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Research Article

A Note on Directional Wavelet Transform: Distributional Boundary Values and Analytic Wavefront Sets

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By using a particular class of directional wavelets (namely, the conical wavelets, which are wavelets strictly supported in a proper convex cone in the k-space of frequencies), in this paper, it is shown that a tempered distribution is obtained as a finite sum of boundary values of analytic functions arising from the complexification of the translational parameter of the wavelet transform. Moreover, we show that for a given distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, the continuous wavelet transform of f with respect to a conical wavelet is defined in such a way that the directional wavelet transform of f yields a function on phase space whose high-frequency singularities are precisely the elements in the analytic wavefront set of f.

1. Introduction

Wavelets, as well as the theory of distributions, have found applications in various fields of pure and applied mathematics, physics, and engineering. The requirements of modern mathematics, mathematical physics and engineering, have brought the necessity to incorporate ideas from wavelet analysis to the distribution theory, and reciprocally. Over the last decades, a number of authors have studied the relationship between wavelets and the theory of distributions for different purposes (see, e.g., [1–17] and references therein). Particularly related to this note, in [5, 7, 11], the authors investigate the singularities of tempered distributions via the wavelet transform. In [5], Moritoh introduced a class of wavelet transform as a continuous and microlocal version of the Littlewood-Paley decomposition and compared the wavefront sets defined by his wavelet transform and

Hörmander's wavefront sets. More recently, Pilipović and Vuletić [11] gave some more precise information concerning the work of Moritoh. They consider a special wavelet transform of Moritoh and gave new definitions of wavefront sets of tempered distributions via that wavelet transform. The major result in [11] is that these wavefront sets are equal to the wavefront sets in the sense of Hörmander in the cases $n = 1, 2, 4, 8$. If $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$, they combined results for dimensions $n = 1, 2, 4, 8$ and characterized wavefront sets in ξ-directions, where ξ are presented as products of nonzero points of $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_s}$, $n_1 + \cdots + n_s = n$, $n_i \in \{1, 2, 4, 8\}$, $i = 1, \ldots, s$. In the paper [7], Navarro introduced an analyzing wavelet in $\mathcal{S}(\mathbb{R}^2)$ and an irreducible group action with the property that the associated wavelet transform of a tempered distribution is singular along the wavefront set of the distribution. The main result relates the notion of the wavefront set and the wavelet transform of distributions in $\mathcal{S}'(\mathbb{R}^2)$. It should be noted that the core of the construction of Navarro (the irreducible group action on $L^2(\mathbb{R}^2)$) parallels that of Kutyniok and Labate [18], where they consider a different notion of wavefront set based on the concept of *continuous shearlet transform*.

In this short note, by using a particular class of directional wavelets (namely, the conical wavelets, which are wavelets strictly supported in a proper convex cone in the k-space of frequencies [19, 20]), it is shown that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is obtained as a finite sum of boundary values of analytic functions arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of f with respect to directional wavelet ψ. Furthermore, we characterize the analytic wavefront set of a tempered distribution in terms of the behavior of its directional wavelet transform. The main results of this work are given in Lemma 3.4 and Theorem 3.5. In Lemma 3.4, we prove that the distributional wavelet transform with respect to the directional wavelet is an analytic function of tempered growth. By using Lemma 3.4, we show that the tempered distributions can be obtained as a finite sum of boundary values of analytic functions of tempered growth (Theorem 3.5). In Section 4, we apply Theorem 3.5 in the study of analytic wavefront set of tempered distributions. We show that, for a given distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, the wavelet transform of f with respect to a conical wavelet is defined in such a way that the directional wavelet transform of f yields a function on phase space whose high-frequency singularities are precisely the elements in the analytic wavefront set of f. In [21], Hörmander introduced the notion of the analytic wavefront set $WF_a(f)$ of f as a subset of the cotangent space $T^*(X) \setminus 0$, whose projection to X coincides with the analytic singular support of f. His definition relies on the use of the Fourier transform of f. In this note, following Nishiwada [22, 23], we present an alternative definition of $WF_a(f)$ in terms of generalized boundary values of analytic functions arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of f with respect to directional wavelet ψ.

