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Loop quantization of a model for $D = 1 + 2$ (anti)de Sitter gravity coupled to topological matter

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Abstract

We present a complete quantization of Lorentzian $D = 1 + 2$ gravity with cosmological constant, coupled to a set of topological matter fields. The approach of loop quantum gravity is used thanks to a partial gauge fixing leaving a residual gauge invariance under a *compact* semi-simple gauge group, namely $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$. A pair of quantum observables is constructed, which are non-trivial despite being gauge-equivalent to zero at the classical level. A semi-classical approximation based on appropriately defined coherent states shows non-vanishing expectation values for them, thus not reproducing their classical behaviour.

Keywords: quantum gravity, topological theories, loop quantization, lower dimensional gravity

1. Introduction

This paper presents a generalization of a previous work [1] where the loop quantum gravity (LQG) quantization of $D = 1 + 2$ gravity with a positive cosmological constant, in the presence of a Barbero–Immirzi-like parameter analogous to the one which may be introduced in the four-dimensional gravitation theory [2, 3] (and first introduced in the three-dimensional theory by the authors of [4]), was performed using a partial gauge fixing procedure leaving the compact $\text{SU}(2)$ group as the residual group of gauge invariance.

$D = 2 + 1$ gravity with a cosmological constant Λ is described by a Lorentz connection ω and a triad e 1-forms, components of an (a)ds connection [5]. (A)dS denotes the $D = 1 + 2$ de Sitter $\text{dS} = \text{SO}(1,3)$ or anti-de Sitter $\text{ADS} = \text{SO}(2,2)$ group, and (a)ds, ds = so

(1,3) or $\mathfrak{ads} = \mathfrak{so}(2,2)$, its Lie algebra. The canonical structure and quantization of this theory have been studied, beyond the pioneering work of Witten [5], in [4, 6–9], among others (see [10] for a general review based on previous literature). A Barbero–Immirzi-like parameter has also been defined in [6] for the three-dimensional theory, although in a different way as in [4], and its role has been discussed in [7, 8] for the classical as well as for the quantum theory³.

The coupling to ‘topological matter’ shown in the present paper will be performed via an extension of the $(\mathfrak{a})\mathfrak{dS}$ Lie algebra which consists in the addition of a multiplet of *non-commuting* generators belonging to the adjoint representation of $(\mathfrak{a})\mathfrak{dS}$, in such a way that the resulting algebra closes on a semi-simple algebra, denoted by $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ for ‘semi-simple extension of $(\mathfrak{a})\mathfrak{dS}$ ’. It results that this extension is a deformation of an algebra introduced by the authors of [11–13] as the extension of $(\mathfrak{a})\mathfrak{dS}$ by *commuting* generators in the adjoint representation. The deformation parameter, λ , will play the role of a coupling constant. This extended algebra possesses four non-degenerate invariant quadratic forms, instead of two for $(\mathfrak{a})\mathfrak{dS}$, which will imply the presence of four independent couplings, three of them being generalized Barbero–Immirzi like parameters.

The theory will be defined as the Chern–Simons theory of a $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ connection, the components of which are the gravity fields: the spin connection ω and the triad e ; and a multiplet of matter fields: 1-forms $\{b, c\}$ transforming in the adjoint representation of $(\mathfrak{a})\mathfrak{dS}$. For suitable choices of the signs of the $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ structure constant parameters Λ and λ , the algebra admits $\mathfrak{so}(4)$ as a compact sub-algebra. We shall restrict ourselves to this family of parametrizations. Moreover, with the same choice of signs, it factorizes as the direct sum of two \mathfrak{dS} sub-algebras, which allows a simpler treatment of the theory, and in particular permits us to use the results of [1] where the pure gravity case, based on the $(A)\mathfrak{dS}$ gauge group, is studied in details.

Loop quantization methods will be applied to the canonical quantization of the model, in the special case of the two-dimensional space sheet topology being that of a cylinder. A partial gauge fixing preserving gauge invariance under $\text{Spin}(4)$, the universal covering of $\text{SO}(4)$, will have to be performed. The constraints will be curvature constraints which can be entirely solved, leaving a physical Hilbert space, with a spin network type basis labelled by pairs of half-integer spins. Finally a pair of quantum observables will be constructed, which are diagonal in the spin-network basis, with a discrete spectrum reminiscent of the area operator spectrum of dimension $1 + 3$ LQG [14]. We will however see that they differ from the latter by their essentially quantum nature, with no classical counterpart—although their expectation values in a semi-classical approximation do not vanish.

The model is presented in section 2 in the canonical formalism, with the derivation of the Hamiltonian and of the constraints. In section 3 we gauge fix the non-compact part of the gauge group, leaving an $\mathfrak{so}(4)$ residual gauge invariance, which allows us, in section 4, to proceed to the quantization using the standard tools of LQG. Observables are constructed in section 5. The appendix is devoted to the definition of the semi-simple extension of a Lie algebra, with application to the extension $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ of the $(\mathfrak{a})\mathfrak{dS}$ algebra together with the study of its compact sub-algebras and factorization properties.

2. A model for (anti)-de Sitter gravity with topological matter

The model is described as a Chern–Simons theory in a $D = 1 + 2$ orientable manifold \mathcal{M} . The gauge group is the ‘semi-simple extension’ $S(A)\mathfrak{dS}$ of the $D = 1 + 2$ de Sitter or anti-de

³ We thank Marc Geiller for informing us on the references [6–8].

Sitter group (A)dS = SO(1,3) or SO(2,2) with the corresponding Lie algebra $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ being described in the appendix. We consider as a basis the six generators $\{J^I, P^I; I = 0, 1, 2\}$ of (A)dS, together with the six extension generators $\{Q^I, R^I; I = 0, 1, 2\}$, satisfying the commutation rules

$$\begin{aligned} [J^I, J^J] &= \epsilon^{IJ}{}_K J^K, & [J^I, P^J] &= \epsilon^{IJ}{}_K P^K, & [P^I, P^J] &= \sigma\Lambda \epsilon^{IJ}{}_K J^K, \\ [J^I, Q^J] &= \epsilon^{IJ}{}_K Q^K, & [J^I, R^J] &= \epsilon^{IJ}{}_K R^K, \\ [P^I, Q^J] &= \epsilon^{IJ}{}_K R^K, & [P^I, R^J] &= \sigma\Lambda \epsilon^{IJ}{}_K Q^K, \\ [Q^I, Q^J] &= \sigma\lambda \epsilon^{IJ}{}_K J^K, & [Q^I, R^J] &= \sigma\lambda \epsilon^{IJ}{}_K P^K, & [R^I, R^J] &= \Lambda\lambda \epsilon^{IJ}{}_K J^K. \end{aligned} \quad (2.1)$$

σ is the $D = 1 + 2$ metric signature⁴, Λ and λ are two arbitrary parameters defining the closure of the algebra, which will play in turn the roles of a cosmological constant and of a coupling constant, as we shall see. The properties of this algebra are described in the appendix.

Remark. The present model is a generalization of the model of [11–13] in the sense that, for $\Lambda = \lambda = 0$, the algebra (2.1) reduces to the Lie algebra of the gauge group I(SO(1,2))—an extension of the Poincaré group ISO(1,2) through Abelian generators⁵.

The field content of the theory is given by the $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ connection 1-form

$$\mathcal{A} = \omega^I J_I + e^I P_I + b^I Q_I + c^I R_I \equiv \sum_{a=1}^{12} \mathcal{A}^a \mathcal{T}_a. \quad (2.2)$$

In order to write an action, we need an $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ -invariant non-degenerate quadratic form. It turns out that in the present case we have four such forms, $K_{\alpha\beta}^i$, (given in equations (A.7) of the appendix) and then the action may be written as a superposition of four Chern–Simons actions for the connection (2.2), each one corresponding to one of these quadratic forms⁶:

$$S = \sum_{i=1}^4 c_i \mathcal{S}_i, \quad \mathcal{S}_i = \int_{\mathcal{M}} K_{\alpha\beta}^i \mathcal{A}^\alpha \left(d\mathcal{A} + \frac{2}{3} \mathcal{A}\mathcal{A} \right)^\beta. \quad (2.3)$$

It is interesting to explicitly write the second term:

$$\mathcal{S}_2 = \int_{\mathcal{M}} \left(e^I F_I(\omega) + \frac{\sigma\Lambda}{6} e^I (e \times e)_I + \sigma\lambda \left(c^I D_\omega b_I + \frac{1}{2} e^I (b \times b)_I + \frac{\sigma\Lambda}{2} e^I (c \times c)_I \right) \right),$$

where⁷

$$F^I(\omega) = d\omega^I + \frac{1}{2}(\omega \times \omega)^I, \quad D_\omega b^I = db^I + (\omega \times b)^I.$$

⁴ The indices I, J, \dots take the values 0, 1, 2. They may be lowered or raised with the metric $\eta_{IJ} = \text{diag}(\sigma, 1, 1)$, $\sigma = \pm 1$ being the signature of the rotation or Lorentz group SO(3) or SO(1,2). The completely antisymmetric tensor ϵ^{IJK} is defined by $\epsilon^{012} = 1$. Note that $\epsilon_{012} = \eta_{0I}\eta_{1J}\eta_{2K}\epsilon^{IJK} = \sigma$. Space–time indices will be denoted later on by greek letters $\mu, \nu, \dots = 0, 1, 2$ or the symbols t, x, y , and space indices by latin letters $a, b, \dots = 1, 2$ or the symbols x, y .

⁵ We use the notation of [13] for the basis generators.

⁶ We do not write explicitly the wedge symbol \wedge for the external products of forms.

