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Critical points of higher order for the normal map of immersions in $\mathbb{R}^d$

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ABSTRACT

We study the critical points of the normal map $\nu : NM \to \mathbb{R}^{k+n}$, where M is an immersed k-dimensional submanifold of $\mathbb{R}^{k+n}$, NM is the normal bundle of M and $\nu(m,u) = m + u$ if $u \in N_m M$. Usually, the image of these critical points is called the focal set. However, in that set there is a subset where the focusing is highest, as happens in the case of curves in $\mathbb{R}^3$ with the curve of the centers of spheres with contact of third order with the curve. We give a definition of r-critical points of a smooth map between manifolds, and apply it to study the 2 and 3-critical points of the normal map in general and the 2-critical points for the case $k = n = 2$ in detail. In the later case we analyze the relation with the strong principal directions of Montaldi (1986) [2].

1. Introduction

Classically the focal set of a differential submanifold is given through the analysis of the singularities of the family of distance squared functions over the submanifold, see [4]. J. Montaldi characterized in [2] the singularities of corank 2 of distance squared functions on surfaces immersed in $\mathbb{R}^4$ as semiumbilic points.

Also, if we consider a differentiable k-dimensional manifold M immersed in $\mathbb{R}^{k+n}$, we know that its focal set can also be interpreted as the image of the critical points of the *normal map* $\nu(m,u) : NM \to \mathbb{R}^{k+n}$ defined by $\nu(m,u) = \pi_N(m,u) + u$, for $u \in N_m M$, where $\pi_N : NM \to M$ denotes the normal bundle.

On the other hand, the concept of curvature ellipse at a point of a surface M immersed in $\mathbb{R}^4$ was treated with full details in [1]. It is defined as the locus of all the end points of the curvature vectors of the normal sections along all the tangent directions to M at a point in it. This ellipse lies in the normal subspace of that point and it is completely determined by the second fundamental. We call *Veronese of curvature* to the natural generalization of the curvature ellipse for higher dimensions of M.

In this work, we describe first the focal set and its geometrical relation to the Veronese of curvature for k-dimensional immersions in $\mathbb{R}^{k+n}$. Then we define the r-critical points of a differential application $f : H \to K$ between two differential manifolds and characterize the 2 and 3-critical points of the normal map. The number of these critical points at $m \in M$ may

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depend on the degenerations of the curvature ellipse and we calculate those numbers in the particular case that M is an immersed surface in $\mathbb{R}^4$.

2. The Veronese of curvature

In the following, M will be a smooth k-dimensional manifold immersed in $\mathbb{R}^{k+n}$. However, since all our study will be local, we shall suppose without loss of generality that M is a regular submanifold of $\mathbb{R}^{k+n}$. Over M we have the tangent bundle $\pi : TM \to M$, and the normal bundle $\pi_N : NM \to M$. Their fibers over $m \in M$ will be denoted by T_mM and $N_mM = (T_mM)^{\perp}$, respectively. We will denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields, and if E is the total space of a vector bundle over M, $\Gamma(E)$ will be the module of its smooth sections. The value of such a section S at $m \in M$ will be denoted S_m.

The usual inner product will be denoted by a dot and the first fundamental form of M by $g \in \Gamma(T^*M \otimes T^*M)$. If $X \in T_m\mathbb{R}^{k+n}$, we will have the decomposition $X = X^{\top} + X^{\perp}$, with $X^{\top} \in T_mM$, $X^{\perp} \in N_mM$, and D_X will stand for the ordinary directional derivative. All vectors of T_mM and of N_mM are considered as elements of $T_m\mathbb{R}^{k+n}$ and frequently identified as usual with points of $\mathbb{R}^{k+n}$.

The second fundamental form will be denoted $\alpha \in \Gamma(NM \otimes T^*M \otimes T^*M)$. If $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(NM)$, we have $Z \cdot \alpha(X, Y) = Z \cdot D_XY = Z \cdot D_YX$. Related to α is the tensor field $\mathcal{A} \in \Gamma(TM \otimes N^*M \otimes T^*M)$ given by $\mathcal{A}_Z(X) = (D_XZ)^{\top}$. Then, $Z \cdot \alpha(X, Y) = -Y \cdot \mathcal{A}_Z(X)$.

Let PT_mM be the projective space of directions in T_mM. The second fundamental form defines a map $\eta_m : PT_mM \to N_mM$, which we call the *Veronese of curvature*, by

$$\eta_m\big([t]\big) = \eta_m(t) = \frac{\alpha_m(t,t)}{t \cdot t}, \quad t \in T_mM\backslash\{0\}.$$

3. The focal set

We will describe in this section and the next the relations between the Veronese of curvature and the focal set of the immersion M in $\mathbb{R}^{k+n}$.

A useful definition of the focal set of M goes as follows.

Definition 3.1. Let ν be the normal map of M. Then the *focal set of* M, denoted here by $\mathcal{F}(M)$, is defined as the set of critical points of ν. Since there will be little risk of confusion, the same name will be used for the image of $\mathcal{F}(M)$ by ν.

The next result is well known:

Proposition 3.2. *The focal set of M is given by*

$$\mathcal{F}(M) = \big\{(m,u) \in NM\colon \det(g_m - u \cdot \alpha_m) = 0,\ \textit{where}\ m = \pi_N(m,u)\big\},$$

where the determinant can be computed by means of any orthonormal basis of T_mM.

