathbb{R})$.

Now, we will prove that $v_n' \to v'$ a.e. on $\mathbb{R}$, as $n\to\infty$. Recall that $u_n' \to u'$, $u_n \to u$ a.e. on $\mathbb{R}$, as $n\to\infty$ (by Lemma 2.2) and $[f^{-1}]'(u_n)$ is continuous, because

$$\left[f^{-1}\right]'(t) = \frac{1}{f'(f(-t))} = \left[1+\beta^{p-1}t^{p(\beta-1)}\right]^{\frac{1}{p}}, \quad \text{as } t \geqslant 0.$$

By using these facts we obtain

$$v_n' = \left[f^{-1}\right]'(u_n)u_n' \to \left[f^{-1}\right]'(u)u' = v' \quad \text{a.e. on } \mathbb{R}, \text{ as } n\to\infty. \qquad \square$$

The next proposition says that every weak solution of Eq. (2.3) is a weak solution of problem (1.1).

Proposition 2.6. *Let $u = f(v)$ ($v = f^{-1}(u)$). Then v is a critical point of the functional J if and only if u is a critical point of the functional I. In addition, if $v \in W^{1,p}(\mathbb{R})$, then $u \in W^{1,p}(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R})$, for some $\alpha \in (0,1)$.*

Proof. We have by Lemma 2.1(b), (c) that $|u|^p \leqslant |f(v)|^p \leqslant |v|^p$ and $|u'|^p \leqslant |f'(v)v'|^p \leqslant |v'|^p$. Hence $\|u\| = [\int_{\mathbb{R}}(|u|^p + |u'|^p)\,dx]^{\frac{1}{p}} \leqslant \|v\| \leqslant \infty$. Since v is a critical point of the functional J, then v is a weak solution for problem (2.3). By regularity theory (see [39]), one has $v \in C^{1,\alpha}(\mathbb{R})$, $\alpha \in (0,1)$, and using the fact that $f \in C^2(\mathbb{R})$, it follows that $u = f(v) \in C^{1,\alpha}(\mathbb{R})$. Thus $u \in W^{1,p}(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R})$.

The proof of the equivalence, that is, v is a critical point of functional J if and only if u is a critical point of functional I, follows as in the proof of Lemma 2.4(b) considering $v = v_n$ and $u = u_n$. $\square$

Now, we will prove that the weak limit of a Palais–Smale sequence for functional J, at the level $c \in \mathbb{R}$, is a weak solution of problem (2.3).

Lemma 2.7. *Let $(v_n) \subset W^{1,p}(\mathbb{R})$ be a Palais–Smale sequence for functional J in the level $c \in \mathbb{R}$. If the sequence (v_n) converges weakly to some $v_0 \in W^{1,p}(\mathbb{R})$, as $n\to\infty$, then v_0 is a weak solution for problem* (2.3).

Proof. Consider an arbitrary function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$. Since $J'(v_n)\cdot\phi \to 0$, as $n\to\infty$, we will prove that $J'(v_0)\cdot\phi = 0$ and by the density argument, we will conclude that $J'(v_0)\cdot w = 0$, for all $w \in W^{1,p}(\mathbb{R})$. Hence $J'(u_0) = 0$ and u_0 is a weak solution of problem (1.1). In fact,

$$0=\lim_{n\to\infty} J'(v_n)\cdot\phi$$
$$=\lim_{n\to\infty}\Big[\int_{\mathbb{R}}\big[|v_n'|^{p-2}v_n'\big]\phi'\,dx-\int_{\mathbb{R}}\big[\big|f(v_n)\big|^{q-2}f(v_n)f'(v_n)\big]\phi\,dx$$
$$+\int_{\mathbb{R}}V(x)\big[\big|f(v_n)\big|^{p-2}f(v_n)f'(v_n)\big]\phi\,dx-\int_{\mathbb{R}}g(x)f'(v_n)\phi\,dx\Big].$$

We denote the above integrals by $J_1(v_n)\cdot\phi$, $J_2(v_n)\cdot\phi$, $J_3(v_n)\cdot\phi$ and $J_4(v_n)\cdot\phi$, respectively. We will prove that $J_i(v_n)\cdot\phi\to J_i(v_0)\cdot\phi$, as $n\to\infty$, for all $i=1,2,3,4$.

We remark that $|v_n'|^{p-2}v_n'$ is bounded in $L^{p'}(\mathbb{R})$ where $p'=\frac{p}{p-1}$. By using this fact, together with Lemma 2.4, we infer that $|v_n'|^{p-2}v_n'\rightharpoonup|v_0'|^{p-2}v_0'$, as $n\to\infty$. Since ϕ' is bounded in $L^p(\mathbb{R})$, by [20, Theorem 13.44] we conclude that $J_1(v_n)\cdot\phi\to J_1(v_0)\cdot\phi$, as $n\to\infty$.

Notice that (v_n) converges weakly to some $v_0\in W^{1,p}(\mathbb{R})$, as $n\to\infty$; then $v_n\to v_0$ a.e. on $\mathbb{R}$, as $n\to\infty$. It follows that $f(v_n)\to f(v_0)$ and $f'(v_n)\to f'(v_0)$ a.e. on $\mathbb{R}$, as $n\to\infty$ (because $f\in C^2(\mathbb{R})$). As before, since $V(x)$ is bounded it is sufficient to prove that $|f(v_n)|^{p-2}f(v_n)f'(v_n)$ is bounded in $L^{p'}(\mathbb{R})$ to conclude that $J_3(v_n)\cdot\phi\to J_3(v_0)\cdot\phi$, as $n\to\infty$. In fact, by Lemma 2.1 we obtain

$$\left|\int_{\mathbb{R}}\big|V(x)\big|f(v_n)\big|^{p-2}f(v_n)f'(v_n)\big|^{\frac{p}{p-1}}\,dx\right|\leqslant M\int_{\mathbb{R}}|v_n|^p\,dx\leqslant M\|v_n\|^p<\infty,$$

where M is a constant.