⁷ We use the notation $(X \times Y)^I \equiv \epsilon_{JK}{}^I X^J Y^K$.

The action S_2 describes a pair of 1-form ‘topological matter’ fields b^I , c^I coupled to a first order gravitation theory described by the spin connection ω^I and the dreibein e^I . Λ is the cosmological constant and λ a coupling constant. With the redefinitions $b^I = \sqrt{|\lambda|} b$ and $c^I = \sqrt{|\lambda|} c$ and taking the limit $\Lambda = \lambda = 0$ one recovers the ‘BCEA’ action of [11–13] as a special case. However, and as it has already been noted by these authors in their particular case, the general case considered in the present paper may lead to equivalent interpretations where the roles of e^I , b^I and c^I as dreibein and matter are permuted. These alternatives are related to the various possible choices for the signs of the parameters Λ and λ . One sees from the discussion made in the appendix, and especially looking at table A.1, that this also corresponds to permutations of the roles of the parameters Λ and λ as cosmological and coupling constants.

Now, since non-vanishing Λ and λ imply the existence of four non-degenerate invariant quadratic forms, one has to consider the general action (2.3). However, as it stands, this action would lead to a rather complicated and non-practical formulation. Substantial simplification arises if one uses the factorization property explained in section A.2.2 of the appendix.

We concentrate from now on to the case of Lorentzian signature $\sigma = -1$ and positive parameters Λ and λ :

$$\text{signs } (\sigma, \Lambda, \lambda) = (-, +, +), \quad (2.4)$$

corresponding to the first line in tables A.1, A.2 and A.3 of the appendix. The cases corresponding to the first, second and third lines of the tables are equivalent. We do not treat the fourth line’s case, where the factorization is not of the form of $(\mathfrak{a})\mathfrak{d}s_+ \oplus (\mathfrak{a})\mathfrak{d}s_-$ ⁸, neither the Riemannian ones ($\sigma = 1$). Thus, in our case, the algebra $\mathfrak{s}(\mathfrak{a})\mathfrak{d}s$ factorizes in two de Sitter sub-algebras $\mathfrak{d}s_{\pm}$ as shown in (A.5). Expanding the connection (2.2) in the factorized basis (A.6), we obtain

$$\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-, \quad \mathcal{A}_{\pm} = \omega_{\pm}^I J_{I\pm} + e_{\pm}^I P_{I\pm} \equiv \sum_{A=1}^6 T_A \mathcal{A}_{\pm}^A, \quad (2.5)$$

with

$$\omega_{\pm}^I = \omega^I \mp \sqrt{\Lambda\lambda} c^I, \quad e_{\pm}^I = \sqrt{\Lambda} e^I \mp \sqrt{\lambda} b^I. \quad (2.6)$$

The action (2.3) is now the sum of two de Sitter Chern–Simons actions

$$S = S_+ + S_- = \kappa_+ \left(S_+^{(1)} - \frac{1}{\gamma_+} S_+^{(2)} \right) + \kappa_- \left(S_-^{(1)} - \frac{1}{\gamma_-} S_-^{(2)} \right), \quad (2.7)$$

where κ_{\pm} and γ_{\pm} are non-zero finite real parameters⁹, and

$$S_{\pm}^{(n)} = - \int_{\mathcal{M}} k_{AB}^{(n)} \left(\mathcal{A}_{\pm}^A \left(d\mathcal{A}_{\pm}^B + \frac{1}{3} (\mathcal{A}_{\pm} \times \mathcal{A}_{\pm})^B \right) \right), \quad n = 1, 2, \quad (2.8)$$

are the actions calculated using the two independent invariant quadratic forms [1, 4, 5] $k^{(n)}$ ($n = 1, 2$) belonging to each of the algebras $\mathfrak{d}s_{\pm}$, as shown in equations (A.8) of the appendix:

⁸ In this case the factorization is $\mathfrak{so}(2,2) \oplus \mathfrak{so}(2,2)$, see table A.3. The maximal compact sub-algebra is the Abelian $\mathfrak{u}(1)^{\oplus 4}$, see table A.2.

⁹ γ_+ and γ_- are two analogues of the Barbero–Immirzi parameter [2] γ in dimension $(1+3)$ LQG, which share with it the property of not appearing in the classical field equations. See also [4] in the context of the dimension $(1+2)$ de Sitter theory.

$$\begin{aligned} k_{J_{\pm}^i, J^j}^{(1)} &= \eta_{IJ}, & k_{P_{\pm}^i, P^j}^{(1)} &= -\eta_{IJ}, \\ k_{J_{\pm}^i, P^j}^{(2)} &= \eta_{IJ}. \end{aligned} \quad (2.9)$$

(We only write the non-vanishing elements.)

Each individual action S_i in (2.3) would lead to the same field equations, and therefore the total action S leads to equations independent of the parameters c_i —in (2.3)—or κ_{\pm} , γ_{\pm} —in (2.7). These equations read simply, in the factorized formulation,

$$\mathcal{F}_{\pm} = 0, \quad \text{with } \mathcal{F}_{\pm} = d\mathcal{A}_{\pm} + \mathcal{A}_{\pm}\mathcal{A}_{\pm}. \quad (2.10)$$

With the signs of its parameters given in (2.4), the gauge algebra $\mathfrak{s}(\mathfrak{a})\mathfrak{d}\mathfrak{s}$ possesses a compact subalgebra $\mathfrak{so}(4)$, as seen in section A.2.1 of the appendix. Its basis generators are listed in the first line of table A.1. With the factorization (A.5), $\mathfrak{so}(4)$ correspondingly factorizes as

$$\mathfrak{so}(4) = \mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-, \quad \text{with } \mathfrak{so}(3)_{\pm} \subset \mathfrak{d}\mathfrak{s}_{\pm}. \quad (2.11)$$

A convenient new basis of the de Sitter sub-algebras $\mathfrak{d}\mathfrak{s}_{\pm}$ is given by the generators L_{\pm}^i and K_{\pm}^i ($i = 1, 2, 3$), with the L_{\pm}^i 's forming a basis of the sub-algebra $\mathfrak{so}(3)_{\pm}$ of $\mathfrak{d}\mathfrak{s}_{\pm} = \mathfrak{so}(3,1)_{\pm}$, and the 'boosts' K_{\pm}^i 's generating the non-compact part of $\mathfrak{d}\mathfrak{s}_{\pm}$. This new basis satisfies the commutation rules¹⁰

$$\left[L_{\pm}^i, L_{\pm}^j \right] = \varepsilon^{ij}{}_k L_{\pm}^k, \quad \left[L_{\pm}^i, K_{\pm}^j \right] = \varepsilon^{ij}{}_k K_{\pm}^k, \quad \left[K_{\pm}^i, K_{\pm}^j \right] = -\varepsilon^{ij}{}_k L_{\pm}^k,$$

and is defined as

$$L_{\pm} = \left(P_{\pm}^2/\sqrt{\Lambda}, -P_{\pm}^1/\sqrt{\Lambda}, J_{\pm}^0 \right), \quad K_{\pm} = \left(J_{\pm}^2, -J_{\pm}^1, -P_{\pm}^0/\sqrt{\Lambda} \right), \quad (2.12)$$

the $\mathfrak{d}\mathfrak{s}_{\pm}$ generators J_{\pm}^i and P_{\pm}^i being expressed in terms of the original generators J^i , P^i , Q^i and R^i by equation (A.6). The expansion of the $\mathfrak{s}(\mathfrak{a})\mathfrak{d}\mathfrak{s}$ connection (2.2) in the basis L_{\pm} , K_{\pm} reads

$$\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-, \quad \mathcal{A}_{\pm} = A_{\pm} \cdot L_{\pm} + B_{\pm} \cdot K_{\pm},$$

with

$$A_{\pm} = \left(\sqrt{\Lambda} e_{\pm}^2, -\sqrt{\Lambda} e_{\pm}^1, -\omega_{\pm}^0 \right), \quad B_{\pm} = \left(\omega_{\pm}^2, -\omega_{\pm}^1, \sqrt{\Lambda} e_{\pm}^0 \right),$$

the components e_{\pm}^i and ω_{\pm}^i being given in (2.6).

We can now write the action S_{\pm} in terms of the new field components as¹¹

$$\begin{aligned} S_{\pm} &= -\frac{\kappa_{\pm}}{2} \int_{\mathbb{R}} dt \left(\int_{\Sigma} \left(\dot{\mathbf{A}}_{\pm} \cdot \left(\mathbf{B}_{\pm} - \frac{1}{\gamma_{\pm}} \mathbf{A}_{\pm} \right) + \dot{\mathbf{B}}_{\pm} \cdot \left(\mathbf{A}_{\pm} + \frac{1}{\gamma_{\pm}} \mathbf{B}_{\pm} \right) \right) \right. \\ &\quad \left. - \mathcal{G}_{\pm}(A_{t\pm}) - \mathcal{G}_{0\pm}(B_{t\pm}) \right), \end{aligned} \quad (2.13)$$

where

$$\mathcal{G}_{\pm}(A_{t\pm}) = \kappa_{\pm} \int_{\Sigma} A_{t\pm}^i(\mathbf{x}) \mathcal{G}_{\pm}^i(\mathbf{x}) = \kappa_{\pm} \int_{\Sigma} A_{t\pm} \cdot \left[\mathbf{D}\mathbf{B}_{\pm} - \frac{1}{\gamma_{\pm}} \left(\mathbf{F}_{\mathbf{A}_{\pm}} - \frac{1}{2} \mathbf{B}_{\pm} \times \mathbf{B}_{\pm} \right) \right], \quad (2.14a)$$

¹⁰ Indices i, j, \dots are raised and lowered by the Euclidean metric δ_{ij} . It will be convenient to adopt a vector-like notation, $A^i B_i = A \cdot B$, $\varepsilon_{ijk} A^j B^k = (A \times B)_i$, etc.