Proof. Let w_i, $i = 1, \dots, n$, be a local orthonormal frame of NM in a neighborhood U of $m \in M$ and let t_i, $i = 1, \dots, k$, be an orthonormal frame of TM in U. By means of the first of those frames we can work with a trivialization of NM on U given by $u \approx (\pi_N(m,u), x_1, \dots, x_n)$, where the x_i are such that $u = \sum_i x_i w_{im}$, being $m = \pi_N(m,u)$. Thus the map ν can be expressed as $\nu(m, x_1, \dots, x_n) = m + \sum_i x_i w_{im}$.

If $X \in T_mM$, then $d\nu(X,0) = X + \sum x_i dw_i(X) = X + \sum x_i D_X w_i$, and $d\nu(0, \partial_{x_i}) = w_i$. Since the w_i are orthonormal, the vanishing of $\det(d\nu)$ is equivalent to the vanishing of the determinant of the orthogonal projection of $d\nu|_{T_mM}$ into T_mM, that is to the vanishing of the determinant of the endomorphism of T_mM given by $X \to (d\nu(X,0))^{\top} = X + \sum x_i (D_X w_i)^{\top}$. The component in t_{jm} of $(d\nu(t_{im}, 0))^{\top}$ is $t_{im} \cdot t_{jm} + \sum_b x_b (D_{t_{im}} w_b) \cdot t_{jm} = g_m(t_{im}, t_{jm}) - u_m \cdot \alpha_m(t_{im}, t_{jm})$, where $u = \sum_b x_b w_{bm}$. That is, the condition is equivalent to the vanishing at m of the matrix with coefficients $(g - u \cdot \alpha)(t_i, t_j)$. $\square$

Proposition 3.3. *Let $\mathcal{F}_m(M) = \mathcal{F}(M) \cap N_mM$. Then the following properties are satisfied*:

1. *If $u : M \to NM$ be a local section in a neighborhood of $m \in M$, then* $\det(g_m - u_m \cdot \alpha_m) = \det(d(\nu \circ u)_m^{\top})$.
2. *If $u \in \mathcal{F}_m(M)$, then there exists $t \in T_mM\backslash\{0\}$, such that $g_m(t) = u \cdot \alpha_m(t) \in T_m^*M$. In this case, we say that u* belongs to the focal set over t. *In the following items, t and u satisfy that property.*
3. $u \cdot \eta_m(t) = 1$. *In particular, $u \neq 0$, $\eta_m(t) \neq 0$.*
4. $\eta_m(t) \notin (d\eta_m)(T_tT_mM) \subset T_{\eta_m(t)}N_mM$, *under the usual identification of N_mM with $T_{\eta_m(t)}N_mM$.*

Proof. 1) If, as before, we take a local frame w_i of NM, we can write $u = \sum_i u^i w_i$. Thus, $\nu \circ u = \mathrm{id} + \sum_i u^i w_i$, whence if $X \in T_mM$, we will have $d(\nu \circ u)_m^{\top}(X) = (X + \sum_i ((D_X u^i) w_{im} + u^i{}_m D_X w_i))^{\top} = X + A_{u_m}(X)$. Therefore,

$$\det d(\zeta \circ u)_m^{\top} = \det\big(d(\nu \circ u)(t_a) \cdot t_b\big) = \det\big(\delta_{ab} - u_m \cdot \alpha_m(t_a, t_b)\big).$$

2) We can look at $g_m - u \cdot \alpha_m$ as a linear map from T_mM to T_m^*M. Since it maps linearly a vector space into another of same dimension and has zero determinant, we conclude that there is some non-zero $t \in T_mM$ such that $(g_m - u \cdot \alpha_m)(t) = 0$.

3) If the 1-form $(g_m - u \cdot \alpha_m)(t)$ acts upon the vector t itself, we get $t \cdot t - u \cdot \alpha_m(t,t) = 0$, whence, by dividing by $t \cdot t \neq 0$, we obtain the claim.

4) Note that here η_m is taken as a map from $T_mM \backslash \{0\}$ to N_mM. For making the calculations easier we can assume that $t \cdot t = 1$. Then, if $X \in T_tT_mM$, we have

$$d\eta_m(X) = \frac{2}{(t \cdot t)^2}\big((t \cdot t)\alpha_m(t,X) - (t \cdot X)\alpha_m(t,t)\big) = 2\big(\alpha_m(t,X) - (t \cdot X)\eta_m(t)\big).$$

Suppose that this is equal to $\eta_m(t)$. By inner multiplication of this with u we get $u \cdot \eta_m(t) = t \cdot t = 1$, while the same multiplication with $d\eta_m(X)$ yields $2(u \cdot \alpha_m(t) - g_m(t))(X) = 0$, which is absurd. $\square$

In general, $\mathcal{F}(M)$ will be a hypersurface of NM, possibly with singularities, whose intersection with each fiber N_mM will be an algebraic hypersurface of degree k. Thus, in the case of a surface $M \subset \mathbb{R}^{2+n}$, the intersection $\mathcal{F}_m(M)$ of $\mathcal{F}(M)$ with N_mM will be a quadric.