Arguing as in the case J_3 and using Remark 2.5 we prove that $|f(v_n)|^{q-2}f(v_n)f'(v_n)$ is bounded in $L^{\frac{q}{q-1}}(\mathbb{R})$. Thus $J_2(v_n)\cdot\phi\to J_2(v_0)\cdot\phi$, as $n\to\infty$.

The convergence of the integral J_4 follows directly by using the Dominate Convergent Theorem, because $g\in L^s(\mathbb{R})$ for some $s>1$, f' is continuous and satisfies Lemma 2.1(b). □

3. Proof of Theorem 1.1

We will prove that there exist real numbers $c<0$ and $R>0$ such that the functional J has a bounded $(\mathrm{PS})_c$ sequence (v_n), at the level c, that is

$$\|v_n\|\leqslant R,\qquad J(v_n)\to c\quad\text{and}\quad J'(v_n)\to 0,\quad\text{as } n\to\infty,$$

where

$$c=\inf\big\{J(v)/v\in W^{1,p}(\mathbb{R})\text{ and }v\in\overline{B}_R\big\},$$
$$\overline{B}_R=\big\{v\in W^{1,p}(\mathbb{R})/\|v\|\leqslant R\big\}.$$

We will show that c is achieved by some $v\in W^{1,p}(\mathbb{R})$ and $J'(v)=0$.

Notice that by definition of c and using that $g\neq 0$, it follows that $c<J(0)=0$.

Using Ekeland's variational principle we get the following result:

Affirmation 3.1. *There exists a sequence $(v_n)\in W^{1,p}(\mathbb{R})\cap\overline{B}_R$ such that*

$$J(v_n)\to c,\qquad J'(v_n)\to 0\quad\text{in }\big(W^{1,p}(\mathbb{R})\big)^*,\ \text{as } n\to\infty,$$

By assuming this affirmation, we have that the sequence (v_n) is bounded in $W^{1,p}(\mathbb{R})\cap\overline{B}_R$ and hence $v_n\rightharpoonup v$ converges weakly and $v_n\to v$ a.e. on $\mathbb{R}$, as $n\to\infty$. By Lemma 2.7 we obtain that v is a solution of problem (2.3) and $J'(v)=0$.

Now we will verify that $J(v)=c$ and $v\geqslant 0$.

We have that (v_n) satisfies

$$c+o(1)=J(v_n)=\frac{1}{p}\int_{\mathbb{R}}\big[\big|v_n'\big|^p+V(x)\big|f(v_n)\big|^p\big]\,dx-\frac{1}{q}\int_{\mathbb{R}}\big|f(v_n)\big|^q\,dx-\int_{\mathbb{R}}g(x)f(v_n)\,dx,$$

and for all $w \in W^{1,p}(\mathbb{R})$,

$$\begin{aligned}0 &= J'(v_n)\cdot w\\ &= \int_{\mathbb{R}} \big[|v_n'|^{p-2}v_n'\big]w'\,dx + \int_{\mathbb{R}} V(x)\big[|f(v_n)|^{p-2}f(v_n)f'(v_n)\big]w\,dx\\ &\quad - \int_{\mathbb{R}} \big[|f(v_n)|^{q-2}f(v_n)f'(v_n)\big]w\,dx - \int_{\mathbb{R}} g(x)f'(v_n)w\,dx.\end{aligned}$$

By choosing $w = w_n = \frac{f(v_n)}{f'(v_n)}$ we obtain

$$w_n' = v_n'\left[1 + \frac{\beta^{p-1}(\beta-1)|f(v_n)|^{p(\beta-1)}}{1+\beta^{p-1}|f(v_n)|^{p(\beta-1)}}\right].$$

We note that w_n is bounded (by Lemma 2.1(e)) and hence

$$\begin{aligned}0 &= J'(v_n)\cdot w_n\\ &= \int_{\mathbb{R}} |v_n'|^p\left[1+\frac{\beta^{p-1}(\beta-1)|f(v_n)|^{p(\beta-1)}}{1+\beta^{p-1}|f(v_n)|^{p(\beta-1)}}\right]dx + \int_{\mathbb{R}} V(x)|f(v_n)|^p\,dx - \int_{\mathbb{R}} |f(v_n)|^q\,dx - \int_{\mathbb{R}} g(x)f(v_n)\,dx.\end{aligned}$$

We conclude that

$$\begin{aligned}c + o(1) = J(v_n) &= \int_{\mathbb{R}} \left(\frac{1}{p} - \frac{1}{q}\left[1+\frac{\beta^{p-1}(\beta-1)|f(v_n)|^{p(\beta-1)}}{1+\beta^{p-1}|f(v_n)|^{p(\beta-1)}}\right]\right)|v_n'|^p\,dx\\ &\quad + \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}} V(x)|f(v_n)|^p\,dx + \left(\frac{1}{q}-1\right)\int_{\mathbb{R}} g(x)f(v_n)\,dx.\end{aligned} \tag{3.1}$$

Now we will prove that $c \geqslant J(v)$. By using this fact, the definition of c and the fact that $v \in \overline{B}_R$, we conclude that $J(v) = c$.