¹¹ Boldface letters represent space objects, e.g. $\mathbf{A} = A_a dx^a = (A_a^i dx^i, i = 1, 2, 3)$, etc.

$$\mathcal{G}_{0\pm}(B_{t\pm}) = \kappa_{\pm} \int_{\Sigma} B_{t\pm}^i(\mathbf{x}) \mathcal{G}_{0\pm}^i(\mathbf{x}) = \kappa_{\pm} \int_{\Sigma} B_{t\pm} \cdot \left[\mathbf{F}_{\mathbf{A}_{\pm}} - \frac{1}{2} \mathbf{B}_{\pm} \times \mathbf{B}_{\pm} + \frac{1}{\gamma_{\pm}} \mathbf{D}\mathbf{B}_{\pm} \right], \quad (2.14b)$$

with $\mathbf{F}_{\mathbf{A}_{\pm}} = \mathbf{d}\mathbf{A}_{\pm} + \frac{1}{2}\mathbf{A}_{\pm} \times \mathbf{A}_{\pm}$ and $\mathbf{D}\mathbf{B}_{\pm} = \mathbf{d}\mathbf{B}_{\pm} + \mathbf{A}_{\pm} \times \mathbf{B}_{\pm}$.

One first notes that the conjugate momenta of $A_{t\pm}^i$ and $B_{t\pm}^i$ are primary constraints, in Dirac's terminology [15], whereas $\mathcal{G}_{\pm}(A_{t\pm})$ and $\mathcal{G}_{0\pm}(B_{t\pm})$ are the secondary constraints ensuring the stability of the former primary constraints. $A_{t\pm}^i$ and $B_{t\pm}^i$ then play the role of Lagrange multipliers. There are still two primary constraints involving the conjugate momenta of the fields $A_{a\pm}^i$ and $B_{a\pm}^i$. They turn out to be of second class, whose solution according to the Dirac–Bergmann algorithm [15] gives rise to the Dirac–Poisson brackets

$$\begin{aligned} \left\{ A_{a\pm}^i(\mathbf{x}), A_{b\pm}^j(\mathbf{x}') \right\} &= \frac{1}{\kappa_{\pm}} \epsilon_{ab} \delta^{ij} \frac{\gamma_{\pm}}{1+\gamma_{\pm}^2} \delta^2(\mathbf{x} - \mathbf{x}'), \\ \left\{ B_{a\pm}^i(\mathbf{x}), A_{b\pm}^j(\mathbf{x}') \right\} &= -\frac{1}{\kappa_{\pm}} \epsilon_{ab} \delta^{ij} \frac{\gamma_{\pm}}{1+\gamma_{\pm}^2} \delta^2(\mathbf{x} - \mathbf{x}'), \\ \left\{ B_{a\pm}^i(\mathbf{x}), B_{b\pm}^j(\mathbf{x}') \right\} &= -\frac{1}{\kappa_{\pm}} \epsilon_{ab} \delta^{ij} \frac{\gamma_{\pm}}{1+\gamma_{\pm}^2} \delta^2(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (2.15)$$

The resulting Hamiltonian turns out to be fully constrained, as it is expected in such a general covariant theory:

$$H = H_+ + H_-, \quad H_{\pm} = \mathcal{G}_{\pm}(A_{t\pm}) + \mathcal{G}_{0\pm}(B_{t\pm}), \quad (2.16)$$

with the constraints \mathcal{G}_{\pm} and $\mathcal{G}_{0\pm}$ as given by (2.14a), (2.14b). These constraints are first class, obey the Dirac–Poisson bracket algebra (we only write the non-zero brackets)

$$\begin{aligned} \left\{ \mathcal{G}_{\pm}(\varepsilon), \mathcal{G}_{\pm}(\varepsilon') \right\} &= \mathcal{G}_{\pm}(\varepsilon \times \varepsilon'), \\ \left\{ \mathcal{G}_{0\pm}(\varepsilon), \mathcal{G}_{\pm}(\varepsilon') \right\} &= \mathcal{G}_{0\pm}(\varepsilon \times \varepsilon'), \\ \left\{ \mathcal{G}_{0\pm}(\varepsilon), \mathcal{G}_{0\pm}(\varepsilon') \right\} &= \sigma \mathcal{G}_{\pm}(\varepsilon \times \varepsilon'), \end{aligned} \quad (2.17)$$

and also generate the gauge transformations under which the theory is invariant:

$$\begin{aligned} \left\{ \mathcal{G}_{\pm}(\varepsilon), \mathbf{A}_{\pm} \right\} &= \mathbf{D}_{\pm}\varepsilon, & \left\{ \mathcal{G}_{\pm}(\varepsilon), \mathbf{B}_{\pm} \right\} &= \mathbf{B}_{\pm} \times \varepsilon; \\ \left\{ \mathcal{G}_{0\pm}(\varepsilon'), \mathbf{A}_{\pm} \right\} &= -\mathbf{B}_{\pm} \times \varepsilon' & \left\{ \mathcal{G}_{0\pm}(\varepsilon'), \mathbf{B}_{\pm} \right\} &= \mathbf{D}_{\pm}\varepsilon' \quad (\mathbf{D}_{\pm} = \mathbf{d} + \mathbf{A}_{\pm} \times). \end{aligned} \quad (2.18)$$

Invariance under these gauge transformations ensures diffeomorphism invariance, up to field equations. Indeed, infinitesimal diffeomorphisms, given by the Lie derivative, can be written as infinitesimal gauge transformations with parameters $(\varepsilon, \varepsilon') = (l_{\xi}\mathbf{A}_{\pm}, l_{\xi}\mathbf{B}_{\pm})$, up to field equations:

$$\begin{aligned} \mathfrak{L}_{\xi} \mathbf{A}_{\mu} &= \mathbf{D}_{\pm}\varepsilon - \mathbf{B}_{\pm} \times \varepsilon' + \text{field equations}, \\ \mathfrak{L}_{\xi} \mathbf{B}_{\pm} &= \mathbf{D}_{\pm}\varepsilon' + \mathbf{B}_{\pm} \times \varepsilon + \text{field equations}, \end{aligned} \quad (2.19)$$

with $\mathfrak{L} = dt_{\xi} + l_{\xi}d$ the Lie derivative.

3. Partial gauge fixing: the axial gauge

In order to be left with a compact gauge group, we partially fix the gauge, fixing the ‘boost’ gauge degrees of freedom, which correspond to the gauge transformations generated by the generators K_{\pm}^i defined in (2.12). This is done imposing new constraints $B_{y\pm}^i \approx 0$, implemented

by the addition of the terms

$$\int_{\Sigma} d^2x \left(\mu_{i+}(\mathbf{x}) B_{y+}^i(\mathbf{x}) + \mu_{i-}(\mathbf{x}) B_{y-}^i(\mathbf{x}) \right)$$

to the Hamiltonian (2.16), with $\mu_{i\pm}$ as Lagrange multiplier fields. This condition represents a genuine gauge fixing, very similar to the well-known axial gauge fixing used in Yang–Mills theories [16].

The six gauge fixing constraints together with the six constraints $\mathcal{G}_{0\pm}^i(\mathbf{x})$ defined in (2.14b) are second class, hence become strong equalities through Dirac's redefinition of the brackets. After insertion of the gauge fixing constraints, the \mathcal{G}^0 constraints read

$$\partial_x A_{y\pm}^i - D_{y\pm} \left(A_{x\pm}^i + \frac{1}{\gamma_{\pm}} B_x^i \right) = 0,$$

and can be solved for $B_{x\pm}^i$ as functionals of $A_{x\pm}^i$ and $A_{y\pm}^i$.

After this partial gauge fixing, we are left with the 12 fields $A_{x\pm}^i, A_{y\pm}^i$ as independent phase space coordinates, and with the remaining invariance under the gauge group $G = \text{SO}(3) \times \text{SO}(3)$. In three-dimensional space–time, the background independent and gauge invariant theory for this group is the G -invariant Chern–Simons theory: it turns out indeed that our partially gauge fixed theory is just this Chern–Simons theory. It is easy to check this explicitly by performing the field redefinitions

$$\mathcal{A}_{x\pm}^i = A_{x\pm}^i - \gamma_{\pm} B_{x\pm}^i, \quad \mathcal{A}_{y\pm}^i = A_{y\pm}^i, \quad (3.1)$$

the new fields obeying the Dirac–Poisson algebra

$$\left\{ \mathcal{A}_{x\pm}^i(\mathbf{x}), \mathcal{A}_{y\pm}^j(\mathbf{x}') \right\}_D = \frac{\gamma_{\pm}}{\kappa_{\pm}} \delta^{ij} \delta^2(\mathbf{x} - \mathbf{x}'), \quad (3.2)$$

where $\{, \}_D$ denotes the Dirac bracket—we have only displayed the non-vanishing brackets. In these variables, the Hamiltonian reads

$$H = H_+ + H_-, \quad H_{\pm} = -\frac{\kappa_{\pm}}{\gamma_{\pm}} \mathcal{G}_{\pm}(A_{t\pm}), \quad (3.3)$$

with the first class constraint \mathcal{G}_{\pm} given by

$$\mathcal{G}_{\pm}(\eta) = \int_{\Sigma} d^2x \left(\eta_{i+}(\mathbf{x}) \mathcal{F}_+^i(\mathbf{x}) + \eta_{i-}(\mathbf{x}) \mathcal{F}_-^i(\mathbf{x}) \right) \approx 0, \quad (3.4)$$

or equivalently by the curvature constraints

$$\mathcal{F}_{\pm}^i(\mathbf{x}) \equiv \partial_x \mathcal{A}_y^i - \partial_y \mathcal{A}_x^i + \epsilon^i{}_{jk} \mathcal{A}_x^j \mathcal{A}_y^k \approx 0. \quad (3.5)$$

The basic Dirac–Poisson brackets (3.2), together with the expressions (3.3), (3.5) for the Hamiltonian show that the theory is reduced to a Chern–Simons theory for the two $\text{so}(3)$ connections \mathcal{A}_{\pm} , which indeed transform as

$$\left\{ \mathcal{G}_{\pm}(\eta_{\pm}), \mathcal{A}_{a\pm}^i \right\}_D = \partial_a \eta_{\pm}^i + \epsilon^i{}_{jk} \mathcal{A}_{a\pm}^j \eta_{\pm}^k,$$

under the gauge transformations induced by the constraints.