2. A Glance at the Wavelets: Definitions and Basic Properties

We shall recall in this section some definitions and basic properties of the wavelets. But before we establish some notation. We will use the standard multi-index notation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \ldots, x_n)$ (resp. $z = (z_1, \ldots, z_n)$), such that $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, $x \geq 0$ means $x_1 \geq 0, \ldots, x_n \geq 0$, $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ and $|x| = |x_1| + \cdots + |x_n|$. Moreover, we define $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of nonnegative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha + \beta$ denotes $(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, $\alpha \geq \beta$ means $(\alpha_1 \geq \beta_1, \ldots, \alpha_n \geq \beta_n)$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and

$$D^\alpha \varphi(x) = \frac{\partial^{|\alpha|} \varphi(x_1, \ldots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}, \ldots, \partial x_n^{\alpha_n}}. \tag{2.1}$$

We consider two n-dimensional spaces—x-space and k-space—with the Fourier transform defined as follows

$$\hat{f}(k) = \mathcal{F}[f(x)](k) = \int_{\mathbb{R}^n} d^n x \; f(x) e^{-i\langle k,x\rangle}, \tag{2.2}$$

while the Fourier inversion formula is

$$f(x) = \mathcal{F}^{-1}\left[\hat{f}(k)\right](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d^n k \; \hat{f}(k) e^{i\langle k,x\rangle}. \tag{2.3}$$

The variable k will always be taken real while x will also be complexified: when it is complex, it will be noted $z = x + iy$.

Definition 2.1. A wavelet is a complex-valued function ψ in $L^2(\mathbb{R}^n, d^n x)$, satisfying the admissibility condition

$$C_\psi = (2\pi)^n \int \frac{d^n k}{|k|^n} \left|\hat{\psi}(k)\right|^2 < \infty, \tag{2.4}$$

where $\hat{\psi}$ is the Fourier transform of ψ.

If ψ is sufficiently regular—enough to take $\psi \in L^1(\mathbb{R}^n, d^n x) \cap L^2(\mathbb{R}^n, d^n x)$—then the admissibility condition above means that

$$\hat{\psi}(0) = 0 \Longleftrightarrow \int d^n x \; \psi(x) = 0. \tag{2.5}$$

Definition 2.2. The continuous wavelet transform of a distribution $f(x) \in \mathcal{S}'(\mathbb{R}^n)$ with respect to some analyzing wavelet ψ is defined as the following convolution:

$$\mathbb{W}_\psi f(x', \ell) = \left\langle f(x), \overline{\psi_{x',\ell}(x)} \right\rangle = \int d^n x \; \overline{\psi_{x',\ell}(x)} f(x), \tag{2.6}$$

where

$$\psi_{x',\ell}(x) = \frac{1}{\ell^n} \psi\left(\ell^{-1}(x - x')\right), \tag{2.7}$$

with $\ell \in \mathbb{R}_+$ as the length scale at which we analyze $f(x)$ and $x' \in \mathbb{R}^n$ as the translation parameter corresponding to the position of the analyzing wavelet ψ.

Remark 2.3. Throughout this article the Fourier transform, the wavelet transform, and boundary values of analytic functions are always interpreted in a distributional sense.

Note that the wavelet transform may also be written as

$$\mathbb{W}_{\psi} f(x', \ell) = \frac{1}{(2\pi)^n} \int d^n k \; \widehat{\mathbb{W}_{\psi} f}(k, \ell) e^{i\langle k, x' \rangle}, \tag{2.8}$$

where

$$\widehat{\mathbb{W}_{\psi} f}(k, \ell) = \widehat{\psi}(\ell k) \widehat{f}(k), \tag{2.9}$$

is the Fourier transform of $\mathbb{W}_{\psi} f$. With this we have the following (see [24, Lemma 8.2.6, page 441] for the one-dimensional space).