We denote the above integrals by $J_1(v_n)$, $J_2(v_n)$ and $J_3(v_n)$. To prove the inequality $c \geqslant J(v)$ we will use Fatou's Lemma in the integrals $J_1(v_n)$ and $J_2(v_n)$. In fact, define

$$h_n^1(x) = \left(\frac{1}{p} - \frac{1}{q}\left[1+\frac{\beta^{p-1}(\beta-1)|f(v_n)|^{p(\beta-1)}}{1+\beta^{p-1}|f(v_n)|^{p(\beta-1)}}\right]\right)|v_n'|^p.$$

Since $v_n \to v$ a.e. on $\mathbb{R}$, $v_n' \to v'$ a.e. on $\mathbb{R}$ and f is continuous we have that

$$h_n^1(x) \to \left(\frac{1}{p} - \frac{1}{q}\left[1+\frac{\beta^{p-1}(\beta-1)|f(v)|^{p(\beta-1)}}{1+\beta^{p-1}|f(v)|^{p(\beta-1)}}\right]\right)|v'|^p \quad \text{a.e. on } \mathbb{R}.$$

Using Remark 2.5 and Lemma 2.1(c) we get

$$\int_{\mathbb{R}} h_n^1(x)\,dx \leqslant \int_{\mathbb{R}} \left(\frac{1}{p}+\frac{1}{q}\big[1+\beta^{p-1}(\beta-1)\|v_n\|^{p(\beta-1)}\big]\right)|v_n'|^p\,dx \leqslant C\|v_n\|^p < \infty$$

and thus $h_n^1 \in L^1(\mathbb{R})$. Setting $a \equiv \beta^{p-1}|f(v)|^{p(\beta-1)}$ we infer that $1 + \frac{(\beta-1)a}{1+a} < 1+\beta-1 = \beta$, and hence

$$h_n^1(x) \geqslant \left(\frac{1}{p} - \frac{1}{q}\beta\right)|v_n'|^p = \frac{q-p\beta}{pq}|v_n'|^p \geqslant 0.$$

By Fatou's Lemma we obtain that

$$\lim_{n\to\infty}\inf J_1(v_n) \geqslant \int_{\mathbb{R}} \left(\frac{1}{p} - \frac{1}{q}\left[1+\frac{\beta^{p-1}(\beta-1)|f(v)|^{p(\beta-1)}}{1+\beta^{p-1}|f(v)|^{p(\beta-1)}}\right]\right)|v'|^p\,dx. \tag{3.2}$$

Now we define

$$h_n^2(x) = \left(\frac{1}{p} - \frac{1}{q}\right) V(x)\left|f(v_n)\right|^p$$

and remark that

$$h_n^2(x) \to \left(\frac{1}{p} - \frac{1}{q}\right) V(x)\left|f(v)\right|^p \quad \text{a.e. on } \mathbb{R},$$

because $v_n \to v$ a.e. on $\mathbb{R}$ and f is continuous. Since

$$h_n^2(x) \geqslant 0 \quad \text{and} \quad h_n^2 \in L^1(\mathbb{R})$$

by Fatou's Lemma we obtain that

$$\liminf_{n\to\infty} J_2(v_n) \geqslant \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}} V(x)\left|f(v)\right|^p dx. \tag{3.3}$$

Note that the proof of the convergence of the integral

$$J_3(v_n) \to J_3(v) \tag{3.4}$$

is direct by applying the Lebesgue Convergence Theorem, because $g \in L^s(\mathbb{R})$ for some $s \geqslant 1$ and f is continuous. Taking the lower limit in the equality (3.1) and using (3.4), (3.3) and (3.2), we conclude that $c \geqslant J(v)$.

Finally, since $g \in C_+$ and J is an even function in v, we can replace v by $|v|$. Then we obtain a non-negative solution for problem (1.1).

Proof of Affirmation 3.1. By Lemma 2.1(d), there exist constants $R_1 > 0$ and $C_1 = C_1(R_1) > 0$ such that

$$\left|f(v)\right| \geqslant C_1|v| \quad \text{if } |v| \leqslant R_1. \tag{3.5}$$

Using the Hölder's inequality together with (3.5), Lemma 2.1(c), the condition (V_0) and Remark 2.5 we obtain

$$J(v) \geqslant \int_{\mathbb{R}} \frac{1}{p}\left[|v'|^p + V_0C_1|v|^p\right] - \frac{C_2}{q}\|v\|^q - C_3|g|_s\|v\|\,dx.$$

Let $C = C(R_1) \equiv \min\{1, V_0C_1\}$. By the Young's inequality, we conclude that

$$J(v) \geqslant \frac{1}{p}\left(C - \epsilon^p\right)\|v\|^p - \frac{C_2}{q}\|v\|^q - C_\epsilon|g|_s^{p'}.$$

Fixing $\epsilon \in (0,1)$ we take real numbers $R < R_1$, $\delta_1 > 0$, $\rho_1 > 0$ such that $J(v) \geqslant \rho_1$ if $\|v\| = R$ and $|g|_s < \delta_1$. Hence J is bounded from below on $\overline{B}_R$. We will prove that J is lower semi-continuous in $\overline{B}_R$.

