Applied Mathematics Letters 25 (2012) 884–889

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Stabilization of mixture of two rigid solids modeling temperature and porosity

Margareth S. Alves [a], Jaime E. Muñoz Rivera [b], Mauricio Sepúlveda [c,*], Octavio P. Vera Villagrán [d]

[a] Departamento de Matemática, Universidade Federal de Viçosa-UFV, 36570-000, Viçosa-MG, Brazil
[b] LNCC, Rua Getulio Vargas 333, Quitandinha-Petrópolis, 25651-070, Rio de Janeiro, RJ, Brazil
[c] CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Chile
[d] Departamento de Matemática, Universidad del Bío-Bío, Collao 1202, Casilla 5-C, Concepción, Chile

ARTICLE INFO

Article history:
Received 20 May 2011
Received in revised form 21 October 2011
Accepted 31 October 2011

Keywords:
Analyticity
Exponential stability
C_0-semigroup
Mixture
Temperature
Porosity

ABSTRACT

In this paper we investigate the asymptotic behavior of solutions to the initial boundary value problem for a mixture of two rigid solids modeling temperature and porosity. Our main result is to establish conditions which ensure the analyticity and the exponential stability of the corresponding semigroup.

1. Introduction

This article is concerned with a special case of a linear theory for binary mixtures of porous viscoelastic materials. The theory of viscoelastic mixtures has been investigated by several authors (see for instance, [1–3] and the references therein). In [1] binary mixtures have been considered where the individual components are modeled as porous Kelvin–Voigt viscoelastic materials and the volume fraction of each constituent was considered as an independent kinematical quantity. The authors assumed that the constituents have a common temperature and that every thermodynamical process that takes place in the mixture satisfies the Clausius–Duhem inequality. At the end of that work, they presented as an application the interaction between the temperature field θ and the porosity fields u and w in a homogeneous and isotropic mixture. In this case, and after some considerations, the equations which govern the fields u, w and θ in the absence of body loads are given by the system

$$\begin{aligned}
&\rho_1 u_{tt} - a_{11}\,\Delta u - a_{12}\,\Delta w - b_{11}\,\Delta u_t - b_{12}\Delta w_t + \alpha\,(u-w) - k_1\,\Delta\theta - \beta_1\,\theta = 0 \quad \text{in } \Omega\times(0,\infty),\\
&\rho_2\, w_{tt} - a_{12}\,\Delta u - a_{22}\,\Delta w - b_{12}\,\Delta u_t - b_{22}\,\Delta w_t - \alpha\,(u-w) - k_2\,\Delta\theta - \beta_2\,\theta = 0 \quad \text{in } \Omega\times(0,\infty),\\
&c\,\theta_t - \kappa\,\Delta\theta + k_1\,\Delta u_t + k_2\,\Delta w_t + \beta_1\,u_t + \beta_2\,w_t = 0 \quad \text{in } \Omega\times(0,\infty),
\end{aligned} \tag{1.1}$$

* Corresponding author.
E-mail addresses: malves@ufv.br (M.S. Alves), rivera@lncc.br (J.E. Muñoz Rivera), mauricio@ing-mat.udec.cl, masepulveda.cortes@gmail.com (M. Sepúlveda), overa@ubiobio.cl (O.P. Vera Villagrán).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.10.044

where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The function $u = u(x,t)$ (and $w = w(x,t)$) represents the fraction field of a constituent and $\theta = \theta(x,t)$ the difference of temperature between the actual state and a reference temperature. We consider the following initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad w(x,0) = w_0, \quad w_t(x,0) = w_1, \quad \theta(x,0) = \theta_0 \quad \text{in } \Omega \\ &u(x,t) = u(x,t) = w(x,t) = w(x,t) = \theta(x,t) = \theta(x,t) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

We assume that $\rho_1, \rho_2, c, \kappa$, and α are positive constants. Since coupling is considered, we consider $\left(\beta_1^2 + \beta_2^2\right)\left(k_1^2 + k_2^2\right) \neq 0$, but the sign of β_i or k_i $(i = 1, 2)$ does not matter in the analysis. The matrix $A = (a_{ij})$ is symmetric and positive definite and $B = (b_{ij}) \neq 0$ is symmetric and non-negative definite, that is, $a_{11} > 0, a_{11}a_{22} - a_{12}^2 > 0$, $b_{11} \geq 0$ and $b_{11}b_{22} - b_{12}^2 \geq 0$. Our purpose in this work is to investigate the stability of the solutions to the system (1.1)–(1.2). The asymptotic behavior, as $t \to \infty$, of solutions to the equations of linear thermoelasticity has been studied by many authors. Obviously, to get these stability results, we consider several restrictions on the constitutive coefficients. In this sense, this system of equations does not intend to model the general problem. We refer to the book of Liu and Zheng [4] for a general survey on these topics. However, we recall that very few contributions have been addressed to study the time behavior of the solutions of nonclassical elastic theories. In this direction we mention the works [3,5–7]. In [8], the authors treat a similar problem for a one-dimensional mixture modeling temperature and porosity and prove the exponential decay of solutions. We note that we cannot expect that this system always decays in a exponential way. For instance, in case that $\beta_1 + \beta_2 = 0$, $k_1 + k_2 = 0$, $\rho_2(a_{11} + a_{12}) = \rho_1(a_{12} + a_{22})$ and $b_{11} + b_{12} = b_{12} + b_{22} = 0$ we can obtain solutions of the form $u = w$ and $\theta = 0$. These solutions are undamped and do not decay to zero. These are very particular cases, but we will see that there are some other cases where the solutions decay, but the decay is not so fast to be controlled by an exponential. Our main result is to obtain conditions over the coefficients of the system (1.1) to ensure the exponential stability as well as the analyticity of the semigroup associated with (1.1)–(1.2). We follow the same line of reasoning adopted in the papers [5,6]. This paper is organized as follows. Section 2 outlines briefly the well-posedness of the system is established. In Section 3, we show the exponential stability of the corresponding semigroup provided that certain conditions are guaranteed. In Section 4, we treat the analyticity of the semigroup. In the last Section 5 we show, for some cases, the lack of exponential stability of the semigroup. Throughout this paper C is a generic constant.