It is interesting to observe that although the theory had been originally defined with four independent parameters $\gamma_{\pm}, \kappa_{\pm}$ corresponding to the four invariant quadratic forms, two of them have been absorbed in the field redefinition (3.1) which left only the two ratios $\gamma_{\pm}/\kappa_{\pm}$ as independent parameters. This is the same feature originally met in four-dimensional LQG [14]. For instance, in the Holst approach (see [3] and the second of [14], where the initial

theory depends on two parameters, namely G_{Newton} and γ_{Immirzi} , after the ‘temporal gauge fixing’ only the product of these parameters appears.

We can summarize the result of this section saying that the original $\mathfrak{so}(4)$ theory is gauge equivalent to a Chern–Simons theory for the $\mathfrak{so}(4)$ connection¹²

$$\mathcal{A} \equiv \sum_{\alpha=1}^6 \mathcal{A}^\alpha \mathcal{T}_\alpha = \mathcal{A}_+ + \mathcal{A}_-, \quad \mathcal{A}_\pm = \mathcal{A}_a^i T_{i\pm} dx^a,$$

with the constraints (3.5), which may be written as

$$\mathcal{F} \equiv \mathcal{F}_+ + \mathcal{F}_- \approx 0.$$

The basis $(\tau_\alpha, \alpha = 1, \dots, 6) = (T_{i+}, T_{i-}, i = 1, 2, 3)$ for the algebra $\mathfrak{so}(4)$ obeys the commutation relations

$$[T_{i\pm}, T_{j\pm}] = \epsilon_{ij}^k T_{k\pm}, \quad [T_{i+}, T_{j-}] = 0.$$

For further use, we normalize the Killing forms of $\mathfrak{so}(4)$ and $\mathfrak{so}(3)_\pm$, denoted by the symbol Tr , as

$$\text{Tr}(\mathcal{X}\mathcal{Y}) \equiv \sum_{\alpha=1}^6 \mathcal{X}^\alpha \mathcal{Y}^\alpha, \quad \mathcal{X}, \mathcal{Y} \in \mathfrak{so}(4); \quad \text{Tr}(XY) \equiv \sum_{i=1}^3 X^i Y^i, \quad X, Y \in \mathfrak{so}(3)_\pm.$$

4. Quantization

We apply to the present model the quantization procedure followed in [1, 17]. We have first to choose the gauge group since we will have to go from the Lie algebra level to the group level. Since the residual gauge invariance left after the partial gauge fixing made in the preceding section is $\mathfrak{so}(4)$, a convenient choice¹³ is the universal covering of $\text{SO}(4)$, namely $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$.

The dynamical field variables $\mathcal{A}_{a\pm}^i$, components of the $\mathfrak{so}(4)$ connection \mathcal{A} defined after the partial gauge fixing, are taken now as operators obeying the commutation rules (we display only the non-vanishing commutators)

$$\left[\hat{\mathcal{A}}_{x\pm}^i(\mathbf{x}), \hat{\mathcal{A}}_{y\pm}^j(\mathbf{x}') \right] = \frac{i\gamma_\pm}{\kappa_\pm} \delta^{ij} \delta^2(\mathbf{x} - \mathbf{x}'), \quad (4.1)$$

where $i, j = 1, 2, 3$ are the $\mathfrak{so}(3)$ indices. The task is to find a representation of this algebra in some kinematical Hilbert space, and then to apply the constraints. We shall therefore consider a space of wave functionals $\Psi[\mathcal{A}_x] = \Psi[\mathcal{A}_{x+}, \mathcal{A}_{x-}]$ where the conjugate variables $\mathcal{A}_{y\pm}$ act as functional derivatives:

$$\mathcal{A}_{y\pm}^i(\mathbf{x}) \Psi[\mathcal{A}_x] = \frac{\gamma_\pm}{i\kappa_\pm} \frac{\delta}{\delta \mathcal{A}_{x\pm}^i(\mathbf{x})} \Psi[\mathcal{A}_x].$$

¹² $\alpha, \beta = 1, \dots, 6$ are $\mathfrak{so}(4) = \mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-$ indices, whereas $i, j = 1, 2, 3$ are $\mathfrak{so}(3)_\pm$ ones.

¹³ In $D = 4$ LQG, where the choice for the gauge group is $\text{SU}(2)$, and not $\text{SO}(3)$, which allows the coupling with fermions. The motivation for our present choice of $\text{Spin}(4)$, and not $\text{SO}(4)$, is similar, although its physical necessity is not as strong.

The quantum version of the curvature constraints (3.5) read

$$\left(i \left(\partial_x \frac{\delta}{\delta \mathcal{A}_{x^\pm}^i} + f^i{}_{jk} \mathcal{A}_{x^\pm}^j \frac{\delta}{\delta \mathcal{A}_{x^\pm}^k} \right) + \frac{\kappa_\pm}{\gamma_\pm} \partial_y \mathcal{A}_{x^\pm}^i \right) \Psi[\mathcal{A}_x] = 0, \quad (4.2)$$

and a particular solution is given by [18]

$$\Psi_0[\mathcal{A}_x] = \exp(2\pi i \alpha_{0+}) \exp(2\pi i \alpha_{0-}), \quad (4.3)$$

with

$$\begin{aligned} \alpha_{0\pm} = & \frac{\kappa_\pm}{6\pi\gamma_\pm} \int_{\tilde{\Sigma}} d^3x \, e^{\mu\nu\rho} \text{Tr} \left(h_\pm^{-1} \partial_\mu h_\pm \, h_\pm^{-1} \partial_\nu h_\pm \, h_\pm^{-1} \partial_\rho h_\pm \right) \\ & - \frac{\kappa_\pm}{2\pi\gamma_\pm} \int_{\Sigma=\partial\tilde{\Sigma}} d^2x \, \text{Tr} \left(\mathcal{A}_{x^\pm} h_\pm^{-1} \partial_y h_\pm \right), \end{aligned} \quad (4.4)$$

where $h_\pm(\mathbf{x})$ is an element of the gauge group $\text{SU}(2)_\pm$, defined as a functional of \mathcal{A}_{x^\pm} by

$$\mathcal{A}_{x^\pm} = h_\pm^{-1} \partial_x h_\pm, \quad (4.5)$$

and where $\tilde{\Sigma}$ is a three-manifold having the space sheet Σ as its border. The first term in (4.4) is the Wess–Zumino–Witten action. The group being non-abelian and compact, the integral over $\tilde{\Sigma}$ is defined up to the addition of a constant $24\pi^2 \nu_\pm$, with $\nu_\pm \in \mathbb{Z}$. This requires that each ratio κ_\pm/γ_\pm must be quantized [19]:

$$\frac{\kappa_\pm}{\gamma_\pm} = \frac{\nu_\pm}{4\pi}. \quad (4.6)$$

The general solution of the constraints then can be written as

$$\Psi[\mathcal{A}_x] = \Psi_0[\mathcal{A}_x] \psi'[\mathcal{A}_x], \quad (4.7)$$

where the reduced wave functional $\psi'[\mathcal{A}_x]$ satisfies

$$\left[i \left(\partial_x \frac{\delta}{\delta \mathcal{A}_{x^\pm}^i} + f^i{}_{jk} \mathcal{A}_{x^\pm}^j \frac{\delta}{\delta \mathcal{A}_{x^\pm}^k} \right) \right] \psi'[\mathcal{A}_x] = 0. \quad (4.8)$$

The latter equations mean that ψ' is invariant under the infinitesimal ‘ x -gauge transformations’

$$\delta \mathcal{A}_{x^\pm}^i = D_{x^\pm} e_\pm^i. \quad (4.9)$$

Following the general lines of loop quantization [14], we introduce holonomies of the $\text{so}(4)$ connection component \mathcal{A}_x as configuration space variables, the reduced wave functionals ψ^{inv} being then functions of them. As in [1, 17] we take as the space sheet Σ a space having the topology of a cylinder, for which we choose coordinates x, y with $0 \leq x \leq 2\pi$ and $-\infty < y < +\infty$. The holonomies are thus defined along oriented paths $c(y)$ at constant y :

$$U(y) = \mathcal{P} \exp \int_{c(y)} \mathcal{A}_x dx = U_+(y) U_-(y), \quad U_\pm(y) = \mathcal{P} \exp \int_{c(y)} \mathcal{A}_{x^\pm}^i(x, y) T_{i^\pm} dx,$$

where \mathcal{P} means path ordering. Anticipating the requirement of the wave functionals having to satisfy the constraints (4.8), which is equivalent to require the invariance under the x -gauge transformations (4.9), we shall restrict ourselves to cycles, i.e., to paths $c(y)$ which are closed. If the cycle $c(y)$ begins and ends at the point (x, y) , the holonomy transforms as