Note that

$$\liminf_{n\to\infty} J(z_n) = \liminf_{n\to\infty}\left\{\frac{1}{p}\int_{\mathbb{R}}\left|z_n'\right|^p dx + \frac{1}{p}\int_{\mathbb{R}} V(x)\left|f(z_n)\right|^p dx - \frac{1}{q}\int_{\mathbb{R}}\left|f(z_n)\right|^q dx - \int_{\mathbb{R}} g(x)f(z_n)\,dx\right\}. \tag{3.6}$$

We denote the above integrals by $J_1(z_n)$, $J_2(z_n)$, $J_3(z_n)$ and $J_4(z_n)$, respectively. Since $|\cdot|_p$ is lower semi-continuous, then

$$\liminf_{n\to\infty} \int_{\mathbb{R}}\left|z_n'\right|^p dx \geqslant \int_{\mathbb{R}} |z'|^p\,dx. \tag{3.7}$$

Using Lemma 2.1(b), the Mean Value Theorem and Remark 2.5 we get

$$\left|\int_{\mathbb{R}}\left|f(z_n) - f(z)\right|^p dx\right| \leqslant M\int_{\mathbb{R}} |z_n - z|^p \leqslant C\|z_n - z\|^p \to 0, \quad \text{as } n \to \infty.$$

Therefore $f(z_n) \to f(z)$ in $L^p(\mathbb{R})$, as $n \to \infty$. Thus, since V is bounded, by combining [11, Theorem IV.9] with the Lebesgue Convergence Theorem, we infer that

$$\int_{\mathbb{R}} V(x)\left|f(z_n)\right|^p dx \to \int_{\mathbb{R}} V(x)\left|f(z)\right|^p dx, \quad \text{as } n \to \infty. \tag{3.8}$$

Similarly, we prove that

$$\int_{\mathbb{R}} \left|f(z_n)\right|^q dx \to \int_{\mathbb{R}} \left|f(z)\right|^q dx, \quad \text{as } n \to \infty, \tag{3.9}$$

and since $g \in L^s(\mathbb{R})$, for some $s \geqslant 1$, we obtain that

$$J_4(z_n) \to \int_{\mathbb{R}} g(x) f(z)\, dx, \quad \text{as } n \to \infty. \tag{3.10}$$

Thus using (3.7)–(3.10) in the equality (3.6) we conclude that

$$\begin{aligned}
\liminf_{n\to\infty} J(z_n) &= \liminf_{n\to\infty} \left\{ \frac{1}{p}\int_{\mathbb{R}} \left|z_n'\right|^p dx + \frac{1}{p}\int_{\mathbb{R}} V(x)\left|f(z_n)\right|^p dx - \frac{1}{q}\int_{\mathbb{R}} \left|f(z_n)\right|^q dx - \int_{\mathbb{R}} g(x) f(z_n)\, dx \right\} \\
&= \liminf_{n\to\infty} \frac{1}{p}\int_{\mathbb{R}} \left|z_n'\right|^p dx + \frac{1}{p}\int_{\mathbb{R}} V(x)\left|f(z)\right|^p dx - \frac{1}{q}\int_{\mathbb{R}} \left|f(z)\right|^q dx - \int_{\mathbb{R}} g(x) f(z)\, dx \\
&\geqslant \frac{1}{p}\int_{\mathbb{R}} |z'|^p dx + \frac{1}{p}\int_{\mathbb{R}} V(x)\left|f(z)\right|^p dx - \frac{1}{q}\int_{\mathbb{R}} \left|f(z)\right|^q dx - \int_{\mathbb{R}} g(x) f(z)\, dx = J(z).
\end{aligned}$$

Hence J is lower semi-continuous in $\overline{B}_R$.

Since J is lower semi-continuous on $\overline{B}_R$, by Ekeland's principle (see Kesavan [23, Lemma 5.3.1]), we obtain that for any positive integer n there exists a sequence (v_n) such that

$$c \leqslant J(v_n) \leqslant c + \frac{1}{n} \tag{3.11}$$

and

$$J(w) \geqslant J(v_n) - \frac{1}{n}\|v_n - w\|, \quad \text{for all } w \in \overline{B}_R. \tag{3.12}$$

We recall that $\|v_n\| < R$, for all n large enough.

Next we will prove that $\|J'(v_n)\|^*_{(W^{1,p}(\mathbb{R}))} \to 0$, as $n \to \infty$. In fact, for any $v \in W^{1,p}(\mathbb{R})$ such that $\|v\| = 1$, we define $w_n = v_n - tv$. For a fixed n we have that $\|w_n\| = \|v_n\| - t < R$ if t is small enough. Using the inequality (3.12) we get

$$J(w_n) = J(v_n + tv) \geqslant J(v_n) - \frac{|t|}{n}\|v\|.$$

Therefore

$$\frac{J(v_n + tv) - J(v_n)}{|t|} \geqslant -\frac{1}{n}\|v\| = -\frac{1}{n}.$$

Letting $t \to 0^+$ we conclude that

$$J'(v_n) \cdot v \geqslant -\frac{1}{n}.$$

Using a similar argument for $t \to 0^-$, we obtain that $J'(v_n) \cdot v \leqslant \frac{1}{n}$ for all $v \in W^{1,p}(\mathbb{R})$ with $\|v\| = 1$. Hence

$$J'(v_n) \to 0, \quad \text{as } n \to \infty,$$

and (v_n) is a $(\mathrm{PS})_c$ sequence. □

4. Preliminary results for second solution

We define the energy functional $I_0 : X \to \mathbb{R}$ by equality (2.1), with $g \equiv 0$, which corresponds to the problem (1.1) without the perturbation g. We know by a result in [2] that this problem has a positive solution $\bar{u} > 0$, such that $I_0(\bar{u}) = m$ where

$$m = \inf_{u \in N} I_0(u)$$

and N is the Nehari's manifold given by

$$N \equiv \{u \in X \setminus \{0\}\colon\ I_0'(u) \cdot u = 0\}.$$

Now we prove that I possesses the Mountain Pass Geometry.