2. The existence of the global solution

In this section, we use the semigroup approach to show the well-posedness of the system. We introduce the face space $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ equipped with the inner product given by

$$\begin{aligned} \langle (u_1, w_1, v_1, \eta_1, \theta_1), (u_2, w_2, v_2, \eta_2, \theta_2)\rangle_{\mathcal{H}} &= a_{11}\langle \nabla u_1, \nabla u_2\rangle + a_{22}\langle \nabla w_1, \nabla w_2\rangle \\ &+ a_{12}\left(\langle \nabla u_1, \nabla w_2\rangle + \langle \nabla w_1, \nabla u_2\rangle\right) + \alpha\langle u_1 - w_1, u_2 - w_2\rangle + \rho_1\langle v_1, v_2\rangle + \rho_2\langle \eta_1, \eta_2\rangle + c\langle \theta_1, \theta_2\rangle \end{aligned}$$

where $\langle u, v\rangle = \int_\Omega u\overline{v}\,dx$, and the induced norms $|\cdot|$ and $\|\cdot\|_{\mathcal{H}}$ which are equivalent to the usual norms in $L^2(\Omega)$ and $\mathcal{H}$, respectively. We also consider the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$

$$\mathcal{A}\begin{pmatrix} u \\ w \\ v \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \eta \\ \dfrac{1}{\rho_1}\Delta(a_{11}u + a_{12}w + b_{11}v + b_{12}\eta + k_1\theta) - \dfrac{\alpha}{\rho_1}(u - w) + \dfrac{\beta_1}{\rho_1}\theta \\ \dfrac{1}{\rho_2}\Delta(a_{12}u + a_{22}w + b_{12}v + b_{22}\eta + k_2\theta) + \dfrac{\alpha}{\rho_2}(u - w) + \dfrac{\beta_2}{\rho_2}\theta \\ \dfrac{1}{c}\Delta(\kappa\theta - k_1 v - k_2\eta) - \dfrac{\beta_1}{c}v - \dfrac{\beta_2}{c}\eta \end{pmatrix}$$

whose domain $\mathcal{D}(\mathcal{A})$ is the subspace of $\mathcal{H}$ consisting of vectors (u, v, w, η, θ) such that $v, \eta, \theta \in H_0^1(\Omega)$, $\kappa\theta - k_1 v - k_2\eta \in H^2(\Omega)$, $a_{11}u + a_{12}w + b_{11}v + b_{12}\eta + k_1\theta \in H^2(\Omega)$, and $a_{12}u + a_{22}w + b_{12}v + b_{22}\eta + k_2\theta \in H^2(\Omega)$. The system (1.1)–(1.2) can be rewritten as the following initial value problem $\frac{d}{dt}U(t) = \mathcal{A}U(t)$, $U(0) = U_0$ for all $t > 0$ with $U(t) = (u, w, u_t, w_t, \theta)^T$ and $U_0 = (u_0, w_0, u_1, w_1, \theta_0)^T$, and the T is used to denote the transpose. We can show that the operator $\mathcal{A}$ is densely definite, dissipative, that is, Re $\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leqslant 0$, for all $U \in \mathcal{D}(\mathcal{A})$, and 0 belongs to the resolvent set of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$ (see [6]). Therefore, using the Lumer–Phillips theorem we conclude that the operator $\mathcal{A}$ generates a C_0-semigroup $S_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$. The following theorem follows.