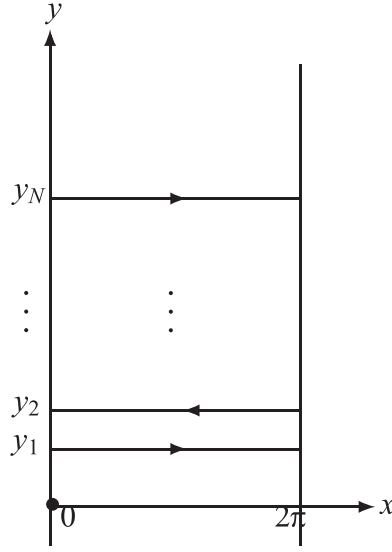


Figure 1. Picture of a graph Γ with N cycles at constant y for $y = y_1, \dots, y_N$.

$$U(y) \mapsto g^{-1}(x, y)U(y)g(x, y), \quad U_{\pm}(y) \mapsto g_{\pm}^{-1}(x, y)U_{\pm}(y)g_{\pm}(x, y), \quad (4.10)$$

where $g = g_+g_- \in \text{Spin}(4)$ and $g_{\pm} \in \text{SU}(2)_{\pm}$ we now define the vector space Cyl as the set of ‘cylindrical’ wave functionals, defined as arbitrary finite linear combinations of wave functionals of the form

$$\Psi_{\Gamma, f}[\mathcal{A}_x] = \Psi_0[\mathcal{A}_x]\psi'_{\Gamma, f}[\mathcal{A}_x], \quad \text{with} \quad \psi'_{\Gamma, f}[\mathcal{A}_x] = f(U(y_1), \dots, U(y_N)),$$

for arbitrary N and arbitrary ‘graphs’ Γ defined as finite sets of N cycles $c(y_n)$ (see figure 1).

Since the cylindrical functionals are functions $\text{Spin}(4) \otimes \dots \otimes \text{Spin}(4) \rightarrow \mathbb{C}$, a scalar product can be defined using the $\text{Spin}(4)$ invariant Haar measure¹⁴:

$$\langle \Gamma, f | \Gamma', f' \rangle = \int \left(\prod_{n=1}^{\tilde{N}} d\mu(g_n) \right) \left(f(g_1, \dots, g_N) {}^* f'(g'_1, \dots, g'_{N'}) \right),$$

with $d\mu(g = g_+g_-) = d\mu_+(g_+)d\mu_-(g_-)$ the normalized Haar measure of $\text{Spin}(4)$ and $\tilde{\Gamma}$ the union of the graphs Γ and Γ' . This internal product allows us to define the non-separable Hilbert space $\overline{\text{Cyl}}$ as the Cauchy completion of Cyl .

An orthonormal basis of $\overline{\text{Cyl}}$, the spin network basis, is provided, thanks to Peter–Weyl’s theorem, by the wave functionals

$$\Psi_{\Gamma, \vec{j}, \vec{\alpha}, \vec{\beta}}[\mathcal{A}_x] = \Psi_0[\mathcal{A}_x] \prod_{n=1}^N \sqrt{2j_n^+ + 1} R_{\alpha_n^+, \beta_n^+}^{j_n^+} \sqrt{2j_n^- + 1} R_{\alpha_n^-, \beta_n^-}^{j_n^-}, \quad (4.11)$$

where we have associated to each cycle $c(y_n)$ the matrix elements of a unitary irreducible representation of $\text{Spin}(4)$, labelled by the half-integer spin pairs (j_n^+, j_n^-) , of the corresponding holonomies:

¹⁴ We use Dirac’s notation, with $\langle \mathcal{A}_x | \Gamma, f \rangle = \Psi_{\Gamma, f}[\mathcal{A}_x]$.

$$R_{\alpha_n^\pm, \beta_n^\pm}^{j_n^\pm} = \left(R^{j_n^\pm} \left(U_\pm(y_n) \right) \right)_{\alpha_n^\pm, \beta_n^\pm}. \quad (4.12)$$

The representation $(0, 0)$, if present for some cycle $c(y_n)$, would yield a vector already present in the set of basis vectors corresponding to the graph obtained from Γ by deleting this cycle. In order to avoid redundancy, we therefore exclude such a representation. The orthonormality conditions read

$$\langle \Gamma, \vec{j}, \vec{\alpha}, \vec{\beta} \mid \Gamma', \vec{j}', \vec{\alpha}', \vec{\beta}' \rangle = \delta_{\Gamma\Gamma'} \delta_{\vec{j}\vec{j}'} \delta_{\vec{\alpha}\vec{\alpha}'} \delta_{\vec{\beta}\vec{\beta}'}.$$

The curvature constraint in the form (4.8) is readily implemented by taking the traces of the representation matrices (4.12), the characters

$$\chi^{j_n^+, j_n^-} = \text{Tr} \left(R^{j_n^+} \right) \text{Tr} \left(R^{j_n^-} \right).$$

We define in this way the Hilbert subspace \mathcal{H}_{kin} of $\overline{\text{Cyl}}$, of orthonormal basis

$$|\Gamma, \vec{j}\rangle, \quad \langle \Gamma, \vec{j} \mid \Gamma', \vec{j}' \rangle = \delta_{\Gamma\Gamma'} \delta_{\vec{j}\vec{j}'},$$

with

$$\langle \mathcal{A}_x \mid \Gamma, \vec{j} \rangle = \Psi_0[\mathcal{A}_x] \prod_{n=1}^N \sqrt{2j_n^+ + 1} \chi^{j_n^+} \sqrt{2j_n^- + 1} \chi^{j_n^-}.$$

\mathcal{H}_{kin} is still non-separable since each vector of its orthonormal basis depends on a set of real numbers y_n characterizing each graph Γ . This defect is due to our particular choice for the class of coordinates x, y adapted to the cylinder's topology. Invariance under general transformations of the y -coordinate—the ‘ y -diffeomorphisms’ in the point of view of active transformations—is not yet fulfilled. In order to implement it, a group averaging over the group of y -diffeomorphisms has to be performed, with the result that two basis vectors $|\Gamma, \vec{j}\rangle$ and $|\Gamma', \vec{j}'\rangle$ corresponding to two graphs which are related to each other by a y -diffeomorphism but sharing the same spin labels, represent the same physical vector

$$|\vec{j}\rangle = |j_1^+, \dots, j_N^+, j_1^-, \dots, j_N^-\rangle, \quad (4.13)$$

element of the orthonormal basis of the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Obviously, the latter Hilbert space is separable¹⁵.

5. Observables

A pair of observables L_\pm which are diagonal in the spin basis (4.13) of $\mathcal{H}_{\text{phys}}$ can be constructed following the lines of [1].

At the classical level, they are given by the expressions

$$L_\pm(b) = \int_b dy \sqrt{\sum_{i=1}^3 W_{y_\pm}^i W_{y_\pm}^i} = \int_b dy \sqrt{\text{Tr} W_{y_\pm}^2},$$

where b is an infinite curve $\{-\infty < y < \infty\}$ at constant x , and

$$W_{y_\pm} = \mathcal{A}_y - h_\pm^{-1} \partial_y h_\pm,$$

with h_\pm given as a non-local functional of \mathcal{A}_\pm as a solution of (4.5). As h_\pm transforms as $h'_\pm = h_\pm g_\pm$ under a $\text{SU}(2)_\pm$ gauge transformation g_\pm , the expression $h_\pm^{-1} \partial_y h_\pm$ transforms as a

¹⁵ See [1, 17] for more details.

connection, hence $\hat{W}_{y_{\pm}}$ is in the adjoint representation and $L_{\pm}(b)$ is gauge invariant: the latter are candidates for observables. It turns out that, as shown in [1], the classical $W_{y_{\pm}}$, hence $L_{\pm}(b)$, are gauge-equivalent to zero, thus physically vanishing. However their quantum counterparts are not, as we show now.

The quantum version of $W_{y_{\pm}}$ reads

$$\hat{W}_{y_{\pm}} = \hat{\mathcal{A}}_y - h_{\pm}^{-1} \partial_y h_{\pm}.$$

In order to give a reliable definition of $\hat{L}_{\pm}(b)$ as quantum operators in the physical Hilbert space, one first introduces a regularization, analogous to the one used to define the area operator of LQG [14]. One begins the construction in the space $\overline{\text{Cyl}}$, then extends it to the kinematical Hilbert space \mathcal{H}_{kin} and finally to the physical Hilbert space $\mathcal{H}_{\text{phys}}$. The regularization consists first in dividing the integration interval b in pieces b_l , small enough for each of them to intersect at most one of the cycles of the graph associated to the basis vector $|\Gamma, \vec{j}, \vec{\alpha}, \vec{\beta}\rangle$ of $\overline{\text{Cyl}}$ on which $\hat{W}_{y_{\pm}}$ acts. Second, one defines the operator $\hat{L}_{\pm}(b)$ as the sum

$$\hat{L}_{\pm}(b) = \sum_l \hat{L}_{\pm}(b_l),$$

where $\hat{L}_{\pm}(b_l)$ is approximated by

$$\hat{L}_{\pm}(b_l) = \sqrt{\sum_{i=1}^3 \int_{b_l} \hat{W}_{y_{\pm}}^i \int_{b_l} \hat{W}_{y_{\pm}}^i}.$$

The result,

$$\hat{L}_{\pm}(b) |\Gamma, \vec{j}, \vec{\alpha}, \vec{\beta}\rangle = \frac{\gamma_{\pm}}{\kappa_{\pm}} \sum_{n=1}^N \sqrt{j_n^{\pm} (j_n^{\pm} + 1)} |\Gamma, \vec{j}, \vec{\alpha}, \vec{\beta}\rangle$$

where the summation is performed on all cycles of the graph Γ , is independent of further refinements of the partition $b = \cup_k b_k$. It is also independent of the location x of the curve b . It only depends on the spins associated to each cocycle of the graph Γ , independently of its location y . This result can therefore be extended to \mathcal{H}_{kin} and then to the physical Hilbert space¹⁶:

$$\forall |\vec{j}\rangle \in \mathcal{H}_{\text{phys}}, \quad \hat{L}_{\pm} |\vec{j}\rangle = \frac{4\pi}{\nu_{\pm}} \sum_{n=1}^N \sqrt{j_n^{\pm} (j_n^{\pm} + 1)} |\vec{j}\rangle, \quad (5.1)$$

where we have used the quantization conditions (4.6), ν_{\pm} being integers.