Lemma 4.1 *(Mountain Pass Geometry). The functional I verifies*

(a) $I(0) = 0$.
(b) *There exist positive constants ρ and R such that $I(u) \geqslant \rho$ if $\|u\| = R$.*
(c) *There exists $z \in W^{1,p}(\mathbb{R})$ such that $I(z) < 0 = I(0)$ if $\|z\| > R$.*

Proof. The item (a) is immediate. Using the Hölder's inequality, the Young's inequality, the condition (V_0) and Remark 2.5 we obtain that

$$I(u) \geqslant \frac{C_1}{p}\|u\|^p - \frac{C_2}{q}\|u\|^q - C_3|g|_s\|u\| \geqslant \left(\frac{C_1}{p} - \frac{\epsilon^p}{p}\right)\|u\|^p - \frac{C_2}{q}\|u\|^q - C_\epsilon\big[|g|_s\big]^{p'}.$$

Fixing $\epsilon \in (0,1)$ we can find $R > 0$, $\delta_1 > 0$ sufficiently small and $\rho > 0$ such that $I(u) \geqslant \rho$ if $\|u\| = R$ and $|g|_s < \delta_1$. Hence the item (b) is proved.

To prove item (c), is sufficient to observe that $I_0(t\bar{u}) \to -\infty$, as $t \to \infty$, because $q > p$. Then there exists $t_0 > 0$ such that $I_0(t\bar{u}) < 0$ if $t \geqslant t_0$. From this fact, together with $g \in C_+$, it follows that

$$I(t\bar{u}) = I_0(t\bar{u}) - t\int_{\mathbb{R}} g(x)u(x)\,dx < 0 \quad \text{if } t \geqslant t_0.$$

Define $z \equiv t\bar{u} \in W^{1,p}(\mathbb{R})$; it follows that for all $t \geqslant t_0$, $\|z\| > R$ and $I(z) < 0$. □

Remark 4.2. It follows from Lemma 4.1, by applying the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [3], that there exists a $(\mathrm{PS})_{c_1}$ sequence $(z_n) \subset W^{1,p}(\mathbb{R})$ such that

$$I(z_n) \to c_1, \qquad I'(z_n) \to 0 \quad \text{in } \big(W^{1,p}(\mathbb{R})\big)^*, \text{ as } n \to \infty,$$

where

$$c_1 = \inf_{h \in \Gamma} \sup_{u \in h} I(u) > 0,$$

with

$$\Gamma = \big\{h \in C\big([0,1], W^{1,p}(\mathbb{R})\big)\colon\ h(0) = 0 \text{ and } h(1) = t_0\bar{u}\big\}.$$

5. Proof of Theorem 1.2

Recall that v_0 is a weak solution for problem (2.3) given by Lemma 2.7 such that $J(v_0) = c < 0$. Then it follows from the proof of Lemma 2.4(a) that $I(u_0) = J(v_0) = c < 0$, where $u_0 = f(v_0)$ is a weak solution of problem (1.1) given by Proposition 2.6.

By Lemma 4.1 and Remark 4.2 there exists a non-negative $(\mathrm{PS})_{c_1}$ sequence $(u_n) \subset W^{1,p}(\mathbb{R})$ (because $g \in C_+$ and I is an even function, then we can replace u_n by $|u_n|$) such that

$$I(u_n) \to c_1, \qquad I'(u_n) \to 0 \quad \text{in } \big(W^{1,p}(\mathbb{R})\big)^*, \text{ as } n \to \infty.$$

We claim that u_n is bounded. In fact, using the Hölder's inequality, the condition (V_0) and Remark 2.5 we obtain that

$$c_1 + \frac{1}{q}\left|I'(u_n)\right|_{(W^{1,p}(\mathbb{R}))^*} \cdot \|u_n\| + 1 \geqslant I(u_n) - \frac{1}{q}I'(u_n)\cdot u_n \geqslant C_1\left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p - C\left(1 - \frac{1}{q}\right)|g|_s\|u_n\|.$$

Hence u_n is bounded in $W^{1,p}(\mathbb{R})$. By Lemma 2.4, v_n is also a bounded $(\mathrm{PS})_{c_1}$ sequence for the functional J. Hence $v_n \rightharpoonup v_1$ weakly in $W^{1,p}(\mathbb{R})$, as $n \to \infty$ and using Lemma 2.7, v_1 is a weak solution of problem (2.3). By Proposition 2.6, $u_1 \equiv f(v_1)$ is a weak solution of problem (1.1). We will prove that $u_0 \neq u_1$ by proving that $I(u_0) \neq I(u_1)$. We do this by arguing as in Rădulescu and Smets [37].

Lemma 5.1. *Let $(u_n) \subset W^{1,p}(\mathbb{R})$ be a Palais–Smale sequence for the functional I in the level $c_1 \in \mathbb{R}$, given above, such that $u_n \rightharpoonup u_1$ weakly in $W^{1,p}(\mathbb{R})$, as $n \to \infty$. Then either $u_n \to u_1$ strongly in $W^{1,p}(\mathbb{R})$ or $c_1 \geqslant m + I(u_1)$.*

Assuming this lemma for while, it follows that either the sequence (u_n) converges strongly in $W^{1,p}(\mathbb{R})$, as $n \to \infty$ and in this case

$$I(u_1) = \lim_{n\to\infty} I(u_n) = c_1 > 0 > c = I(u);$$

or

$$c_1 = \lim_{n\to\infty} I(u_n) \geqslant I(u_1) + m.$$

If we suppose that $I(u_1) = I(u_0) = c$ we obtain that $c_1 \geqslant c + m$, which contradicts the lemma below and Theorem 1.2 is proved by choosing $0 < \epsilon < \delta$ where $\delta = \min\{\delta_1, \delta_2\}$ and δ_2 is also defined in the following lemma.