4. Focal set and the inverted pedal of the Veronese of curvature

Definition 4.1. Let P be a smooth manifold and $\mu : P \to \mathbb{R}^n$ a smooth map. For each $p \in P$, let $\mathrm{ped}_\mu(p)$ be the nearest point to the origin among those of the affine subspace tangent to $\mu(P)$ at $\mu(p)$, i.e. $\{\mu(p) + d\mu(X)\colon\ X \in T_pP\}$. The resulting map $\mathrm{ped}_\mu : P \to \mathbb{R}^n$ is called the *pedal map of* μ. Let $\tilde{P}_\mu = \{p \in P\colon\ \mathrm{ped}_\mu(p) \neq 0\}$. If $R : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is the inversion with respect to the hypersphere with center 0 and unit radius, the composition $R \circ \mathrm{ped}_\mu : \tilde{P} \to \mathbb{R}^n \backslash \{0\}$ (and sometimes, also its image) will be called the *inverted pedal of* μ.

Let us show the relation between the focal set of and the inverted pedal of η at $m \in M$. We are here interested solely in the study of $\mathcal{F}_m(M)$. This justifies the use of the following simplified notation in this section:

$$T = T_mM \backslash \{0\}, \qquad N = N_mM, \qquad \alpha = \alpha_m, \qquad \eta = \eta_m, \qquad \mathcal{F} = \mathcal{F}_m(M), \qquad g = g_m.$$

Proposition 4.2. *Let $z \in N$ be a point in the inverted pedal of η. Then $z \in \mathcal{F}$.*

Proof. Note that here η is taken as a map from T to N. Let $t \in T$ and let $0 \neq z = \eta(t) + d\eta(X)$, with $X \in T_t(T)$, a point in the pedal of η so that $R(z)$ belongs to the inverted pedal of η. We must have $z \cdot d\eta(T_tT) = d(z \cdot \eta)(T_tT) = 0$. In particular, $z \cdot d\eta(X) = 0$, whence $z \cdot z = z \cdot \eta(t) \neq 0$. Also, $d(z \cdot \eta)_t = 0$. Hence t is a critical point of the map $t \mapsto z \cdot \eta(t)$. But one sees easily that this entails

$$g(t,t) z \cdot \alpha\big(t, t'\big) - \big(t \cdot t'\big) z \cdot \alpha(t,t) = 0$$

for any $t' \in T_tT$, i.e. $(z \cdot \alpha - z \cdot \eta(t) g)(t) = 0$, and this requires the vanishing of $\det(z \cdot \alpha - z \cdot \eta(t) g)$. By dividing that determinant by $(-z \cdot \eta(t))^k$, we conclude that

$$\det\left(g - \frac{z}{z \cdot \eta(t)} \cdot \alpha\right) = 0,$$

that is

$$\frac{z}{z \cdot \eta(t)} = \frac{z}{z \cdot z} = R(z) \in \mathcal{F}. \qquad \square$$

Let us see whether there is some form of converse of this. We put $B_t = R(\mathrm{ped}_\eta(t)) + \alpha(t,T)^\perp$. Thus, B_t is an affine subspace of N passing by $R(\mathrm{ped}_\eta(t))$. Note that $\alpha(t,T)$ is the vector space generated by $\eta(t)$ and the tangent space of the Veronese of curvature at $\eta(t)$. Let $\tilde{T} = \{t \in T\colon\ \mathrm{ped}_\eta(t) \neq 0\}$.

Theorem 4.3. *$\mathcal{F}$ is the union of the inverted pedal of η with $\bigcup_{t \in \tilde{T}} B_t$.*

Proof. Let $x \in N$ be a point in $\mathcal{F}$. Then $\det(g - x \cdot \alpha) = 0$. Let $t \in T$, $t \cdot t = 1$, be such that $g(t) = x \cdot \alpha(t)$. We know that then $x \cdot \eta(t) = 1$ and $\eta(t) \notin (d\eta)(T_tT)$. Let $z = \mathrm{ped}_\eta(t)$; if $z = 0$ we would have $\eta(t) \in (d\eta_p)(T_tT)$, which is absurd. As we have seen before we will have $(z \cdot z) g(t) = z \cdot \alpha(t)$, from which we obtain $g(t) = R(z) \cdot \alpha(t)$. Therefore $(R(z) - x) \cdot \alpha(t) = 0$, that is $R(z) - x \in B_t$. Hence we can write this in the form

$$x = R(z) + u, \quad u \in \alpha(t,T)^\perp. \qquad \square$$

This describes completely $\mathcal{F}(M)$. Note that $\dim \alpha(t,T) \leqslant k$. Hence, if for example $k = 2$ (M is thus a surface) and $n = 2$ then generically the dimension of $\mathcal{F}_m$ will be that of $\eta(PT_mM)$, that is one; thus, $\mathcal{F}_p$ will be a conic. If $n = 3$, it will be generically a ruled quadric surface.

5. *r*-Critical points of smooth maps

In this section we recall first the well-known notion of r-tangent bundle and define r-critical points. We shall use the letter π to denote the natural map $\pi : TM \to M$ for any smooth manifold M.