Theorem 2.1. *For any $U_0 \in \mathcal{H}$, there exists a unique solution $U(t) = (u, w, u_t, w_t, \theta)$ of (1.1)–(1.2) satisfying $u, w \in C([0,\infty[: H_0^1(\Omega)) \cap C^1([0,\infty[: L^2(\Omega))$, $\theta \in C([0,\infty[: L^2(\Omega)) \cap L^2(]0,\infty[: H_0^1(\Omega))$. If $U_0 \in \mathcal{D}(\mathcal{A})$ then $u, w \in C^1([0,\infty[: H_0^1(\Omega)) \cap C^2([0,\infty[: L^2(\Omega))$, $\theta \in C([0,\infty[: H_0^1(\Omega)) \cap C^1([0,\infty[: L^2(\Omega))$, and*

$$\begin{aligned} &a_{11}u + a_{12}w + b_{11}u_t + b_{12}w_t + k_1\theta \in C([0,\infty[: H^2(\Omega)) \\ &a_{12}u + a_{22}w + b_{12}u_t + b_{22}w_t + k_2\theta \in C([0,\infty[: H^2(\Omega)) \\ &\kappa\theta - k_1 u_t - k_2 w_t \in C([0,\infty[: H^2(\Omega)). \end{aligned}$$

3. Exponential stability

We denote by C_P the Poincaré constant. To simplify the notation in the product, in the next theorems we consider $\Xi = \frac{\alpha(b_{12}+b_{11})(\rho_1 b_{12}-\rho_2 b_{11})}{\rho_2 b_{11}(a_{11}b_{12}-a_{12}b_{11})+\rho_1 b_{12}(a_{12}b_{12}-a_{22}b_{11})}$. Our main tool is the following theorem established in [9] (see also [10,11]).

Theorem 3.1. *Let $\mathcal{S}(t)$ be a C_0-semigroup of contractions of linear operators on a Hilbert space X with infinitesimal generator $\mathcal{A}$. Then $\mathcal{S}(t)$ is exponentially stable if, and only if, $i\mathbb{R} \subset \rho(\mathcal{A})$ and*

$$\limsup_{|\lambda|\to\infty} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(X)} < \infty. \tag{3.1}$$

Our starting point in order to show the exponential stability is the following lemma

Lemma 3.2. *We suppose that one of the following items holds.*

(a) $\beta_2 b_{11} = \beta_1 b_{12}$, $\beta_2 b_{12} = \beta_1 b_{22}$, *and* $k_2 b_{11} \neq k_1 b_{12}$, *or* $k_2 b_{12} \neq k_1 b_{22}$;
(b) Ξ *is not an eigenvalue of the operator* $-\Delta$;
(c) $\beta_1 = \varrho k_1$, $\beta_2 = \varrho k_2$, $\varrho \neq 0$, *and* $\varrho < \frac{1}{C_P}$, *and* $k_2 b_{11} \neq k_1 b_{12}$, *or* $k_2 b_{12} \neq k_1 b_{22}$.

Then $i\mathbb{R} \subset \rho(\mathcal{A})$.

Proof. We show this result by a contradiction argument. Following the arguments given in [4] (see also Ref. [8]), the proof consists of the following steps:

Step 1. Since $0 \in \rho(\mathcal{A})$, for any real number λ with $\|\lambda\mathcal{A}^{-1}\| < 1$, the linear bounded operator $i\lambda\mathcal{A}^{-1} - I$ is invertible. Therefore $i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$ is invertible and its inverse belongs to $\mathcal{L}(\mathcal{H})$, that is, $i\lambda \in \rho(\mathcal{A})$. Moreover, $\|(i\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $\left(-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1}\right)$.

Step 2. If $\sup\left\{\|(i\lambda I - \mathcal{A})^{-1}\| : |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\right\} = M < \infty$, then for $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$ and $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < M^{-1}$, we have $\|(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1}\| < 1$, therefore the operator $i\lambda I - \mathcal{A} = (i\lambda_0 I - \mathcal{A})\,(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1})$ is invertible with its inverse in $\mathcal{L}(\mathcal{H})$, that is, $i\lambda \in \rho(\mathcal{A})$. Since λ_0 is arbitrary we can conclude that $\left\{i\lambda : |\lambda| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\right\} \subset \rho(\mathcal{A})$ and the function $\|(i\lambda I - \mathcal{A})^{-1}\|$ is continuous in the interval $\left(-\|\mathcal{A}^{-1}\|^{-1} - M^{-1}, \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\right)$.

Step 3. It follows from item (3.1) that if $i\mathbb{R} \subset \rho(A)$ is not true, then there exists $\omega \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\|^{-1} \leq |\omega|$ such that $\{i\lambda : |\lambda| < |\omega|\} \subset \rho(\mathcal{A})$ and $\sup\left\{\|(i\lambda I - \mathcal{A})^{-1}\| : |\lambda| < |\omega|\right\} = \infty$. Therefore, there exists a sequence of real numbers $(\lambda_\nu)_{\nu\in\mathbb{N}}$ with $\lambda_\nu \to \omega$ when $\nu \to \infty$ and $|\lambda_\nu| < |\omega|$, for all $\nu \in \mathbb{N}$, and sequences of vector functions $U_\nu = (u_\nu, w_\nu, v_\nu, \eta_\nu, \theta_\nu) \in \mathcal{D}(\mathcal{A})$, $F_\nu = (f_\nu, g_\nu, h_\nu, p_\nu, q_\nu) \in \mathcal{H}$, such that $(i\lambda_\nu I - \mathcal{A})U_\nu = F_\nu$ and $\|U_\nu\|_{\mathcal{H}} = 1$, for all $\nu \in \mathbb{N}$, and $F_\nu \to 0$ in $\mathcal{H}$ when $\nu \to \infty$. Hence,

$$\mathrm{Re}\langle(i\lambda_\nu I - \mathcal{A})U_\nu, U_\nu\rangle_{\mathcal{H}} = b_{11}|\nabla v_\nu|^2 + b_{22}|\nabla\eta_\nu|^2 + 2b_{12}\mathrm{Re}\langle\nabla v_\nu, \nabla\eta_\nu\rangle + \kappa|\nabla\theta_\nu|^2 \to 0 \quad \text{as } \nu \to \infty;$$