Lemma 5.2. *Let c, c_1 and m defined previously and a function $f \in C_+$ satisfying $|f|_{L^s(\mathbb{R})} = 1$. Then there exist real numbers $R > 0$ and $\delta_2 = \delta_2(R)$ such that $c_1 < c + m$ for all functions $g = \epsilon f$ whenever $\epsilon < \delta_2$. (In fact, R is given in the proof of Affirmation* 3.1.)

Proof of Lemma 5.1. By Lemma 2.7 and Proposition 2.6, if $u_n \to u_1$, as $n \to \infty$, by the continuity of the functional I we obtain that u_1 is a critical point of I and $I(u_1) = c_1$.

On the other hand, if the sequence (u_n) does not converge strongly to u_1 in $W^{1,p}(\mathbb{R})$, we define $z_n = u_n - u_1$ and we obtain that $z_n \rightharpoonup 0$ weakly in $W^{1,p}(\mathbb{R})$, as $n \to \infty$. Thus we can assume that

$$\|z_n\| \to \gamma > 0, \quad \text{as } n \to \infty. \tag{5.1}$$

Using that $u_n \rightharpoonup u_1$ weakly in $W^{1,p}(\mathbb{R})$, as $n \to \infty$, Remark 2.5 and by [20, Theorem 13.44] we conclude that

$$\int_{\mathbb{R}} g(x) z_n(x)\,dx \to 0, \quad \text{as } n \to \infty.$$

Thus

$$I(z_n) = I_0(z_n) + o(1).$$

We recall Proposition 4.1 in [2] (for $p = 2$ see [34]), that is,

$$\liminf_{n\to\infty}\left|u_n' u_n^{\beta-1}\right|_p^p \geqslant \liminf_{n\to\infty}\left|z_n' z_n^{\beta-1}\right|_p^p + \left|u_1' u_1^{\beta-1}\right|_p^p, \tag{5.2}$$

as well as, the identities

$$\|u_n\|^p - \|u_1\|^p - \|z_n\|^p = o(1), \text{ as } n \to \infty, \tag{5.3}$$

$$|u_n|_q^q - |u_1|_q^q - |z_n|_q^q = o(1), \text{ as } n \to \infty, \tag{5.4}$$

given by Brezis and Lieb [12]. From (5.2)–(5.4) it follows that

$$\liminf_{n\to\infty}\left[I(u_n) - I(u_1) - I(z_n)\right] \geqslant 0.$$

Hence taking a subsequence if necessary,

$$\lim_{n\to\infty}\big[I(u_n)-I(u_1)-I(z_n)\big]\geqslant 0,$$

and we conclude that

$$c_1+o(1)=I(u_n)\geqslant I(u_1)+I(z_n)+o(1)=I_0(z_n)+I(u_1)+o(1). \tag{5.5}$$

Similarly, using Lemma 2.7 and Proposition 2.6, we obtain

$$o(1)=I'(u_n)\cdot u_n\geqslant I'(u_1)\cdot u_1+I'(z_n)\cdot z_n+o(1)=I_0'(z_n)\cdot z_n+o(1).$$

Then

$$I_0'(z_n)\cdot z_n\leqslant o(1). \tag{5.6}$$

If $\lim_{n\to\infty} I_0'(z_n)\cdot z_n<0$, then, for n large enough, by [2, Lemma 2.1(b)] there exists $\lambda_n\in\,]0,1[$ such that $\lambda_n z_n\in N$. Furthermore one has that

$$\limsup_{n\to\infty}\lambda_n<1. \tag{5.7}$$

Otherwise if $\limsup_{n\to\infty}\lambda_n=1$, then along a subsequence, we have that $\lambda_n\to 1$, as $n\to\infty$. Hence

$$I_0'(z_n)\cdot z_n=I_0'(\lambda_n z_n)\cdot\lambda_n z_n+o(1).$$

Thus $\lim_{n\to\infty} I_0'(z_n)\cdot z_n=0$. But this is a contradiction. Therefore $\limsup_{n\to\infty}\lambda_n<1$.

Since that $\lambda_n z_n\in N$, by (5.5) we have that

$$\begin{aligned}c_1+o(1)&=I(u_n)\geqslant I_0(z_n)+I(u_1)+o(1)\\&=\frac{\beta-1}{p\beta}\lambda_n^{-p}\int_{\mathbb{R}}\frac{1}{p}\big[\big|\lambda_n z_n'\big|^p+V(x)|\lambda_n z_n|^p\big]+\lambda_n^{-q}\frac{q-p\beta}{pq\beta}|\lambda_n z_n|_q^q+I(u_1)+o(1).\end{aligned}$$

Using the inequality (5.7) and taking the lim sup in the above inequality we obtain

$$c_1\geqslant I_0(\lambda_n z_n)|_N+I(u_1)+o(1)=m+I(u_1),$$

and Lemma 5.1 is proved.