5.1. A semi-classical analysis

The question one may have in mind is about the semi-classical behaviour of the observables \hat{L}_{\pm} . In order to define ‘semi-classicality’, we shall rely on the Fock-like structure of the physical Hilbert space. The discussion can be done separately for each of both observables L_{\pm} . In order to simplify the argument, we specialize on L_+ and its action on the subspace spanned by the basis vectors $|j_1^+, \dots, j_N^+\rangle$, omitting from now on the index $+$. The present discussion applies as it stands to the model studied in [1]. The generalization to the full theory is straightforward.

We want to compute the expectation value of \hat{L} in a suitably defined coherent state. Let us begin by defining ‘creation operators’ a_j^{\dagger} , which maps a basis vector of N spins to a vector

¹⁶ See [1, 17] for more details.

of $N + 1$ spins, completely symmetric in its arguments:

$$a_j^\dagger |j_1, \dots, j_N\rangle = S |j, j_1, \dots, j_N\rangle \equiv |j, j_1, \dots, j_N\rangle_S,$$

where S is the symmetrization operator. a_j^\dagger creates a spin j and projects on the Hilbert subspace $\mathcal{H}_{\text{phys}}^S$ spanned by the symmetric basis vectors. The index j takes the values $\frac{1}{2}, 1, \frac{3}{2}, \dots$. The ‘annihilation operators a_j are defined by hermitian conjugation, with the result:

$$a_j |j_1, \dots, j_N\rangle = \sum_{n=1}^N \delta_{j j_n} |j_1, \dots, \check{j}_n, \dots, j_N\rangle_S,$$

where \check{X} means that X is omitted. In particular, $a_j |0\rangle = 0$. Note that the symmetric basis vector $|j_1, \dots, j_N\rangle_S = a_{j_1}^\dagger \dots a_{j_N}^\dagger |0\rangle$ have the norm¹⁷

$$\| |j_1, \dots, j_N\rangle_S \|^2 = \sum_{\text{perm } \pi} \delta_{\pi(j_1)}^{j_1} \dots \delta_{\pi(j_N)}^{j_N}.$$

The subspace $\mathcal{H}_{\text{phys}}^S$ has the structure of a bosonic Fock space, the creation and annihilation operators obeying the canonical commutation rules (observe the presence of the symmetrization operator)

$$[a_j, a_{j'}^\dagger] = \delta_{jj'} S, \quad [a_j, a_{j'}] = 0, \quad [a_j^\dagger, a_{j'}^\dagger] = 0.$$

The observable \hat{L} , obeying the eigenvalue equations (see (5.1))

$$\hat{L} |j_1, \dots, j_N\rangle = \frac{4\pi}{\nu} \sum_{n=1}^N \sqrt{j_n(j_n + 1)} |j_1, \dots, j_N\rangle,$$

has the following commutation relations with the creation and annihilation operators:

$$[\hat{L}, a_j^\dagger] = \frac{4\pi}{\nu} \sqrt{j(j+1)} a_j^\dagger, \quad [\hat{L}, a_j] = -\frac{4\pi}{\nu} \sqrt{j(j+1)} a_j.$$

‘Coherent states’ $|\lambda\rangle$ will be defined as usually to be the eigenvectors of the annihilation operators a_j :

$$a_j |\lambda\rangle = \lambda_j |\lambda\rangle,$$

where λ represents an infinite series of complex numbers $\lambda_j \in \mathbb{C}$ submitted to the square summability condition

$$|\lambda|^2 \equiv \sum_j \lambda_j \lambda_j^* < \infty. \quad (5.2)$$

These eigenvectors read

$$|\lambda\rangle = e^{-|\lambda|^2/2} \exp\left(\sum_j \lambda_j a_j^\dagger\right) |0\rangle = e^{-|\lambda|^2/2} \sum_{K=0}^{\infty} \frac{1}{K!} |\psi_K\rangle_S, \quad (5.3)$$

¹⁷ To the contrary of the basis vectors $|\dots\rangle$ defined in section 4, the present vectors $|\dots\rangle_S$ are not normalized.

where

$$|\psi_K\rangle_S = \left(\sum_j \lambda_j a_j^\dagger \right)^K |0\rangle,$$

is the contribution of the vectors of K spins to $|\lambda\rangle$. The normalization $\langle\lambda|\lambda\rangle = 1$ holds.

The system being in the state described by $|\lambda\rangle$, the probability to measure the spin values j_1, \dots, j_N (in any order) is given by

$$P_\lambda(j_1, \dots, j_N) = \left| \langle j_1, \dots, j_N | \lambda \rangle \right|^2 / \left\| |j_1, \dots, j_N\rangle_S \right\|^2.$$

From this one can calculate the probability of having N spins of any values:

$$P_\lambda(N) = \sum_{j_1, \dots, j_N} P_\lambda(j_1, \dots, j_N) = e^{-|\lambda|^2} \sum_{j_1, \dots, j_N} \frac{|\lambda_{j_1}|^2 \dots |\lambda_{j_N}|^2}{\left\| |j_1, \dots, j_N\rangle_S \right\|^2}.$$

This probability distribution is a multi-dimensional generalization of a Poisson distribution. In fact, in the very special case of the λ_j being all zero excepted for one value j_0 of j , since $\left\| |j_0, \dots, j_0\rangle_S \right\|^2 = N!$, we get the Poisson distribution

$$P_\lambda(N) = e^{-|\lambda_{j_0}|^2} \frac{|\lambda_{j_0}|^{2N}}{N!},$$

an essential characteristic of standard coherent states [20]. Another characteristic of coherent states is their (over)completeness. This is recovered in the one spin case:

$$\frac{1}{\pi} \int d^2\lambda_{j_0} |\lambda_{j_0}\rangle \langle\lambda_{j_0}| = \text{Id},$$

whereas, in the general case—the integration being done over all λ_j —the identity operator in the right-hand side is replaced by the projector S on $\mathcal{H}_{\text{phys}}^S$.

Having thus a reasonable definition of the coherent states, we can now compute the expectation value of \hat{L} in the state defined by $|\lambda\rangle$. We first compute its matrix elements $s\langle\psi_K|\hat{L}|\psi_{K'}\rangle_S$. They obviously vanish for $K \neq K'$. The diagonal ones, denoted by

$$L_K \equiv s\langle\psi_K|\hat{L}|\psi_K\rangle_S,$$

are seen to obey the recurrence relations

$$L_K = |\lambda|^2 K L_{K-1} + K! |\lambda|^{2(K-1)} L_1, \quad K \geq 2,$$

with

$$L_1 = \langle\psi_1|\hat{L}|\psi_1\rangle = \frac{4\pi}{\nu} \sum_j \sqrt{j(j+1)} |\lambda_j|^2, \quad (5.4)$$

and, of course, $L_0 = \langle 0|\hat{L}|0\rangle = 0$. The solution of the recurrence relation is given by

$$L_K = K K! |\lambda|^{2(K-1)} L_1,$$

valid for $K \geq 0$. Finally, inserting this solution into

$$\langle \lambda | \hat{L} | \lambda \rangle = \sum_K e^{-|\lambda|^2} \frac{1}{K!^2} L_K,$$

(see (5.3)), we obtain

$$\langle \lambda | \hat{L} | \lambda \rangle = L_1, \quad (5.5)$$

with L_1 given by (5.4), i.e., a mean value of the spectrum (5.1), with weights given by the $|\lambda_j|^2$.

We observe that vectors with an arbitrary number of spins contribute, altogether cancelling the decreasing exponential factor.

As a result, we have thus found that the semi-classical behaviour of both observables is not trivial, in contrast to the triviality of their corresponding classical version.

6. Conclusion

We have proceeded to the loop quantization of $D = 1 + 2$ gravity with a cosmological constant and a coupling with topological matter fields defined via a semi-simple extension of the de Sitter or anti-de Sitter group. The resulting theory has four free real parameters corresponding to the four different non-degenerate quadratic forms which have been used to construct the classical action. But the quantum theory depends only on two independent parameter's ratios, and in fact on two integers, ν_+ and ν_- , due to a topological quantization condition. An orthonormal spin-network basis has been constructed, the basis vectors being the eigenvectors of two global observables with eigenvalues very similar to those of the area operator in $(1 + 3)$ -dimensional LQG.