The r-tangent bundle $\pi^r : T^rM \to M$, for any non-negative integer r, is defined recursively as follows: the 0-tangent bundle is $\mathrm{id} : M \to M$, the 1-tangent bundle is the tangent bundle. Suppose that we have defined $\pi^r : T^rM \to M$ and also a bundle $\pi^r_{r-1} : T^rM \to T^{r-1}M$ such that $\pi^r = \pi^{r-1} \circ \pi^r_{r-1}$, where $\pi = \pi^1_0$. Then we define the next total space

$$T^{r+1}M = \{X \in T(T^rM):\ \pi(X) = d\pi^r_{r-1}(X)\}$$

and the maps $\pi^{r+1} = \pi^r \circ \pi$ and $\pi^{r+1}_r(X) = \pi(X) = d\pi^r_{r-1}(X)$. It is easy to see that T^rM is the bundle of r-jets of curves in M. That is, if $X \in T^rM$, there is a curve $\gamma : I \to M$, where I is an open neighborhood of $0 \in \mathbb{R}$, such that $\gamma(0) = m$, and X stores all of the information of $\gamma(0), \gamma'(0), \dots, \gamma^{(r)}(0)$. In fact, assume that $\gamma^{(r)}$ is a smooth map from I to T^rM and that $\pi^r_{r-1} \circ \gamma^{(r)} = \gamma^{(r-1)}$. Then, define $\gamma^{(r+1)} = \gamma^{(r)\prime} = d\gamma^{(r)} \circ \mathbf{1} : I \to TT^rM$, where $\mathbf{1} : \mathbb{R} \to T\mathbb{R}$ is the unit vector field. Therefore

$$\pi(\gamma^{(r+1)}) = (\pi \circ d\gamma^{(r)} \circ \mathbf{1}) = \gamma^{(r)}$$

and $d\pi^r_{r-1} \circ \gamma^{(r+1)} = d(\pi^r_{r-1} \circ \gamma^{(r)}) \circ \mathbf{1} = d\gamma^{(r-1)} \circ \mathbf{1} = \gamma^{(r-1)\prime} = \gamma^{(r)}$. Hence, $\gamma^{(r+1)}(0) \in T^{r+1}M$.

If $A \in T^rM$, we shall denote by T^{r+1}_AM the fiber of π^{r+1}_r over A. Let $VT^rM = \{X \in T(T^rM):\ d\pi^r_{r-1}(X) = 0\}$ be the *vertical bundle* over T^rM, which is a vector bundle whose fibre upon $X \in T^rM$ is denoted V_XT^rM. Let $A \in T^rM$ and $X, Y \in T^r_AM$. Then $d\pi^r_{r-1}(X - Y) = d\pi^r_{r-1}(X) - d\pi^r_{r-1}(Y) = \pi(X) - \pi(Y) = 0$. Hence, $X - Y \in V_AT^rM$. Therefore the fibre T^{r+1}_AM is an affine space. Its dimension is that of M. We define recursively the subset $O^rM \subset T^rM$ as follows. First, O^1M is the image of the zero section of $\pi : TM \to M$, i.e. $O^0_m = \pi(O^1_m)$. Assume that $r > 1$ and that $O^{r-1}M$ has been defined so that the intersection of $O^{r-1}M$ with the fibre T^{r-1}_mM, for $m \in M$, is exactly one point denoted O^{r-1}_mM and that $\pi^{r-1}_{r-2}(O^{r-1}_mM) = O^{r-2}_mM$. Then, O^r_mM is by definition the zero of the vector space $T_{O^{r-1}_mM}T^rM$. Note that $\pi(O^r_mM) = O^{r-1}_mM$ and that $d\pi^{r-1}_{r-2}(O^r_mM) = \{0 \in T_{O^{r-2}_mM}(T^{r-1}M)\} = O^{r-1}_mM$. Hence, $O^r_mM \in T^rM$ and $\pi^r_{r-1}(O^r_mM) = O^{r-1}_mM$.

Finally, if $f : M \to N$ is a smooth map between manifolds, it induces a smooth map $f^{(r)} : T^rM \to T^rN$ given by $f^{(r)}(\gamma^{(r)}(0)) = (f \circ \gamma)^{(r)}(0)$ for any smooth curve $\gamma : I \to M$, where $0 \in I$. If $g : N \to P$ is another smooth map, then $(g \circ f)^{(r)} = g^{(r)} \circ f^{(r)}$.

Definition 5.1. Let $f : H \to K$ be a smooth map between manifolds. We say that $X \in T^1H = TH$ is *a* 1*-critical point of* f if $X \neq 0$ and $f^{(1)}(X) \in O^1K$ that is if $df(X) = 0$. And if $A \in T^rH$ we say that it is an *r-critical point of* f if $\pi^r_{r-1}(A)$ is an $(r-1)$-critical point of f and $f^{(r)}(A) \in O^rK$.

6. 2-Critical points of ν for immersions in $\mathbb{R}^{k+n}$

Now, we will study the 2-critical points of the normal map $\nu : NM \to \mathbb{R}^{k+n}$. We use the following notations for the different bundles that we consider: $\pi : TM \to M$, $\pi_N : NM \to M$, $\pi_1 : T(NM) \to NM$, $\pi_T : T(T(NM)) \to T(NM)$, $\pi_2 : T^2(NM) \to T(NM)$.