Lemma 2.4. *For any $f \in L^2(\mathbb{R}^n)$, we have*

$$\begin{aligned} \mathcal{F}\left[\mathbb{W}_{\psi} f(x', \ell)\right](k, \ell) &= \int_{\mathbb{R}^n} d^n x \; \mathbb{W}_{\psi} f(x', \ell) e^{-i\langle k, x' \rangle} \\ &= \widehat{\psi}(\ell k) \widehat{f}(k). \end{aligned} \tag{2.10}$$

Remark 2.5. The domain of a wavelet transform is usually the L^2 space, but the Lemma 2.4 can be extended to $\mathcal{S}'(\mathbb{R}^n)$, which is the dual space of $\mathcal{S}(\mathbb{R}^n)$. In particular, see Remark 3.2, the class of conical wavelets belongs to the space $\mathcal{S}(\mathbb{R}^n)$. This allows us to define the directional wavelet transform of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ in such a way that the wavelet transform of f yields a function on phase space whose high-frequency singularities are precisely the elements in the analytic wavefront set of f.

3. Distributional Boundary Values

In this section, the directional wavelet transform is used to show that analytic functions which satisfy a tempered growth condition obtain distributional boundary values in $\mathcal{S}'(\mathbb{R}^n)$. Before that, in order to define a directional wavelet, we need some terminology and simple facts concerning cones.

An open set $C \subset \mathbb{R}^n$ is called a cone if C (unless specified otherwise, all cones will have their vertices at zero) is invariant under positive homotheties; that is, if for all $\lambda > 0$, $\lambda C \subset C$. A cone C is an open connected cone if C is an open connected set. Moreover, C is called convex if $C + C \subset C$ and *proper* if it contains no any straight line (observe that if C is a cone, then C is proper if and only if $x \in C$ and $x \neq 0$ implie $-x \notin C$). A cone C' is called compact in C—we write $C' \Subset C$—if the projection $\mathrm{pr}\overline{C}' \overset{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathrm{pr}C \overset{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in x-space, we associate with C a closed convex cone C^* in k space which is the set $C^* = \{k \in \mathbb{R}^n \mid \langle k, x \rangle \geq 0, \forall x \in C\}$. The cone C^* is called the *dual cone* of C.

Definition 3.1. A wavelet $\psi(x)$ is said to be *directional* if the effective support of its Fourier transform $\widehat{\psi}(k)$ is contained in a convex and proper cone in the k-space of frequencies, with vertex at the origin, or a finite union of disjoint such cones; in that case, one will usually call ψ multidirectional.

Remark 3.2. According to [19], for us to reach a genuinely directional wavelet $\psi(x)$, it suffices to consider a smooth function $\widehat{\psi}(k)$ with support in the strictly convex cone C^* in the k space of frequencies having an arbitrary large number of vanishing moments on the *boundary* of the supporting cone and behaving inside C^* as $P(k_1,\ldots,k_n)e^{-\langle k,y\rangle}$, where $y \in C$and $P(\cdot)$ denotes a polynomial in n variables. In this case, the resulting directional wavelet is also called *conical*. We also observe that, for all $y \in C$, the exponential $e^{-\langle k,y\rangle}$, considered as a function of $k \in C^*$, belongs to the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$. Then, $P(k_1,\ldots,k_n)e^{-\langle k,y\rangle} \in \mathcal{S}(\mathbb{R}^n)$ too. Since the inverse Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$, it follows that directional wavelet $\psi(x) \in \mathcal{S}(\mathbb{R}^n)$.

In particular, from the above remark, it follows that for a directional wavelet (2.8) can be rewritten as

$$\mathbb{W}_\psi f(z,\ell) = \frac{1}{(2\pi)^n}\int d^n k\ P(\ell k)\widehat{f}(k)e^{i\langle k,z\rangle}, \tag{3.1}$$

where we have introduced the complex variable $z = x + i\ell y \in \mathbb{R}^n + iC$. Remember that here x corresponds to the location of the continuous wavelet transform of f with respect to ψ. Consequently, it is clear from (3.1) that the directional wavelet transform, $\mathbb{W}_\psi f(x,\ell)$, of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is naturally extensible as an analytic function in a domain in $\mathbb{C}^n$, for each arbitrary but fixed $\ell > 0$. Still, since C^* is a regular set [25, pages 98, 99], it follows that $\operatorname{supp}(\widehat{\mathbb{W}_\psi f}) = \operatorname{supp}(\widehat{\psi}) \subseteq C^*$.