The novel feature resulting from the present study—as well as from the previous paper [1]—is that these observables are a pure quantum effect, their classical counterparts being vanishing. Looking for a semi-classical behaviour of them, we have found however that their mean values in some suitably defined coherent states $|\lambda_j; j = \frac{1}{2}, 1, \frac{3}{2}, \dots\rangle$ are non-zero, but equal to weighted mean values of their spectra with weights $|\lambda_j|^2$. Thus the semi-classical limit, as we have defined it, does not reproduce the classical behaviour. It seems that we are in the presence of quantum observables that have no classical counterpart—which does not preclude a non-trivial semi-classical limit. We do not know of any similar result in the literature. Also, comparison with other quantization schemes is difficult since, to the contrary of the example studied in the present paper, the space sheet topologies generally considered are compact (see e.g. [10, 11]). Moreover, there, the quantization is performed after the constraints have been solved at the classical level, with the states being described by wave functions depending on the moduli of the space sheet. It would be highly desirable to investigate other quantization schemes in the same topological situation.

We have specialized, in the present paper, on a particular topology. It would also be very desirable to consider other cases in order to investigate how topology influences the nature of the quantum observables.

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Appendix. The algebra $\mathfrak{s}(\mathfrak{a})\mathfrak{d}\mathfrak{s}$

A.1. Semi-simple extension of a semi-simple Lie algebra

Let \mathcal{G} be the Lie algebra of a semi-simple Lie group G , and $\{T_\alpha, \alpha = 1, \dots, d\}$ a basis of \mathcal{G} , with the commutation relations

$$[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma. \quad (\text{A.1})$$

Let us consider a set of operators $\{S_A, A = 1, \dots, D\}$ in a dimension D representation of \mathcal{G} , i.e., transforming as

$$[S_B, T_\alpha] = R_{\alpha B}{}^C S_C$$

under the action of the basis generators of \mathcal{G} , $R_{\alpha B}{}^C$ being the elements of the matrix representing T_α .

We define the *semi-simple extension* $\mathcal{S}\mathcal{G}$ of \mathcal{G} through the representation R_α the algebra spanned by the operators $\{T_\alpha, S_A\}$, whereby the commutation relations above are completed by

$$[S_A, S_B] = C_{AB}{}^\gamma T_\gamma.$$

A necessary and sufficient condition for this extension to exist is the fulfilment of the Jacobi identities involving the new structure constants $C_{AB}{}^\gamma$:

$$\begin{aligned} f_{\alpha\delta}{}^\epsilon C_{BC}{}^\delta + R_{\alpha C}{}^D C_{BD}{}^\epsilon + R_{\alpha B}{}^D C_{DC}{}^\delta &= 0, \\ C_{AB}{}^\delta R_{\delta C}{}^E + C_{BC}{}^\delta R_{\delta A}{}^E + C_{CA}{}^\delta R_{\delta B}{}^E &= 0. \end{aligned}$$

The first equation states that $C_{AB}{}^\gamma$ must be an invariant mixed tensor, whereas the second one is a cocycle condition it must fulfil.

A special case is provided by $S = \{S_A, A = \alpha = 1, \dots, d\}$ being a vector in the adjoint representation. Then $R_{\alpha B}{}^C = f_{\alpha\beta}{}^\gamma$ and the cocycle condition is obviously fulfilled by $C_{AB}{}^\gamma = C f_{\alpha\beta}{}^\gamma$ with C an arbitrary real number. This will be the case of interest in the present paper, the full commutator algebra for the basis generators of $\mathcal{S}\mathcal{G}$ being summarized by

$$[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma, \quad [S_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma, \quad [S_\alpha, S_\beta] = C f_{\alpha\beta}{}^\gamma T_\gamma. \quad (\text{A.2})$$

One notes that, if $C > 0$, then the algebra factorizes as $\mathcal{S}\mathcal{G} = \mathcal{G}^+ + \mathcal{G}^-$, the generators of each factor \mathcal{G}^\pm being defined by

$$T_\alpha^\pm = \frac{1}{2} \left(T_\alpha \pm \frac{1}{\sqrt{C}} S_\alpha \right), \quad (\text{A.3})$$

and obeying the same commutation rules as in (A.1).

A.2. Properties of the semi-simple extension of $(\mathfrak{a})\mathfrak{d}\mathfrak{s}$

The $(\mathfrak{a})\mathfrak{d}\mathfrak{s}$ algebra being given by the commutation rules written in the first line of (2.1) for the basis generators \mathbf{J}, \mathbf{P} , its semi-simple extension $\mathfrak{s}(\mathfrak{a})\mathfrak{d}\mathfrak{s}$ through the adjoint representation vector \mathbf{Q}, \mathbf{R} is defined by (A.2), the result being given by the full system of commutators (2.1). The second and third lines of (2.1) correspond to the second equation of (A.2), whereas the last line of (2.1) represents the cocycle condition given by the third equation of (A.2). The

Table A.1. Signs of the Killing form eigenvalues in function of the signs of the parameters σ , Λ , λ .

Signs of σ , Λ , λ	J^0	J^1	J^2	P^0	P^1	P^2	Q^0	Q^1	Q^2	R^0	R^1	R^2
- , + , +	-	+	+	+	-	-	+	-	-	-	+	+
- , + , -	-	+	+	+	-	-	-	+	+	+	-	-
- , - , +	-	+	+	-	+	+	+	-	-	+	-	-
- , - , -	-	+	+	-	+	+	-	+	+	-	+	+
+ , + , +	+	+	+	+	+	+	+	+	+	+	+	+
+ , + , -	+	+	+	+	+	+	-	-	-	-	-	-
+ , - , +	+	+	+	-	-	-	+	+	+	-	-	-
+ , - , -	+	+	+	-	-	-	-	-	-	+	+	+

Table A.2. Compact sub-algebras and their basis of generators in function of the signs of the parameters σ , Λ , λ .

Signs of σ , Λ , λ	Compact subalgebras	Basis of generators
- , + , +	so(4)	$J^0, P^1, P^2; R^0, Q^1, Q^2$
- , + , -	so(4)	$J^0, P^1, P^2; Q^0, R^1, R^2$
- , - , +	so(4)	$J^0, Q^1, Q^2; P^0, R^1, R^2$
- , - , -	$u(1) \oplus u(1) \oplus u(1) \oplus u(1)$	$J^I, P^I, Q^I, R^I; I = 0 \text{ or } 1 \text{ or } 2$
+ , + , +	$so(4) \oplus so(4)$	$J^I, P^I, Q^I, R^I; I = 0, 1, 2$
+ , + , -	so(4)	$J^I, P^I; I = 0, 1, 2$
+ , - , +	so(4)	$J^I, Q^I; I = 0, 1, 2$
+ , - , -	so(4)	$J^I, R^I; I = 0, 1, 2$

closure parameter C is now represented by the parameter λ (multiplied by the signature σ) appearing in the last line of (2.1)¹⁸.

A.2.1. Maximal compact sub-algebras. For the purpose of the gauge fixing proposed in the main text, we are interested in finding the maximal compact sub-algebras of $\mathfrak{s}(\mathfrak{a})\mathfrak{d}\mathfrak{s}$. These sub-algebras are spanned by subsets L of the 12 basis generators, such that their Killing forms are positive or negative definite.

Ordering the generators of $\mathfrak{s}(\mathfrak{a})\mathfrak{d}\mathfrak{s}$ as

$$\{\mathcal{T}_\alpha, \alpha = 1, \dots, 12\} = \{J^0, J^1, J^2; P^0, P^1, P^2; Q^0, Q^1, Q^2; R^0, R^1, R^2\},$$

and writing their commutation relations as $[\mathcal{T}_\alpha, \mathcal{T}_\beta] = F_{\alpha\beta}{}^\gamma \mathcal{T}_\gamma$, the Killing form K reads

$$K_{\alpha\beta} = -\frac{\sigma}{2} F_{\alpha\gamma}{}^\delta F_{\beta\delta}{}^\gamma = \text{diag}(\sigma, 1, 1; \Lambda, \sigma\Lambda, \sigma\Lambda; \lambda, \sigma\lambda, \sigma\lambda; \sigma\Lambda\lambda, \Lambda\lambda, \Lambda\lambda). \quad (\text{A.4})$$

Which are the maximal compact subgroups that will depend on the values of the parameters σ , Λ , λ . For instance, with the signs of σ , Λ , λ being - , + , + , the signs of the Killing form eigenvalues are

$$(-, +, +; +, -, -; +, -, -; -, +, +).$$

¹⁸ One notes that the $\mathfrak{a})\mathfrak{d}\mathfrak{s}$ algebra itself, spanned by the generators \mathbf{J}, \mathbf{P} , is the semi-simple extension of the Lorentz algebra through the 'translation' vector \mathbf{P} , the closure parameter being the cosmological constant Λ .

Table A.3. Factorization properties of s(a)ds in function of the signs of the parameters σ, Λ, λ .