Let $i^*(T\mathbb{R}^{k+n}) \to M$ be the induced bundle of the bundle $T\mathbb{R}^{k+n} \to \mathbb{R}^{k+n}$, by the inclusion $i : M \to \mathbb{R}^{k+n}$. The sections of this bundle are differentiable applications of the form $Z : M \to \mathbb{R}^{k+n}$. We can decompose each application in a unique way in two smooth summands $Z = Z^\top + Z^\perp$, where $Z^\top \in \mathfrak{X}(M)$ and $Z^\perp \in \Gamma(NM)$.

We define the operator $\nabla : \mathfrak{X}(M) \times \Gamma(i^*(T\mathbb{R}^{k+n})) \to \Gamma(i^*(T\mathbb{R}^{k+n}))$ putting $\nabla_X Z = (D_X Z^\top)^\top + (D_X Z^\perp)^\perp$, where $(D_X Z)_m = dZ(X_m)$ with the usual identification of $T_{Z_m}\mathbb{R}^{k+n}$ with $\mathbb{R}^{k+n}$. One can verify easily that ∇ is a linear connection in the vector bundle $i^*(T\mathbb{R}^{k+n}) \to M$, which preserves the dot product.

Also we can decompose the dual subbundle of $i^*(T\mathbb{R}^{k+n}) \to M$, in two subbundles, respectively isomorphic to the dual of the bundle TM, denoted T^*M, and to the dual of NM, denoted N^*M. This means, among other things, that we can look to tensor fields such as g or α as tensor fields over the vector bundle $i^*(T\mathbb{R}^{k+n})$. That is, if for instance $X, Y \in \Gamma(i^*(T\mathbb{R}^{k+n}))$ and $\beta \in \Gamma((i^*(T\mathbb{R}^{k+n}))^*)$ we will have

$$\beta(\alpha(X, Y)) := \beta(\alpha(X^\top, Y^\top)), \qquad g(X, Y) := g(X^\top, Y^\top).$$

Then, we extend ∇ to sections of those subbundles in the usual manner, and to tensor fields that are sections of the tensor product of copies of the bundles TM, T^*M, NM and N^*M, that can be seen as a section of the tensorial product of copies of $i^*(T\mathbb{R}^{k+n}) \to M$ by its dual. This extension of ∇ will preserve also the dot product. For instance, if $W \in \mathfrak{X}(M)$ we have

$$\begin{aligned} D_W(Z \cdot \alpha(X, Y)) &= W(Z \cdot \alpha(X, Y)) = \nabla_W(Z \cdot \alpha(X, Y)) = (\nabla_W Z) \cdot \alpha(X, Y) \\ &\quad + Z \cdot (\nabla_W \alpha)(X, Y) + Z \cdot \alpha(\nabla_W X, Y) + Z \cdot \alpha(X, \nabla_W Y). \end{aligned}$$

The connection ∇ is determined by a metric connection in the principal fiber bundle of adapted frames of $i^*(T\mathbb{R}^{k+n}) \to M$, that is frames as $(t_1, \dots, t_k, w_1, \dots, w_n)$ where $(t_1, \dots, t_k)$ is a frame of TM and $(w_1, \dots, w_n)$ is one of NM.

Definition 6.1. Let $m \in M$, $v \in T_mM$ and $U \in N_mM$. In the following we will say that *there is a* 2-*critical point of* ν *over* (v, U), *or that* (v, U) *admits a* 2-*critical point of* ν if there is some 2-critical point $X^2 \in T^2(NM)$ of ν such that $U = \pi_2(X^2)$ and $v = (\pi_1^2 \circ \pi_N^{(2)})(X^2)$. In other terms, if $X^2 = u''(0)$, being $u : I \to NM$ a smooth curve, then $v = (\pi_N \circ u)'(0)$ and $U = u(0)$. In the same manner we say that $v \in T_mM$ *admits a* 2-*critical point of* ν if there is $U \in N_mM$ such that (v, U) admits a 2-critical point of ν.

The following theorem characterizes the properties that must have a pair (v, U) as above in order to admit a 2-critical point of ν. In it we shall suppress the subindex m, that means evaluation at m, whenever this will not cause confusion, for instance when there appears any of the symbols U or v.

Theorem 6.2. *Let $m \in M$, $v \in T_mM$ and $U \in N_mM$. Then, (v, U) admits a 2-critical point iff the following conditions are satisfied*:

a) *$v \neq 0$ and $g(v, .) - U \cdot \alpha(v, .) = 0$, i.e. U belongs to the focal set over v.*
b) *$U \cdot (\nabla_v \alpha)(v, v) = 0$.*
c) *Let $(t_1, \ldots, t_k)$ be an orthonormal basis of T_mM such that t_1 and v are parallel. Then the following linear system, whose unknowns are the components of $x = x^2 t_2 + \cdots + x^k t_k$, has a solution*:

$$g(x, t_j) - U \cdot \alpha(x, t_j) = U \cdot (\nabla_v \alpha)(v, t_j), \quad j = 2, \ldots, k.$$

Proof. Suppose that $X^2 \in T^2(NM)$ can be written as $X^2 = \tilde{u}''(0)$, where $\tilde{u} : I \to NM$ is a smooth curve such that, if $\gamma := \pi_N \circ \tilde{u}$, we have $\tilde{u}(0) = U$ and $\gamma'(0) = v$. We have $\pi_2 \circ \tilde{u}'' = \tilde{u}'$ and $\pi_N \circ \tilde{u}' = \tilde{u}$. Let $X := \pi_2(X^2)$. For $t \in I$, $\tilde{u}(t) \in N_{\gamma(t)}N \subset \mathbb{R}^{k+n}$. Hence we may thing also of $\tilde{u}$ as a map from I to $\mathbb{R}^{k+n}$, and this justifies the use of the following notations:

1. A will denote $X^2 = \tilde{u}''(0)$ as element of $\mathbb{R}^{k+n}$.
2. V will denote $\pi_1^2(X^2) = \tilde{u}'(0)$ as element of $\mathbb{R}^{k+n}$.
3. a will denote $\pi_N^{(2)}(X^2) = \gamma''(0)$ as element of $\mathbb{R}^{k+n}$.