Let Ω be an open subset in $\mathbb{C}^n$. Then we shall denote by $\mathcal{O}(\Omega)$ the space of analytic functions in Ω. Let C be a proper open convex cone, and let C' be an arbitrary compact cone contained in C. Denote by $T(C')$ the subset of $\mathbb{C}^n$ consisting of all elements whose imaginary parts lie in C'. $T(C')$ is referred to as a tube domain. We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C') = \{z = x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C', |y| < \delta\}, \tag{3.2}$$

where $\delta > 0$ is an arbitrary number.

Definition 3.3. Let Z be a complex neighbourhood of $\mathbb{R}^n$. An analytic function $f(z) \in \mathcal{O}(Z \cap T(C'))$ is said to be of tempered growth if there are an integer α and a constant $\mathbf{M}$ depending on C' such that

$$|f(x+iy)| \le \mathbf{M}(C')|y|^{-\alpha}, \tag{3.3}$$

for all point $z = x + iy$ in $Z \cap T(C')$.

Lemma 3.4. *For an arbitrary but fixed scale $\ell \in \mathbb{R}_+$, assume that the function $\mathbb{W}_\psi f(z,\ell)$, arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to directional wavelet ψ, is analytic in $Z \cap T(C')$. Then $\mathbb{W}_\psi f(z,\ell)$ is of tempered growth as a function of z.*

Proof. We start considering the formula

$$\mathbb{W}_{\psi} f(x + i\ell y, \ell) = \frac{1}{(2\pi)^n} \int_{C^*} d^n k \; P(\ell k) \hat{f}(k) e^{i\langle k,z\rangle}. \tag{3.4}$$

By Theorem 7.13 of [26], since $P(\ell k)$ is a polynomial and $\hat{f}(k)$ a tempered distribution, then $P(\ell k)\hat{f}(k)$ is also tempered. Note that, we can write

$$P(\ell k) = P_m(\ell k) + P_{m-1}(\ell k) + \cdots + P_0, \tag{3.5}$$

where P_j is homogeneous of degree j and $P_m(\ell k) \neq 0$ when $k \neq 0$. It follows that for some constant $\mathbf{M}$, we have that $|P(\ell k)| \leq \mathbf{M}\ell^m |P(k)|$. This implies that $|P(\ell k)\hat{f}(k)| \leq \mathbf{M}\ell^m |P(k)\hat{f}(k)|$. Still, the character tempered of $P(k)\hat{f}(k)$ implies that there exist an integer N and a constant $\mathbf{M}_1$ such that $P(k)\hat{f}(k)$ satisfies the estimate

$$\left|P(k)\hat{f}(k)\right| \leq \mathbf{M}_1 (1 + |k|)^N. \tag{3.6}$$

Hence,

$$\left|P(\ell k)\hat{f}(k)\right| \leq \ell^m \mathbf{M}_2 (1 + |k|)^N. \tag{3.7}$$

Using the binomial theorem, the above estimate can be rewritten as

$$\left|P(\ell k)\hat{f}(k)\right| \leq \ell^m \mathbf{M}_2 \sum_{j=0}^{N} c_j |k|^j. \tag{3.8}$$

Now, let C' be a cone, such that $C' \Subset C$. Then there exists $c > 0$ so that $\langle k, \ell y\rangle \geq c|k||\ell y|$, for all $k \in C^*$ and for all $y \in C'$. Hence for $x + i\ell y \in \mathbb{R}^n + iC'$,

$$\begin{aligned} \left|\mathbb{W}_{\psi} f(x + i\ell y, \ell)\right| &\leq \frac{\ell^m}{(2\pi)^n} \int_{C^*} d^n k \left|P(k)\hat{f}(k)\right| e^{-\langle k,\ell y\rangle} \\ &\leq \frac{\ell^m \mathbf{M}_2}{(2\pi)^n} \sum_{j=0}^{N} c_j \int_{C^*} d^n k \; |k|^j e^{-c|k||ly|}. \end{aligned} \tag{3.9}$$