Case I. If B is positive definite: $\kappa|\nabla\theta_\nu|^2 + \frac{\det B}{2b_{22}}|\nabla v_\nu|^2 + \frac{\det B}{2b_{11}}|\nabla\eta_\nu|^2 \to 0$, then $\lim_{\nu\to\infty}\|U_\nu\|_{\mathcal{H}} = 0$.
Case II. If B is singular. We suppose $b_{11} > 0$ (the case $b_{11} = 0$ and $b_{22} > 0$ is similar), we obtain

$$\kappa|\nabla\theta_\nu|^2 + \frac{1}{b_{11}}|\nabla(b_{11}v_\nu + b_{12}\eta_\nu)|^2 \to 0 \quad \text{as } \nu \to \infty. \tag{3.2}$$

It follows that $\theta_\nu \to 0$ and $b_{11}v_\nu + b_{12}\eta_\nu \to 0$ in $H_0^1(\Omega)$. Since $(u_\nu)_{\nu\in\mathbb{N}}$ and $(w_\nu)_{\nu\in\mathbb{N}}$ are sequences bounded in $H_0^1(\Omega)$, there exist subsequences, still denoted by $(u_\nu)_{\nu\in\mathbb{N}}$ and $(w_\nu)_{\nu\in\mathbb{N}}$, such that $u_\nu \to u$ and $w_\nu \to w$ in $L^2(\Omega)$. It follows that $v_\nu \to v$, $\eta_\nu \to \eta$ and $b_{11}v_\nu + b_{12}\eta_\nu \to b_{11}v + b_{12}\eta$ in $L^2(\Omega)$, and by (3.2) we have $b_{11}v + b_{12}\eta = 0$ and $b_{11}u + b_{12}w = 0$. On the other hand,

$$i\lambda_\nu\rho_1 v_\nu - \Delta(a_{11}u_\nu + a_{12}w_\nu + b_{11}v_\nu + b_{12}\eta_\nu + k_1\theta_\nu) + \alpha(u_\nu - w_\nu) - \beta_1\theta_\nu = \rho_1 h_\nu \to 0 \quad \text{in } L^2. \tag{3.3}$$

We apply the basic energy estimate, and by compactness arguments we conclude that the sequence $(a_{11}u_\nu + a_{12}w_\nu + b_{11}v_\nu + b_{12}\eta_\nu + k_1\theta_\nu)_{\nu\in\mathbb{N}}$ converge in $H_0^1(\Omega)$. In a similar way we have the same convergence of $(a_{12}u_\nu + a_{22}w_\nu + b_{12}v_\nu + b_{22}\eta_\nu + k_2\theta_\nu)_{\nu\in\mathbb{N}}$. Therefore, using (3.2) we obtain that $(a_{11}u_\nu + a_{12}w_\nu)_{\nu\in\mathbb{N}}$ and $(a_{12}u_\nu + a_{22}w_\nu)_{\nu\in\mathbb{N}}$ converge in $H_0^1(\Omega)$. Since (a_{ij}) is positive definite, it follows that $u_\nu \to u$, $w_\nu \to w$, $v_\nu \to v$, $\eta_\nu \to \eta$ in $H_0^1(\Omega)$.

(a) Using the Cauchy–Schwarz and Young inequalities we get

$$\frac{1}{2}|\nabla(k_1v_\nu + k_2\eta_\nu)|^2 \leqslant (|cq_\nu| + |\lambda_\nu|\,|c\theta_\nu| + C\,|b_{11}v_\nu + b_{12}\eta_\nu|)\,|k_1v_\nu + k_2\eta_\nu| + \kappa^2\,|\nabla\theta_\nu|^2.$$

Since the sequence $(k_1v_\nu + k_2\eta_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, using (3.2) we conclude that $k_1v_\nu + k_2\eta_\nu \to 0$ in $H_0^1(\Omega)$, and then $k_1v + k_2\eta = 0$. Since $k_1b_{12} \neq k_2b_{11}$, we have, $b_{11}v + b_{12}\eta = 0$ and $b_{11}u + b_{12}w = 0$, that $u = w = v = \eta = 0$. Therefore, $\lim_{\nu\to\infty}\|U_\nu\|_{\mathcal{H}} = 0$ and we have a contradiction.