This is taken in the same sense as when one speaks of the acceleration of a moving particle as a vector in $\mathbb{R}^3$, without telling that it is the acceleration of the particle at the point m and velocity v.

Suppose that X^2 is 2-critical. Then X must be 1-critical, that is

$$d\nu(X) = d\nu\big(\tilde{u}'(0)\big) = (\nu \circ \tilde{u})'(0) = (\gamma + \tilde{u})'(0) = v + V = 0.$$

Since X cannot vanish we have $V \neq 0$, whence $v = \gamma'(0) \neq 0$. This means that me may suppose that $\gamma : I \to NM$ is an immersion and this implies that there is a smooth section u of NM in a neighborhood of m such that $\tilde{u} = u \circ \gamma$ in a neighborhood of 0 and $U = u_m$. We deduce that $\tilde{u}' = du \circ \gamma' = D_{\gamma'}u$, so that $V = \tilde{u}'(0) = D_v u$, and $A = \tilde{u}''(0) = D_v D_{\gamma'} u$. Moreover $\nu^{(2)}(X^2) = \nu^{(2)}(\tilde{u}''(0)) = (\gamma + u \circ \gamma)''(0) = 0$. Therefore, X is a 2-critical point of f iff the following conditions are satisfied:

1) $v \neq 0$ and $v + D_v u = 0$, or equivalently $v + V = 0$.
2) $a + D_v(D_{\gamma'}u) = 0$ or equivalently $a + A = 0$.

The first condition can be separated in two parts. The normal part says that $\nabla_v u = 0$ and the tangent part that $v + (D_v u)^\top = 0$. Multiplying the tangent part by a vector $y \in T_mM$, we obtain $g(v, y) - U \cdot \alpha(v, y) = 0$. In other words, we obtain the following conditions:

a) $v \neq 0$ and $g(v, \cdot) - U \cdot \alpha(v, \cdot) = 0$,
1.2) $\nabla_v u = V^\perp = 0$.

Now, we study condition 2). Denote by Y the parallel transport of y along γ. Then, we have $\nabla_{\gamma'} Y = (D_{\gamma'} Y)^\top = 0$, so that

$$Y \cdot D_v(D_{\gamma'}u) = D_v(Y \cdot D_{\gamma'}u) - (D_v Y) \cdot D_v u = -D_v\big(u \cdot \alpha\big(Y, \gamma'\big)\big) + (D_v Y) \cdot v,$$

by 1). Since Y is parallel, the second term vanishes. Having in mind that $\nabla_v u = 0$ by 1.2) an applying 2) we get

$$y \cdot a - U \cdot (\nabla_v \alpha)(y, v) - U \cdot \alpha(\nabla_v Y, v) - U \cdot \alpha(y, a) \tag{6.1}$$

$$= g(a, y) - U \cdot \alpha(a, y) - U \cdot (\nabla_v \alpha)(v, y) = 0, \tag{6.2}$$

i.e.

$$g\big(a^\top, \cdot\big) - u \cdot \alpha\big(a^\top, \cdot\big) = U \cdot (\nabla_v \alpha)(v, \cdot).$$

Let us put $a^\top = pv + b$, with $b \cdot v = 0$. Then 2) reads $g(b,.) - U \cdot \alpha(b,\cdot) = U \cdot (\nabla_v \alpha)(v,\cdot)$ by condition a). If this 1-form acts over v, and we consider condition a), we obtain:

$$\text{b)}\quad U \cdot (\nabla_v \alpha)(v,v) = 0. \tag{6.3}$$

And if that 1-form acts upon the vectors t_j, $j = 2,\dots,k$, we get condition c).

We have $a^\perp = (D_v \gamma')^\perp = \alpha(v,v)$. Hence the normal part of 2) is:

$$2.2)\quad \alpha(v,v) + A^\perp = 0.$$

Suppose that the system a), b), c) is satisfied by v, U and $x = x^2 t_2 + \cdots + x^k t_k$. It is enough to prove that we may choose γ and $\tilde{u}$ such that $\gamma'(0) = v$, $\gamma''(0)^\top = x$, $\tilde{u}(0) = U$, and that 1.2) and 2.2) are satisfied. The conditions on γ can always be satisfied because $v \in T_m M$ and the tangent part of $\gamma''(0)$ may be arbitrary. As for $\tilde{u}$, we take first a parallel orthonormal reference of NM along γ, $(u_1,\dots,u_n)$. Thus we can write $u(t) = \sum_i p^i(t) u_i(t)$ with the conditions $\sum_i p^i(0) u_i(0) = u$ and $p^{i'}(0) = 0$, $i = 1,\dots,n$. Then $u'(0) = \sum_i p^i(0) u_i'(0)$. Since the u_i are parallel, this implies that $u'(0)^\perp = 0$, which is condition 1.2).