Following Schwartz [25, Proposition 32, page 39], we get the following:

$$\begin{aligned} \left|\mathbb{W}_{\psi} f(x + i\ell y, \ell)\right| &\leq \frac{\ell^m \mathbf{M}_2}{(2\pi)^n} \sum_{j=0}^{N} c_j \sigma^{n-1} \int_0^\infty dt \; t^{n+j-1} e^{-c|\ell y|t} \\ &\leq \mathbf{M}_3(C') \left|\ell y\right|^{-(n+N)}, \end{aligned} \tag{3.10}$$

where σ^{n-1} is the area of the unit sphere in $\mathbb{R}^n$. □

Now, let C be an open cone of the form $C = \cup_{j=1}^{m} C_j$, $m < \infty$, where each C_j is an proper open convex cone. If we write $C' \Subset C$, we mean that $C' = \cup_{j=1}^{m} C'_j$ with $C'_j \Subset C_j$. Furthermore, we define by $C_j^* = \{k \in \mathbb{R}^n \mid \langle k, x\rangle \geq 0, \forall x \in C_j\}$ the dual cones of C_j, such that the dual cones C_j^*, $j = 1, \ldots, m$, have the properties

$$\begin{gathered}\mathbb{R}^n \setminus \left(\cup_{j=1}^{m} C_j^*\right), \\ C_j^* \cap C_k^*, \quad j \neq k, j, k = 1, \ldots, m,\end{gathered} \tag{3.11}$$

that are sets of Lebesgue measure zero. Moreover, assume that $P(\ell k)\widehat{f}(k)$ can be written as $P(\ell k)\widehat{f}(k) = \sum_{j=1}^{m} \lambda_j(k) P(\ell k)\widehat{f}(k)$, where $\lambda_j(k)$ denotes the characteristic function of C_j^*, $j = 1, \ldots, m$. We shall consider the asymptotic property of $\mathbb{W}_\psi f(z,\ell)$ as $\ell \to 0$ for $z = x + i\ell y \in Z \cap T(C')$.

Theorem 3.5. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then f can be expressed as a finite sum*

$$f = \sum_{j=1}^{m} b_{C'_j}\left(\mathbb{W}_\psi f_j(z,\ell)\right), \tag{3.12}$$

where each $\mathbb{W}_\psi f_j(z,\ell)$, arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of $f_j \in \mathcal{S}'(\mathbb{R}^n)$ with respect to directional wavelet ψ, is analytic in $Z \cap T(C'_j)$ and of tempered growth, and where $b_{C'_j}(\mathbb{W}_\psi f_j(z,\ell))$ denotes the boundary value in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. The proof that each $\mathbb{W}_\psi f_j(z,\ell)$ is of tempered growth is obtained by the similar way as in Lemma 3.4. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Choose now $y_0 \in C'_j$ and write $y = \ell y_0$ (this defines a half-line for $0 \neq y_0 \in C'_j$). Note that with $y = \ell y_0$ in (3.1), then $\ell \to 0$ when $y \to 0$. Thus, we have

$$\begin{aligned}\left\langle b_{C'_j}(\mathbb{W}_\psi f_j), \varphi \right\rangle &= \lim_{y \to 0} \int_{\mathbb{R}^n} d^n x \mathbb{W}_\psi f_j(x + iy, \ell)\varphi(x) \\ &= \lim_{y \to 0} \int_{\mathbb{R}^n} d^n x \left(\frac{1}{(2\pi)^n} \int_{C_j^*} d^n k \lambda_j(k) P(\ell k)\widehat{f}(k) e^{i\langle k, x+iy\rangle} \right)\varphi(x) \\ &= \lim_{y \to 0} \frac{1}{(2\pi)^n} \int_{C_j^*} d^n k \lambda_j(k) P(\ell k)\widehat{f}(k)\widehat{\varphi}(-k) e^{-\langle k, y\rangle} \\ &= \frac{1}{(2\pi)^n} \int_{C_j^*} d^n k \lambda_j(k)\widehat{f}(k)\widehat{\varphi}(-k) \\ &= \langle f_j, \varphi\rangle.\end{aligned} \tag{3.13}$$