(b) It results that $\rho_1\,\omega^2 u + a_{11}\,\Delta u + a_{12}\,\Delta w - \alpha\,(u-w) = 0$ and $\rho_2\,\omega^2 w + a_{12}\,\Delta u + a_{22}\,\Delta w + \alpha\,(u-w) = 0$ in $L^2(\Omega)$. On the other hand, it results by $b_{11}v + b_{12}\,\eta = 0$ and $b_{11}u + b_{12}w = 0$, that $u = -\frac{b_{12}}{b_{11}}w$. Thus w verifies $-\Delta w = \Xi\, w$, and it follows that $w = 0$ and $u = v = \eta = 0$. If $b_{12} = 0$, by $b_{11}v + b_{12}\,\eta = 0$ and $b_{11}u + b_{12}w = 0$, we obtain that $u = v = 0$ and then $a_{12}\,\Delta w + \alpha w = 0$. Using the hypothesis we conclude that $w = 0$ and $\eta = 0$. Therefore $\lim_{\nu\to\infty}\|U_\nu\|_{\mathcal{H}} = 0$ and we have a contradiction.

(c) We use similar arguments to show that $v = \eta = 0$, and then $\lim_{\nu\to\infty}\|U_\nu\|_{\mathcal{H}} = 0$. □

Theorem 3.3. *Under the hypothesis of Lemma 3.2, the C_0-semigroup $\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable, that is, there exist positive constants M and μ such that $\|\mathcal{S}_{\mathcal{A}}(t)\|_{\mathcal{L}(\mathcal{H})} \leqslant M\exp(-\mu t)$.*

Proof. In view of Theorem 3.1 and Lemma 3.2 it is sufficient to prove (3.1). Given $\lambda \in \mathbb{R}$ and $F = (f, g, h, p, q) \in \mathcal{H}$, let $U = (u, w, v, \eta, \theta) \in \mathcal{D}(\mathcal{A})$ be the solution of $(i\lambda I - \mathcal{A})U = F$. That is,

$$i\lambda u - v = f \quad \text{in } H_0^1(\Omega), \quad \text{and} \quad i\lambda w - \eta = g \quad \text{in } H_0^1(\Omega) \tag{3.4}$$

$$i\lambda\,\rho_1 v - \Delta(a_{11}u + a_{12}w + b_{11}v + b_{12}\,\eta + k_1\,\theta) + \alpha\,(u-w) - \beta_1\,\theta = \rho_1 h \quad \text{in } L^2(\Omega) \tag{3.5}$$

$$i\lambda\,\rho_2\,\eta - \Delta(a_{12}u + a_{22}w + b_{12}v + b_{22}\,\eta + k_2\,\theta) - \alpha\,(u-w) - \beta_2\,\theta = \rho_2 p \quad \text{in } L^2(\Omega) \tag{3.6}$$

$$i\lambda c\,\theta + \beta_1 v + \beta_2\,\eta - \Delta(\kappa\,\theta - k_1 v - k_2\,\eta) = cq \quad \text{in } L^2(\Omega). \tag{3.7}$$

Since $\mathrm{Re}\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = \mathrm{Re}\langle F, U\rangle_{\mathcal{H}}$, there exists a positive constant C such that

$$|\nabla\theta|^2 + |\nabla(b_{11}v + b_{12}\,\eta)|^2 \leqslant C\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}\,. \tag{3.8}$$

Taking the inner product of (3.5) with u and (3.6) with w, adding these identities, using (3.4), the Young and Cauchy–Schwarz inequalities, we obtain

$$\frac{\det A}{2a_{22}}|\nabla u|^2 + \frac{\det A}{2a_{11}}|\nabla w|^2 \leqslant \rho_1|v|^2 + \rho_2|\eta|^2 + C|\nabla\theta|\,(|\nabla u| + |\nabla w|) + |\nabla(b_{11}v + b_{12}\,\eta)|\,|\nabla u| + \rho_1\,(|v|\,|f| + |h|\,|u|) + |\nabla(b_{12}v + b_{22}\,\eta)|\,|\nabla w| + \rho_2\,(|\eta|\,|g| + |p|\,|w|)\,. \tag{3.9}$$

(a) Multiplying (3.7) by $(\overline{k_1 v + k_2\,\eta})$, integrating over Ω and using the Gauss Theorem, Young and Cauchy–Schwarz inequalities, and (3.8), we obtain

$$|\nabla(k_1 v + k_2\,\eta)|^2 \leqslant C\,|\langle\theta, \lambda i\,(k_1 v + k_2\,\eta)\rangle| + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.10}$$

Multiplying Eqs. (3.5) and (3.6) by $k_1 u/\rho_1$ and $k_2 w/\rho_2$ respectively, adding the result, taking the inner product of θ with $i\lambda(k_1 v + k_2\,\eta)$; in $L^2(\Omega)$ and by (3.8), it follows that

$$|\langle\theta, \lambda i\,(k_1 v + k_2\,\eta)\rangle| \leqslant C|\nabla\theta|\,(|\nabla u| + |\nabla w|) + C\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}. \tag{3.11}$$

Substituting (3.11) in (3.10), since $k_1 b_{12} \neq k_2 b_{11}$, and using (3.8) we conclude that

$$|\nabla v|^2 + |\nabla\eta|^2 \leqslant C\,|\nabla\theta|\,(|\nabla u| + |\nabla w|) + C\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}. \tag{3.12}$$

By (3.8), (3.9) and (3.12) we obtain $|\nabla u|^2 + |\nabla w|^2 \leqslant C\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}$. Using (3.8) and (3.12) we get $\|(i\lambda I - \mathcal{A})^{-1}F\| \leqslant C\,\|F\|_{\mathcal{H}}$.