Condition 2.2) can be written as $u''(0)^\perp = -\alpha_(v,v)$ or equivalently as $u_i(0) \cdot u''(0) = -u_i(0) \cdot \alpha(v,v)$, $i = 1,\dots,n$. Since we have

$$u''(0) = \sum_i \bigl(p^{i''}(0) u_i(0) + p^i(0) u_i''(0)\bigr),$$

this condition means that 2.2) is satisfied by choosing

$$p^{i''}(0) = -u_i(0) \cdot \alpha(v,v) - \sum_j p^j(0) u_i(0) \cdot u_j''(0).$$

Hence, our claim is true. □

6.1. 2-Critical points of surfaces in $\mathbb{R}^4$

In this section we study the 2-critical points of the normal map for surfaces in $\mathbb{R}^4$. Since it is obvious from the preceding theorem that there is a 2-critical point over (v,U) with $v \neq 0$ iff there is one over $(v/|v|,u)$ we may assume that $g(v,v) = 1$. We shall denote by $J : T_m M \to T_m M$ the rotation of 90°.

Proposition 6.3. *The pair (v,U), where $0 \neq v \in T_m M$ and $U \in N_m M$, admits a 2-critical point of ν iff the following conditions are satisfied*:

a) $U \cdot \alpha(v,v) = v \cdot v$, $U \cdot \alpha(v,Jv) = 0$;
b) $U \cdot (\nabla_v \alpha)(v,v) = 0$;
c) $U \cdot \alpha(Jv,Jv) \neq v \cdot v$ *or* $U \cdot (\nabla_v \alpha)(v,Jv) = 0$.

Proof. A vector $x \in T_m M$ that is orthogonal to v may be written as qJv, for some $q \in \mathbb{R}$. Also J is an isometry, whence $g(Jv,Jv) = v \cdot v$. Thus, the conditions of Theorem 6.2 for (v,U) are now

a′) $g(v,.) - U \cdot \alpha(v,.) = 0$;
b) $U \cdot (\nabla_v \alpha)(v,v) = 0$;
c′) $q(v \cdot v - U \cdot \alpha(Jv,Jv)) = U \cdot (\nabla_v \alpha)(v,Jv)$ has a solution for $q \in \mathbb{R}$.

Then, a) is equivalent to a′) because (v,Jv) is a basis of $T_m M$. Obviously c) is equivalent to c′). □

Remark 6.4. Note that conditions a) and b) are the same that characterize the strong principal directions defined by Montaldi in [2]. Note also that c′) is true generically, so that generically the question whether a direction v admits 2-critical points is answered by ascertaining that v satisfies the equation

$$\det\bigl(\alpha(v,Jv), (\nabla_v \alpha)(v,v)\bigr) = 0, \tag{6.4}$$

which leads to a polynomial equation of fifth degree. For its effective computation see [3].

Conditions a′) or a), mean that U is a focal point corresponding to v and this can be analyzed easily by using the description of the focal locus by means of the inverted pedal. Let us fix the names of some special points. A point m where the curvature ellipse consists of the origin only is called *planar*. If it reduces to a point it is called *umbilic*. If the ellipse lies in a line passing by the origin, it is called *of inflection*. If it is a segment, *semiumbilic*. Note that an inflection point is semiumbilic, an umbilic is of inflection, etc.

For simplicity, in the following we shall assume that v is a unit vector. If m is not semiumbilic, that is its curvature ellipse does not collapse to a segment, then for any v we have $\alpha(v, Jv) \neq 0$. In fact, that vector is the derivative of $\eta(v)$ (its velocity) with respect to t when $v(t) = t_1 \cos t + t_2 \sin t$, where (t_1, t_2) is an orthonormal basis of T_mM, and that velocity only vanishes at the cusp points of the ellipse when it degenerates to a segment. Then, a) and b) can be satisfied only when v is a solution of Eq. (6.4). Suppose that then the first alternative of c) does not hold, that is $v \cdot v - U \cdot \alpha(Jv, Jv) = 0$. Then we have $g(Jv, .) - U \cdot \alpha(Jv, .) = 0$, as it is easily verified. But this means that U is the focal point corresponding to v and to Jv, and this means that the tangent line to the ellipse at $\eta(v)$ and $\eta(Jv)$ are the same. Since $\eta(v)$ and $\eta(Jv)$ are opposite with respect to the center of the ellipse, we conclude that the ellipse is a segment against the hypothesis. The conclusion is that *if m is not semiumbilic, then v determines a strong principal direction iff it admits a 2-critical point, or equivalently iff it satisfies Eq.* (6.4).