Now, we will study the case where $\lim_{n\to\infty} I_0'(z_n)\cdot z_n=0$. Since the inequality (5.5) is satisfied, it is sufficient to prove that

$$I_0(z_n)\geqslant m+o(1).$$

This result follows by adapting some arguments used in [37] (see also [1,5]). In fact, define

$$\chi_n\equiv I_0'(z_n)\cdot z_n,\qquad \varphi_n\equiv\int_{\mathbb{R}}\big[\big|z_n'\big|^p+V(x)|z_n|^p\big],$$

$$\theta_n\equiv\int_{\mathbb{R}}\beta^pK_0|z_n|^{p(\beta-1)}\big|z_n'\big|^p\quad\text{and}\quad\psi_n\equiv\int_{\mathbb{R}}|z_n|^q.$$

Therefore

$$0=\lim_{n\to\infty}I_0'(z_n)\cdot z_n=\lim_{n\to\infty}\chi_n=\lim_{n\to\infty}[\varphi_n+\theta_n-\psi_n]. \tag{5.8}$$

Using that V is bounded, $z_n\in W^{1,p}(\mathbb{R})$, Remark 2.5, we conclude that

$$\varphi_n,\psi_n,\theta_n \text{ are finite non-negative numbers} \tag{5.9}$$

and by (V_0) one has that

$$\varphi_n\geqslant C\|z_n\|\geqslant C\gamma>0. \tag{5.10}$$

To conclude the proof of this lemma we need the following result:

Affirmation 5.3. *There exists a sequence* $(t_n) \subset \mathbb{R}$ *such that*

$$\lim_{n\to\infty} t_n = 1 \quad \textit{and} \quad I_0'(t_n z_n)\cdot t_n z_n = 0.$$

Assuming the affirmation for a while, we obtain

$$\lim_{n\to\infty}\bigl[I_0(z_n) - I_0(t_n z_n)\bigr] = \frac{1}{p}\bigl(1-t_n^p\bigr) + K_0\frac{\beta^{p-1}}{p}\bigl(1-t_n^{p\beta}\bigr)\bigl\|z_n^{\beta-1}z_n'\bigr\|_p^p - \frac{1}{q}\bigl(1-t_n^q\bigr)\|z_n\|_q^q = 0.$$

Since $t_n z_n \in N$ we get

$$I_0(z_n) = I_0(t_n z_n) + o(1) \geqslant m + o(1).$$

This completes our proof. □

Proof of Affirmation 5.3. Let $t = 1+\tau$, where $\tau > 0$ is small enough. Using the definitions of $\chi_n, \varphi_n, \theta_n$ and ψ_n we obtain

$$\begin{aligned}
I_0'\bigl((1+\tau)z_n\bigr)\cdot(1+\tau)z_n &= (1+\tau)^p\varphi_n + (1+\tau)^{p\beta}\theta_n - (1+\tau)^q\psi_n\\
&= \bigl[(1+\tau)^p - (1+\tau)^q\bigr]\varphi_n + \bigl[(1+\tau)^{p\beta} - (1+\tau)^q\bigr]\theta_n - (1+\tau)^q\chi_n\\
&= \tau\bigl[(p-q)\varphi_n + (p\beta-q)\theta_n\bigr] + \varphi_n o(\tau) + \theta_n o(\tau) + (1+\tau)^q\chi_n.
\end{aligned}$$

Define

$$\tau_n \equiv \frac{K|\chi_n|}{\varphi_n(q-p)+\theta_n(q-p\beta)}, \quad \text{where } K>1 \text{ is a constant.}$$

We can assume that $\tau_n > 0$. Using (5.8)–(5.10), we infer that

$$\lim_{n\to\infty}\tau_n = 0.$$

Note that

$$\begin{aligned}
I_0'\bigl((1+\tau_n)z_n\bigr)\cdot(1+\tau_n)z_n &= \frac{K|\chi_n|}{\varphi_n(q-p)+\theta_n(q-p\beta)}\bigl[(p-q)\varphi_n + (p\beta-q)\theta_n\bigr]\\
&\quad + \varphi_n o(\tau_n) + \theta_n o(\tau_n) + \left[1 + \frac{K|\chi_n|q}{\varphi_n(q-p)+\theta_n(q-p\beta)} + o(\tau_n)\right]\chi_n\\
&= -K|\chi_n| + \chi_n + \frac{K|\chi_n|q}{\varphi_n(q-p)+\theta_n(q-p\beta)} + \varphi_n o(\tau_n) + \theta_n o(\tau_n) + \chi_n o(\tau_n).
\end{aligned}$$

Using that $\lim_{n\to\infty}\tau_n = 0$, (5.9), (5.10) and that $K>1$ we conclude that

$$I_0'\bigl((1+\tau_n)z_n\bigr)\cdot(1+\tau_n)z_n < 0 \quad \text{for } n \text{ large enough.}$$

Similarly,

$$I_0'\bigl((1-\tau_n)z_n\bigr)\cdot(1-\tau_n)z_n > 0 \quad \text{for } n \text{ large enough.}$$

Hence there exists t_n between $(1-\tau_n, 1+\tau_n)$ such that

$$I_0'(t_n z_n)\cdot t_n z_n = 0. \qquad \square$$

Proof of Lemma 5.2. The proof of the lemma follows by using the next affirmation.

Affirmation 5.4. $\sup_{t\geqslant 0} I(t\bar{u}) < m + c$.