(b) From (3.5) and (3.8) we obtain $|\nabla(b_{11}u + b_{12}w)|^2 \leqslant C\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}$, for $|\lambda| > 1$. Performing the inner product between (3.5) and u, v in $H_0^1(\Omega)$, and using $b_{12}^2 = b_{11}b_{22}$, we get

$$|\nabla u|^2 + |\nabla w|^2 \leqslant C\left(|\nabla(b_{11}v + b_{12}\,\eta)|\,\|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{1}{|\lambda|}\|U\|_{\mathcal{H}}^2 + \frac{1}{|\lambda|}\;\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}\right). \tag{3.13}$$

Combining (3.8), the inner product of (3.5) with u, and (3.6) with w, respectively, and using (3.13) we obtain $\left(1 - \frac{C}{|\lambda|}\right)\|U\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}$, for $|\lambda| > 1$. Thus $\|(i\,\lambda\, I - A)^{-1}\,F\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}$ when $|\lambda|$ is large enough. Since the function $\lambda \mapsto \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}$ is continuous, we conclude.

(c) By similar arguments taking the inner product in $L^2(\Omega)$ of (3.7) with $k_1 v + k_2\,\eta$, using $(\beta_1,\ \beta_2) = \varrho\,(k_1, k_2)$, and (3.11) we conclude $|\nabla(k_1 v + k_2\,\eta)|^2 \leqslant C|\nabla\theta|\,(|\nabla u| + |\nabla w|) + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Since $k_1\,b_{12} \neq k_2\,b_{11}$, we obtain (3.12). □

4. Analyticity

We recall the following result (see [4]): Let $\mathcal{S}(t)$ be a C_0-semigroup of contractions of linear operators in a Hilbert space X with infinitesimal generator $\mathcal{A}$. Suppose that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then, $\mathcal{S}(t)$ is analytic if and only if $\limsup_{|\lambda|\to\infty}\|\lambda\,(i\,\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)}g < \infty$. It follows from Lemma 3.2 that the imaginary axis is contained in $\rho(\mathcal{A})$. In the next theorem of this section, we will show that there is a positive constant C, independent on λ, such that $|\lambda|\,\|(i\lambda I - \mathcal{A})^{-1}\| \leqslant C$, $\forall\lambda \in \mathbb{R}$.

Theorem 4.1. *Suppose that item* (a) *or* (c) *of Lemma 3.2 occurs. Then the semigroup $\mathcal{S}_{\mathcal{A}}(t)$ is analytic.*

Proof. Given $\lambda \in \mathbb{R}$ and $F = (f, g, h, p, q) \in \mathcal{H}$, let $U = (u, w, v, \eta, \theta) \in \mathcal{D}(\mathcal{A})$ be the solution of $(i\lambda I - \mathcal{A})U = F$. In Theorem 3.3 we proved (see (3.8) and (3.12)) that there exists $C > 0$ such that

$$|\nabla\theta|^2 + |\nabla u|^2 + |\nabla w|^2 + |\nabla v|^2 + |\nabla\eta|^2 \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.1}$$

Since $\operatorname{Im}\langle(i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = \operatorname{Im}\langle F, U\rangle_{\mathcal{H}}$ we have $\lambda\|U\|^2_{\mathcal{H}} \leqslant |\operatorname{Im}\langle\mathcal{A}U, U\rangle_{\mathcal{H}}| + \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$, with

$$\begin{aligned}\operatorname{Im}\langle\mathcal{A}U, U\rangle_{\mathcal{H}} = 2i \operatorname{Im}\big(&a_{11}\langle\nabla v, \nabla u\rangle + a_{12}\langle\nabla v, \nabla w\rangle + \alpha\langle v - \eta, u - w\rangle + a_{12}\langle\nabla\eta, \nabla u\rangle \\ &+ a_{22}\langle\nabla\eta, \nabla w\rangle - \beta_1\langle\theta, v\rangle - \beta_2\langle\theta, \eta\rangle + k_2\langle\nabla\theta, \nabla\eta\rangle + k_1\langle\nabla\theta, \nabla v\rangle\big).\end{aligned} \tag{4.2}$$

By (4.1)–(4.2) we conclude that $|\operatorname{Im}\langle\mathcal{A}U, U\rangle_{\mathcal{H}}| \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$, and then $\lambda\|U\|^2_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$, for all $\lambda \in \mathbb{R}$. The proof is complete. □

5. About the lack of exponential stability

In this section we will show that there are cases where the lack of exponential stability of the semigroup occur. To show the lack of exponential stability we will show that the condition (3.1) of Theorem 3.1 does not hold. To do this, it is sufficient to show the existence of sequences $F_\nu \in \mathcal{H}$ and $\xi_\nu \in \mathbb{R}$ such that $(F_\nu)_{\nu\in\mathbb{N}}$ is bounded, $|\xi_\nu| \to \infty$ and $\|(i\xi_\nu I - \mathcal{A})^{-1}F_\nu\| \to \infty$ when $\nu \to \infty$. We denote by $\varphi_\nu \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\lambda_\nu \in \mathbb{R}$ the sequences of eigenvectors and eigenvalues, respectively, of the operator $-\Delta$, that is, $-\Delta\varphi_\nu = \lambda_\nu\varphi_\nu$ in Ω, with $\lambda_\nu \to \infty$ as $\nu \to \infty$ and such that $(\varphi_\nu)_{\nu\in\mathbb{N}}$ is a orthonormal basis of $L^2(\Omega)$.