Hence, using the linearity of $f \in \mathcal{S}'(\mathbb{R}^n)$ and the assumptions (3.11), we obtain that

$$\langle f, \varphi \rangle = \sum_{j=1}^{m} \langle f_j, \varphi \rangle = \sum_{j=1}^{m} \left\langle b_{C'_j}(\mathbb{W}_\psi f_j), \varphi \right\rangle. \tag{3.14}$$

Thus, the limit of each $\mathbb{W}_\psi f_j(z,\ell)$ as $C'_j \ni y \to 0$ exists in $\mathcal{S}'(\mathbb{R}^n)$; that is, f admits the distributional boundary value $\sum_{j=1}^{m} b_{C'_j}(\mathbb{W}_\psi f_j)$ in the sense of weak convergence. But from [27, Corollary 1, page 358], the latter implies strong convergence since $\mathcal{S}(\mathbb{R}^n)$ is Montel. □

4. Analytic Wavefront Set

From what we have seen in the previous section, a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is obtained as a finite sum of boundary values of analytic functions $\mathbb{W}_\psi f_j(z,\ell)$, with $j = (1,\ldots,m)$, arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of f_j with respect to directional wavelet ψ, and where a tempered growth condition was described to characterize such boundary values. We now translate growth condition in terms of the analytic wavefront set.

Definition 4.1. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, such that $f = b_C(\mathbb{W}_\psi f(z,\ell))$, where $b_C(\mathbb{W}_\psi f(z,\ell))$ denotes the strong boundary value in $\mathcal{S}'(\mathbb{R}^n)$ of an analytic function $\mathbb{W}_\psi f(z,\ell)$ arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of f with respect to directional wavelet ψ. Let $q = (x_0, k_0)$. Then, $q \notin WF_a(f)$ if and only if there exist $\mathbf{M}(C')$ and N for which we have the estimate

$$\left|\mathbb{W}_\psi f(z,\ell)\right| \leq \mathbf{M}(C')\left|\ell y\right|^{-N}, \quad z = x + i\ell y \in Z \cap T(C'). \tag{4.1}$$

$WF_a(f)$ is called analytic wavefront set of f.

Proposition 4.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be the boundary value, in the distributional sense, of a function $\mathbb{W}_\psi f(z,\ell)$ analytic in $Z \cap T(C')$, arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of f with respect to directional wavelet ψ and which satisfies the estimate* (3.3)*. Then near x_0 the fibre $WF_a(f)|_{x_0}$ is contained in C^*.*

Proof. Let $WF_a(f)|_{x_0} = \{k \in (\mathbb{R}^n \setminus 0) \mid (x_0, k) \in WF_a(f)\}$ be the fibre over x_0. Let $\{C_j^*\}_{j\in L}$ be a finite covering of closed properly convex cones of C^*. Decompose $P(\ell k)\widehat{f}(k)$ as follows:

$$P(\ell k)\widehat{f}(k) = \sum_{j=1}^{m} \lambda_j(k) P(\ell k)\widehat{f}(k), \tag{4.2}$$

where $\lambda_j(k)$ denotes the characteristic function of C_j^*, $j \in L$. Then, by Theorem 3.5, the decomposition (4.2) will induce a representation of f in the form of a sum of boundary values of functions $\mathbb{W}_\psi f_j(z,\ell)$, such that $\mathbb{W}_\psi f_j(z,\ell) \to f_j$ in the strong topology of $\mathcal{S}'(\mathbb{R}^n)$ as

$y \to 0$, $y \in C'_j \Subset C_j$. According to Lemma 3.4, the family of functions $\mathbb{W}_\psi f_j(z,\ell)$ satisfies the following estimate:

$$\left|\mathbb{W}_\psi f_j(z,\ell)\right| \le \mathbf{M}(C')\left|\ell y\right|^{-N}, \qquad z = x + i\ell y \in Z \cap T(C'), \tag{4.3}$$

unless $\langle k, Y\rangle < 0$ for $k \in C_j^*$ and $Y \in -C'_j$, with $|Y| < \delta$. Then, the cones of "bad" directions responsible for the singularities of these boundary values are contained in the dual cones of the base cones. So we have the inclusion

$$WF_a(f) \subset \mathbb{R}^n \times \left(\cup_j C_j^*\right). \tag{4.4}$$

Then, by making a refinement of the covering and shrinking it to C^*, we obtain the desired result. □