Proof. We need to prove that $c + m > 0$ for $\delta_1 > 0$ and $R > 0$ given in the proof of Theorem 1.1. In fact, let u be a solution of the problem (1.1) obtained by Theorem 1.1. Since $I'(u) \cdot u = 0$ one has

$$c = I(u) = \left[\frac{1}{p} - \frac{1}{q}\right] \int_{\mathbb{R}} \left[|u'|^p + V(x)|u|^p\right] + K_0\beta^{p-1}\left[\frac{q-p\beta}{pq}\right]\left|u^{\beta-1}u'\right|_p^p - \left[1 - \frac{1}{q}\right] \int_{\mathbb{R}} g(x)u(x)\,dx.$$

Using the fact that the second term is positive together with the condition (V_0), the Hölder's inequality, Remark 2.5, we conclude that

$$c \geqslant \left[\frac{1}{p} - \frac{1}{q}\right] C\|u\|^p - \left[1 - \frac{1}{q}\right]|g|_s\|u\|,$$

where C is a constant. Applying the Young's inequality, we find that

$$c \geqslant \left[\frac{1}{p} - \frac{1}{q}\right] C\|u\|^p - \frac{\lambda^p}{p}\|u\|^p - \frac{M}{p'\lambda^{p'}}\left[1 - \frac{1}{q}\right]^{p'}|g|_s^{p'} \quad \left(p' = \frac{p}{p-1}\right).$$

Now taking the constant $\lambda \equiv (1 - \frac{p}{q})^{\frac{1}{p}}$ and arguing as [37] (see also [5]) one has that $\frac{\lambda^p}{p} = \frac{1}{p} - \frac{1}{q}$ and $\frac{M}{p'\lambda^{p'}}[1 - \frac{1}{q}]^{p'} = \frac{M}{p'}[\frac{q-1}{q}]^{p'}\frac{1}{(1-\frac{p}{q})^{1/(p-1)}} \equiv \mu > 0$. Then

$$c \geqslant -\mu|g|_s^{p'}. \tag{5.11}$$

Choosing $|g|_s$ sufficiently small, we obtain that the negative real number c is close enough to zero and since $m > 0$ we conclude that $c + m > 0$.

As $c + m > 0 = I(0)$ and the functional I is continuous, there exist $t' > 0$ and $\epsilon' > 0$ (which is uniformly with respect to all g satisfying $0 < |g|_s < \epsilon'$), such that

$$\sup_{t \in [0,t']} I(t\bar{u}) < m + c \quad \text{if } |g|_s < \epsilon' < \delta_1.$$

Thus, to conclude Affirmation 5.4 it is enough to prove that

$$\sup_{t \geqslant t'} I(t\bar{u}) < m + c \quad \text{if } |g|_s \text{ is sufficiently small.}$$

But, since $I(\bar{u}) = m$, by a property of the Nehari's manifold, namely, $I_0(\lambda u) \leqslant I_0(u)$, for every $u \in N$ and for $\lambda > 0$, we have that

$$I(t\bar{u}) \leqslant \sup_{t \geqslant t'} I(t\bar{u}) \leqslant m - t' \int_{\mathbb{R}} g(x)\bar{u}(x)\,dx = m - t'\epsilon a_0,$$

where $a_0 = \int_{\mathbb{R}} f(x)\bar{u}(x)\,dx$ is positive because $f \in C_+$, $|f|_s = 1$ and $g = \epsilon f$. From (5.11) and from the fact that $|g|_s = \epsilon^s|f|_s = \epsilon^s < \epsilon$ with $\epsilon \leqslant \epsilon''$ it follows that

$$c > -\mu\epsilon^{p'}.$$

Choosing $\epsilon'' > 0$ sufficiently small such that $-t'\epsilon a_0 < -\mu\epsilon^{p'}$ for all $\epsilon < \epsilon''$ we conclude that

$$I(t\bar{u}) < m + c.$$

To finish the proof it is enough to take $\delta_2 = \min\{\epsilon', \epsilon''\}$. □

Remark 5.5. The weak non-negative solution of problem (1.1), $u \in C^{1,\alpha}(\mathbb{R})$ with $\alpha \in (0, 1)$, is strictly positive in $\mathbb{R}$. This fact can be proved by applying Vázquez's result [41, Theorem 5].

In fact, let $u \geqslant 0$ be a solution of problem (1.1). By changing variable, $u = f(v)$, we obtain $v \geqslant 0$ and v verifies the following equation:

$$-\left(|v'|^{p-2}v'\right)' + \left[V(x)\left|f(v)\right|^{p-2}f(v)\right]f'(v) = f'(v)\left[\left|f(v)\right|^{q-2}f(v)\right] + g(x) \quad x \in \mathbb{R}.$$

Notice that $u \neq 0$, thus $v \neq 0$. Since $g \in C_+$, then $g \geqslant 0$ a.e. in $\mathbb{R}$ and we obtain that

$$-\left(|v'|^{p-2}v'\right)' + \left[V(x)\left|f(v)\right|^{p-2}f(v)\right]f'(v) \geqslant 0 \quad \text{a.e. in } \mathbb{R}.$$

Define, as in Vázquez [41, Theorem 5],

$$\beta(s) \equiv \left[\left|f(s)\right|^{p-2}f(s)\right]f'(s).$$

By Lemma 2.1(c) and (e) we have

$$\left[\beta(s)s\right]^{-\frac{1}{p}} \geqslant \left[\frac{1}{\beta}f(s)\left|f(s)\right|^{p-2}f(s)\right]^{-\frac{1}{p}} = \frac{1}{f(s)} \geqslant \frac{1}{|s|}.$$

Therefore

$$\int_0^1 \left[\beta(s)s\right]^{-\frac{1}{p}}\,ds = +\infty.$$

Then, by applying Maximum Principle due to Vázquez we obtain $v > 0$ in $\mathbb{R}$. Therefore u is strictly positive in $\mathbb{R}$. □

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