Theorem 5.1. *Suppose that $\rho_2 b_{11}(a_{11}b_{12} - a_{12}b_{11}) = \rho_1 b_{12}(a_{22}b_{11} - a_{12}b_{12})$, $k_2 b_{11} = k_1 b_{12}$, $k_2 b_{12} = k_1 b_{22}$ and $\beta_1 b_{12} = \beta_2 b_{11}$. Additionally, we assume that $a_{11}b_{12} - a_{12}b_{11}$ have the same sign that b_{12}. Then $\mathcal{S}_{\mathcal{A}}(t)$ is not exponentially stable.*

Proof. First of all, we assume that $b_{12} \neq 0$ and $b_{12} + b_{11} \neq 0$. For each $\nu \in \mathbb{N}$, we take $F_\nu = (0, 0, a\rho_1^{-1}\varphi_\nu, b\rho_2^{-1}\varphi_\nu, 0) \in \mathcal{H}$, with $a, b \in \mathbb{R}$, and we denote by $U_\nu = (u_\nu, w_\nu, v_\nu, \eta_\nu, \theta_\nu)$ the solution of the resolvent equation $(i\lambda I - \mathcal{A})U_\nu = F_\nu$, $\lambda \in \mathbb{R}$. For each $\nu \in \mathbb{N}$, the solutions of the resolvent equation are of the form $u_\nu = A_\nu\varphi_\nu$, $w_\nu = B_\nu\varphi_\nu$ and $\theta_\nu = C_\nu\varphi_\nu$. Thus, we get the system

$$v_\nu = i\lambda u_\nu, \qquad \eta_\nu = i\lambda w_\nu, \tag{5.1}$$

$$-\rho_1\lambda^2 A_\nu + \lambda_\nu(a_{11} + i\lambda b_{11})A_\nu + \lambda_\nu(a_{12} + i\lambda b_{12})B_\nu + \lambda_\nu k_1 C_\nu + \alpha(A_\nu - B_\nu) - \beta_1 C_\nu = a, \tag{5.2}$$

$$-\rho_2\lambda^2 B_\nu + \lambda_\nu(a_{12} + i\lambda b_{12})A_\nu + \lambda_\nu(a_{22} + i\lambda b_{22})B_\nu + \lambda_\nu k_2 C_\nu - \alpha(A_\nu - B_\nu) - \beta_2 C_\nu = b, \tag{5.3}$$

$$(ic\lambda + \kappa\lambda_\nu)C_\nu + i\lambda(\beta_1 - k_1\lambda_\nu)A_\nu + i\lambda(\beta_2 - k_2\lambda_\nu)B_\nu = 0. \tag{5.4}$$

Multiplying (5.2) by b_{12} and (5.3) by b_{11}, subtracting the results we get

$$\begin{aligned}&\left(-\lambda^2 + \frac{(a_{11}b_{12} - a_{12}b_{11})\lambda_\nu}{\rho_1 b_{12}}\right)(\rho_1 b_{12}A_\nu - \rho_2 b_{11}B_\nu) + \alpha(b_{12} + b_{11})(A_\nu - B_\nu) \\ &\quad + [\lambda_\nu(k_1b_{12} - k_2b_{11}) - (\beta_1b_{12} - \beta_2b_{11})]C_\nu = ab_{12} - bb_{11}.\end{aligned} \tag{5.5}$$

Taking $a = \alpha$ and $b = -\alpha$ in (5.3) and (5.4), respectively, we obtain

$$\left(-\lambda^2 + \frac{(a_{11}b_{12} - a_{12}b_{11})\lambda_\nu}{\rho_1 b_{12}}\right)(\rho_1 b_{12}A_\nu - \rho_2 b_{11}B_\nu) + \alpha(b_{12} + b_{11})(A_\nu - B_\nu) = \alpha(b_{12} + b_{11}). \tag{5.6}$$