Signs of σ, Λ, λ	Factorization	Generators
$-, +, +$	$\text{so}(1, 3)_+ \oplus \text{so}(1, 3)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \mp \frac{R^I}{\sqrt{\Lambda\lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{P^I}{\sqrt{\Lambda}} \pm \frac{Q^I}{\sqrt{\lambda}})$
$-, +, -$	$\text{so}(1, 3)_+ \oplus \text{so}(1, 3)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \pm \frac{Q^I}{\sqrt{-\lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{P^I}{\sqrt{\Lambda}} \pm \frac{R^I}{\sqrt{-\Lambda\lambda}})$
$-, -, +$	$\text{so}(1, 3)_+ \oplus \text{so}(1, 3)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \pm \frac{P^I}{\sqrt{-\Lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{Q^I}{\sqrt{\lambda}} \pm \frac{R^I}{\sqrt{-\Lambda\lambda}})$
$-, -, -$	$\text{so}(2, 2)_+ \oplus \text{so}(2, 2)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \pm \frac{P^I}{\sqrt{-\Lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{Q^I}{\sqrt{-\lambda}} \pm \frac{R^I}{\sqrt{\Lambda\lambda}})$
$+, +, +$	$\text{so}(4)_+ \oplus \text{so}(4)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \pm \frac{P^I}{\sqrt{\Lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{Q^I}{\sqrt{\lambda}} \pm \frac{R^I}{\sqrt{\Lambda\lambda}})$
$+, +, -$	$\text{so}(1, 3)_+ \oplus \text{so}(1, 3)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \pm \frac{P^I}{\sqrt{\Lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{Q^I}{\sqrt{-\lambda}} \pm \frac{R^I}{\sqrt{-\Lambda\lambda}})$
$+, -, +$	$\text{so}(1, 3)_+ \oplus \text{so}(1, 3)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \pm \frac{Q^I}{\sqrt{\lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{P^I}{\sqrt{-\Lambda}} \pm \frac{R^I}{\sqrt{-\Lambda\lambda}})$
$+, -, -$	$\text{so}(1, 3)_+ \oplus \text{so}(1, 3)_-$	$J_{\pm}^I = \frac{1}{2}(J^I \mp \frac{R^I}{\sqrt{\Lambda\lambda}}), \quad P_{\pm}^I = \frac{1}{2}(\frac{P^I}{\sqrt{-\Lambda}} \pm \frac{Q^I}{\sqrt{-\lambda}})$

One sees that the six negative eigenvalues correspond to the six generators $J^0, P^1, P^2; R^0, Q^1, Q^2$, which span a compact sub-algebra which is easily identified as $\text{so}(4)$. Note that the generators corresponding to the six positive eigenvalues do not span a sub-algebra. We conclude that the maximal compact sub-algebra, in this case, is $\text{so}(4)$. Similar reasoning hold for the other cases. The results are displayed in table A.2, based on the Killing eigenvalue's signs shown in table A.1, for each possible choice of signs of σ, Λ, λ . In table A.2, we only show one possibility of $\text{so}(4)$ sub-algebra for each choice of signs. But, in the first line for instance, there is another $\text{so}(4)$ sub-algebra spanned by $\{J^0, Q^1, Q^2; R^0, P^1, P^2\}$, with similar alternatives for the other case. It may be useful to note that we have the following sub-algebras, $\text{so}(3)$ or $\text{so}(1,2)$ depending of the signs of σ, Λ, λ :

$$\begin{aligned} \{J^0, P^1, P^2\}: [J^0, P^1] &= P^2, \quad [P^1, P^2] = \Lambda J^0, \quad [P^2, J^0] = P^1, \\ \{J^0, Q^1, Q^2\}: [J^0, Q^1] &= Q^2, \quad [Q^1, Q^2] = \lambda J^0, \quad [Q^2, J^0] = Q^1, \\ \{J^0, R^1, R^2\}: [J^0, R^1] &= R^2, \quad [R^1, R^2] = \sigma\Lambda\lambda J^0, \quad [R^2, J^0] = R^1. \end{aligned}$$

Remark: Going back to our example—corresponding to the first line of each table—we remark that we have a de Sitter sub-algebra $\text{so}(1,3)$ spanned by \mathbf{J}, \mathbf{P} , with positive cosmological constant $\Lambda > 0$, another one spanned by \mathbf{J}, \mathbf{Q} , with positive cosmological constant $\lambda > 0$, and also an anti-de Sitter¹⁹ sub-algebra $\text{so}(2,2)$ spanned by \mathbf{J}, \mathbf{R} , with negative cosmological constant $\sigma\Lambda\lambda < 0$. A similar remark applies to the other choices for the signs of the parameters σ, Λ, λ .

A.2.2. Factorization. We check now that, in the three cases displayed in the first three lines of tables A.1 and A.2, the s(a)ds algebra factorizes in two de Sitter algebras $\text{so}(1,3)$. To see this explicitly, let us consider first the case of the first line of these tables, and define two triplets of generators

¹⁹ This sub-algebra possesses an own sub-algebra $\text{so}(3)$ spanned by J^0, P^1, P^1 , which shows that it is really de Sitter $\text{so}(1,3)$, and not anti-de Sitter $\text{so}(2,2)$.

$$(X^i, i = 1, 2, 3) \equiv (J^0, P^1, P^2), \quad (Y^i, i = 1, 2, 3) \equiv (-P^0, J^1, J^2),$$

where we use the normalized $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ generators

$$J^I = J^I, \quad P^I = P^I/\sqrt{|\Lambda|}, \quad Q^I = Q^I/\sqrt{|\lambda|}, \quad R^I = R^I/\sqrt{|\Lambda\lambda|}.$$

The X 's and Y 's obey the canonical $\mathfrak{so}(1,3)$ commutation relations

$$[X^i, X^j] = \epsilon^{ijk}X^k, \quad [X^i, Y^j] = \epsilon^{ijk}Y^k, \quad [Y^i, Y^j] = -\epsilon^{ijk}X^k.$$

Thus $(J^I, I = 0, 1, 2; P^I, I = 0, 1, 2) \equiv (T_\alpha, \alpha = 1, \dots, 6)$ form just another basis for the same algebra $\mathfrak{so}(1,3)$, with commutation relations which one may write as

$$[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma.$$

Now it is a matter of checking that $(-R^I; Q^I \equiv (S_\alpha, \alpha = 1, \dots, 6))$ obey together with the T 's the commutation relations

$$[T_\alpha, S_\beta] = f_{\alpha\beta}{}^\gamma S_\gamma, \quad [S_\alpha, S_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma.$$

The first ones show that the S 's span the adjoint representation of $\mathfrak{so}(3,1)$, and the second ones express the closure of the algebra $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ generated by the 12 generators T_α, S_α . This set of commutation rules is of the type shown in (A.2), with $C = 1$. C being positive, the factorization (A.3) holds: the $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ algebra splits in two de Sitter factors:

$$\mathfrak{s}(\mathfrak{a})\mathfrak{dS} = \mathfrak{ds}_+ \oplus \mathfrak{ds}_- = \mathfrak{so}(1, 3)_+ \oplus \mathfrak{so}(1, 3)_-, \quad (\text{A.5})$$

generated by

$$J_\pm^I = \frac{1}{2} \left(J^I \mp \frac{R^I}{\sqrt{\Lambda\lambda}} \right), \quad P_\pm^I = \frac{1}{2} \left(\frac{P^I}{\sqrt{\Lambda}} \pm \frac{Q^I}{\sqrt{\lambda}} \right) \quad (\text{A.6})$$

with the commutation rules

$$[J_\pm^I, J_\pm^J] = \epsilon^{IJ}{}_{K} J_\pm^K, \quad [J_\pm^I, P_\pm^J] = \epsilon^{IJ}{}_{K} P_\pm^K, \quad [P_\pm^I, P_\pm^J] = -\epsilon^{IJ}{}_{K} J_\pm^K.$$

The cases corresponding to the second or third lines of tables A.1 and A.2 are equivalent and can be deduced from the first case by interchanging the Q 's with the R 's, or the P 's with the R 's, respectively. The Riemannian cases ($\sigma = 1$) displayed in the three last lines of the tables follow equivalent patterns. The explicit results are summarized in table A.3.

A.2.3. Invariant quadratic forms. The $\mathfrak{s}(\mathfrak{a})\mathfrak{dS}$ algebra has four quadratic Casimir operators $C_i = C_i^{\alpha\beta} \mathcal{T}_\alpha \mathcal{T}_\beta$, $i = 1, 2, 3, 4$:

$$C_1 = J^I J_I + \frac{P^I P_I}{\sigma\Lambda} + \frac{Q^I Q_I}{\sigma\lambda} + \frac{R^I R_I}{\Lambda\lambda},$$

$$C_2 = J^I P_I + \frac{Q^I R_I}{\sigma\lambda}, \quad C_3 = J^I Q_I + \frac{P^I R_I}{\sigma\Lambda}, \quad C_4 = J^I R_I + P^I Q_I,$$

to which correspond four invariant quadratic forms K_{ab}^i proportional to the inverse matrices $(C^{-1})_{ab}^i$ (we only write their non-vanishing components):

$$\begin{aligned}
K_{J^I, J^I}^1 &= \eta_{IJ}, & K_{P^I, P^I}^1 &= \sigma\Lambda\eta_{IJ}, & K_{Q^I, Q^I}^1 &= \sigma\lambda\eta_{IJ}, & K_{R^I, R^I}^1 &= \Lambda\lambda\eta_{IJ}, \\
K_{J^I, P^I}^2 &= \eta_{IJ}, & K_{Q^I, R^I}^2 &= \sigma\lambda\eta_{IJ}, \\
K_{J^I, Q^I}^3 &= \eta_{IJ}, & K_{P^I, R^I}^3 &= \sigma\Lambda\eta_{IJ}, \\
K_{J^I, R^I}^4 &= \eta_{IJ}, & K_{P^I, Q^I}^4 &= \eta_{IJ},
\end{aligned} \tag{A.7}$$

all—the fourth one excepted—being non-degenerate only if both Λ and λ are non-vanishing. We note that the first one is the Killing form (A.4)

All this is a generalization of the case of the (a)ds algebra, which has two invariant quadratic forms [5]. If we choose the (a)ds basis J^I and P^I , which obeys the commutation rules of the first line of (2.1), the two quadratic forms read:

$$\begin{aligned}
k_{J^I, J^I}^1 &= \eta_{IJ}, & k_{P^I, P^I}^1 &= \sigma\Lambda\eta_{IJ}, \\
k_{J^I, P^I}^2 &= \eta_{IJ},
\end{aligned} \tag{A.8}$$

the first one being the Killing form of (a)ds, non-degenerate if $\Lambda \neq 0$.

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