Remark 4.3. It is remarked that in [21] the fiber over x_0, $WF_a(f)|_{x_0}$, is completely characterized by sequences of type $f_N = \phi_N f$, where $\{\phi_N\}$ is a bounded sequence in $C_0^\infty(X)$ which is equal to 1 in a common neighborhood of x_0 and satisfies the following estimate:

$$\left|D^{\alpha+\beta}\phi_N\right| \le C_\alpha (CN)^{|\beta|}, \quad \text{if } |\beta| \le N. \tag{4.5}$$

For the existence of such functions, we refer to Lemma 2.2 in [21].

We can meet Definition 4.1 and Proposition 4.2 in the following proposition.

Proposition 4.4. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $(x_0, k_0) \in T^*(\mathbb{R}^n) \setminus 0$, where $T^*(\mathbb{R}^n) \setminus 0 := \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. Then $(x_0, k_0) \notin WF_a(f)$ if and only if there exists a finite family $\{C_j\}$ of proper open convex cones in $\mathbb{R}^n$, a complex neighborhood Z of x_0 in $\mathbb{C}^n$ and a decomposition of f*

$$f = \sum_{j=1}^{m} b_{C'_j}\left(\mathbb{W}_\psi f_j(z,\ell)\right), \tag{4.6}$$

with each $\mathbb{W}_\psi f_j(z,\ell) \in \mathcal{O}(Z \cap T(C'_j))$ being of tempered growth and analytic near to x_0 for every j satisfying $C'_j \subset \{y \mid \langle k_0, y\rangle \ge 0\}$, and where $b_{C'_j}(\mathbb{W}_\psi f_j(z,\ell))$ denotes the boundary value, in the distributional sense, of analytic functions $\mathbb{W}_\psi f_j(z,\ell)$ arising from the complexification of the variable x corresponding to the location of the continuous wavelet transform of f with respect to directional wavelet ψ.

Note that the above proposition shows that a decomposition of a tempered distribution f into a sum of boundary values of analytic functions is equivalent to a decomposition of analytic wavefront set of f since the fibre $WF_a(f)|_{x_0}$ is contained in $\cup_j C_j^*$. Moreover, the decomposition (4.6) is carried out in the space of C^∞ functions, provided that f is C^∞.

Finally, we recall that in [28] Hörmander defined the wavefront set, $WF(f)$, for a distribution as the set of points in the cotangent space which must be characteristic for every pseudodifferential operator P such that $Pf \in C^\infty$. It is clear that $WF(f) \subset WF_a(f)$.

Following Nishiwada [22] another characterization of $WF(f)$ can be obtained based on the above results.

Proposition 4.5. *Let U be an open set in $\mathbb{R}^n$, $(x_0, k_0) \in T^*(U) \setminus 0$ and $f \in \mathcal{S}'(U)$. Then $(x_0, k_0) \notin WF(f)$ if there exists a finite family $\{C_j\}$ of proper open convex cones in $\mathbb{R}^n$, with $j = 1, \ldots, m$, a complex neighborhood Z of x_0 and a decomposition of f near x_0*

$$f = \sum_{j=1}^{m} b_{C'_j}\left(\mathbb{W}_\psi f_j(z, \ell)\right) \quad \text{in } U, \tag{4.7}$$

with $\mathbb{W}_\psi f_j(z, \ell) \in \mathcal{O}(Z \cap T(C'_j))$ being of tempered growth, such that $b_{C'_j}(\mathbb{W}_\psi f_j(z, \ell)) \in C^\infty$ near x_0 for every j with $C'_j \subset \{y \mid \langle k, \ell y \rangle \geq 0\}$.

Proof. The proof is similar to the proof of the first part of Theorem 3.4 in [23]. □

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