Taking $\lambda = \xi_\nu = \sqrt{\frac{a_{11}b_{12} - a_{12}b_{11}}{\rho_1 b_{12}}\lambda_\nu}$, it results by (5.6) that $A_\nu = 1 + B_\nu$. Replacing in (5.4) we get $C_\nu = -\frac{i\xi_\nu(\beta_1 - k_1\lambda_\nu)}{\kappa\lambda_\nu + ic\xi_\nu} - \frac{i\xi_\nu[(\beta_1+\beta_2) - (k_1+k_2)\lambda_\nu]}{\kappa\lambda_\nu + ic\xi_\nu}B_\nu$. Replacing in (5.2), we have $B_\nu = \frac{P_\nu + i\lambda_\nu\xi_\nu Q_\nu}{R_\nu + i\lambda_\nu\xi_\nu S_\nu}$ where

$$\begin{aligned}P_\nu &= \left(\rho_1\xi_\nu^2 - a_{11}\lambda_\nu\right)\left(\kappa^2\lambda_\nu^2 + c^2\xi_\nu^2\right) - c\,\xi_\nu^2(k_1\lambda_\nu - \beta_1)^2, \\ Q_\nu &= -\kappa(k_1\lambda_\nu - \beta_1)^2 - b_{11}(\kappa^2\lambda_\nu^2 + c^2\xi_\nu^2), \\ R_\nu &= \left[-\rho_1\xi_\nu^2 + (a_{11} + a_{12})\lambda_\nu\right](\kappa^2\lambda_\nu^2 + c^2\xi_\nu^2) - c\xi_\nu^2(k_1\lambda_\nu - \beta_1)[\beta_1 + \beta_2 - (k_1 + k_2)\lambda_\nu], \\ S_\nu &= (b_{11} + b_{12})(\kappa^2\lambda_\nu^2 + c^2\xi_\nu^2) + \kappa(\beta_1 - k_1\lambda_\nu)[\beta_1 + \beta_2 - (k_1 + k_2)\lambda_\nu].\end{aligned}$$

We conclude that $\lim_{\nu\to\infty}\|\eta_\nu\| = \lim_{\nu\to\infty}\xi_\nu|B_\nu| = \infty$ and therefore

$$\lim_{\nu\to\infty}\|U_\nu\|_{\mathcal{H}} = \infty.$$

Now, assume that $b_{12} = 0$. In this case $b_{22} = 0$ and by hypothesis of the proposition we must have $a_{12} = k_2 = \beta_2 = 0$. Taking $a = \alpha + 1$, $b = -\alpha$ in (5.2) and (5.3), respectively, it follows that

$$-\rho_1\lambda^2 A_\nu + \lambda_\nu(a_{11} + i\lambda b_{11})A_\nu + \alpha\,(A_\nu - B_\nu) + (\lambda_\nu k_1 - \beta_1)\,C_\nu = \alpha + 1 \tag{5.7}$$

$$\left(-\rho_2\,\lambda^2 + a_{22}\,\lambda_\nu\right)B_\nu - \alpha(A_\nu - B_\nu) = -\alpha, \quad \text{and} \quad (ic\lambda + \kappa\,\lambda_\nu)C_\nu + i\lambda(\beta_1 - k_1\lambda_\nu)A_\nu = 0. \tag{5.8}$$

Taking $\lambda = \xi_\nu = \sqrt{\frac{a_{22}}{\rho_2}\lambda_\nu}$; in (5.7)–(5.8) we obtain $B_\nu = A_\nu - 1$. The proof of the theorem is complete. $\square$

Acknowledgments

The third author is partially supported by Fondecyt project 1110540 and Basal, CMM, Universidad de Chile. The second author is supported by Faperj project E26/102.812/2008. This work was done while the fourth author visited LNCC/MCT and Universidade Federal de Viçosa and he thanks the fellowship of LNCC/PCI.

References

[1] D. Ieşan, R. Quintanilla, A theory of porous thermoviscoelastic mixtures, J. Thermal Stresses 30 (2007) 693–714.
[2] D. Ieşan, L. Nappa, On the theory of viscoelastic mixtures and stability, Math. Mech. Solids 13 (2008) 55–80.
[3] F. Martínez, R. Quintanilla, Some qualitative results for the linear theory of binary mixtures of thermoelastic solids, Collect. Math. 46 (1995) 263–277.
[4] Z. Liu, S. Zheng, Semigroups Associated with Dissipative Systems, in: CRC Research Notes in Mathematics, vol. 398, Chapman & Hall, 1999.
[5] M.S. Alves, J.E. Muñoz Rivera, R. Quintanilla, Exponential decay in a thermoelastic mixture of solids, Internat. J. Solids Structures 46 (2009) 1659–1666.
[6] M.S. Alves, J.E. Muñoz Rivera, M. Sepúlveda, O. Vera Villagrán, Exponential stability in thermoviscoelastic mixtures of solids, Internat. J. Solids Structures 46 (2009) 4151–4162.
[7] M.S. Alves, J.E. Muñoz Rivera, M. Sepúlveda, O. Vera Villagrán, Analyticity of semigroups associated with thermoviscoelastic mixtures of solids, J. Thermal Stresses 32 (2009) 986–1004.
[8] M. Alves, J.E. Muñoz Rivera, M. Sepúlveda, O. Vera, Stabilization of a system modelling temperature and porosity fields in a Kevin–Voigt type mixture, Acta Mech. 219 (2011) 145–167.
[9] L.M. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, Trans. Amer. Math. Soc. 236 (1978) 385–394.
[10] F.L. Huang, Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations 1 (1985) 43–56.
[11] J. Prüss, On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. 284 (2) (1984) 847–857.