If m is semiumbilic but not a point of inflection, then for all v such that $\eta(v)$ is not an extremal point of the ellipse we have $\alpha(v, Jv) \neq 0$, so that then there is a solution to a) and b) iff (6.4) is true. But then $\eta(v) = \eta(Jv)$ and $\alpha(v, Jv) = \alpha(Jv, J^2v)$; hence, the point U in $\mathcal{F}_m$ associated to v is the same as that associated to Jv. We conclude that the first alternative of c) is false. Thus, *v admits a 2-critical point iff, in addition to* (6.4)*, the second alternative of* c) *is true*. If $\eta(v)$ is an extremal point of the ellipse, then $\alpha(v, Jv) = 0$, and $\alpha(v, v) \neq \alpha(Jv, Jv)$ because m is not umbilic. Therefore (v, U) satisfy a) and b) iff $U \cdot (\nabla_v\alpha)(v, v) = 0$ and $U \cdot \alpha(v, v) = 1$. Suppose that the first alternative of c) is false. Then U is a focal point associated both to v and Jv. Therefore, as a consequence of Theorem 4.3, U is exactly the point of intersection of two lines, one being the line orthogonal to $\eta(v)$ passing by the inverse of $\eta(v)$, and the other is the same but relative to $\eta(Jv)$. If then the second alternative of c) does not hold, v does not admit a 2-critical point.

Assume now that m is an inflection non-umbilic point. Then if $\eta(v)$ is not an extreme point, the line tangent to the ellipse at $\eta(v)$ passes by the origin, so that v cannot have an associated focal point, and a fortiori a 2-critical point. If $\eta(v)$ is an extremal point and $\eta(v) = 0$ then it cannot be a strong principal direction nor admit a 2-critical point. If it is extremal but different from the origin, then, as in the case of semiumbilics, it satisfies a) and b) iff $U \cdot (\nabla_v\alpha)(v, v) = 0$ and $U \cdot \alpha(v, v) = 1$. Then, since $\alpha(Jv, Jv)$ is collinear with $\alpha(v, v)$, the first alternative of condition c) only fails if both vectors are equal, that is if the point is umbilic, against the hypothesis. Therefore, *if m is an inflection non-umbilic point, there are no strong principal directions over m, nor 2-critical points over v*.

Suppose that m is umbilic but not planar. Then $\alpha = c \otimes g$, with $0 \neq c \in N_mM$. Thus the vector U must lie in the line orthogonal to c that passes by the inverse of c and be orthogonal to $(\nabla_v\alpha)(v, v)$. The first alternative of c) will fail always, so that (v, U) will admit a 2-critical point if in addition U is orthogonal to $(\nabla_v\alpha)(v, Jv)$, and this requires that this vector and $(\nabla_v\alpha)(v, v)$ be parallel.

Finally, if m is planar, it does not admit strong principal directions nor 2-critical points.

7. 3-Critical points of ν for immersions in $\mathbb{R}^{k+n}$

Now, we will study the 3-critical points of the normal map. In addition to the notation of the preceding section, we will use $\pi_3 : T^3(NM) \to T^2(NM)$.

Let $X^3 \in T^3(NM)$ be a 3-critical point of ν. Then $X^2 = \pi_3(X^3)$ is a 2-critical point, so that we may use the same notation as before plus the following:

1. B will denote $X^3 = \tilde{u}'''(0)$ as element of $\mathbb{R}^{k+n}$.
2. b will denote $\pi_N^{(3)} X^3 = \gamma'''(0)$ as element of $\mathbb{R}^{k+n}$.

We will say that ν admits a 3-critical point over (v, a, U), where $v \in T_mM$, $a \in T_m^2M$ and $U \in N_mM$ if there is a 3-critical point of ν, $X^3 \in T^3(NM)$, such that $(\pi_1^3 \circ \pi_N^{(3)})(X^3) = v$, $(\pi_2^3 \circ \pi_N^{(3)})(X^3) = a$ and $\pi_3(X^3) = U$.

Theorem 7.1. *Let $m \in M$, $v \in T_mM$, $a \in T_m^2M$ and $U \in N_mM$. Let us choose any orthonormal basis of T_mM, $(t_1, \dots, t_k)$ such that t_1 and v are parallel. Then, ν admits a 3-critical point over (v, a, U) iff the following conditions are satisfied*:

a) $v \neq 0$ *and* $g(v, .) - U \cdot \alpha(v, .) = 0$, *i.e. U belongs to the focal set over v.*
b) $U \cdot (\nabla_v\alpha)(v, v) = 0$.
c) $g(x, a) - U \cdot \alpha(x, a) = U \cdot (\nabla_v\alpha)(v, a)$.
d) $2U \cdot (\nabla_v\alpha)(v, a) + U \cdot (\nabla^2\alpha)(v, v, v, v) + U \cdot (\nabla_{a^\top}\alpha)(v, v) = 0$.
e) *The following linear system, whose unknowns are the components of $y = y^2t_2 + \dots + y^kt_k$ has a solution*:

$$g(y, t_j) - U \cdot \alpha(y, t_j) = -\alpha(v, v) \cdot \alpha(v, t_j) + U \cdot \big(\nabla^2\alpha\big)(v, v, v, t_j) + U \cdot (\nabla_{a^\top}\alpha)(v, t_j) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)}v, t_j) + 2U \cdot (\nabla_v\alpha)\big(a^\top, t_j\big).$$

Proof. Note that c) is the same as Eq. (6.2). Thus, the condition that we must require in addition to a), b) and c) is

$$\nu^{(3)}\big(\tilde{u}'''(0)\big) = (\gamma + u \circ \gamma)'''(0) = b + D_vD_{\gamma'}D_{\gamma'}u = b + B = 0.$$

Let us compute the normal part of $b+B$. If $w \in N_mM$ and W is the parallel transport of w along γ, we will have

$$\begin{aligned} w \cdot b = w \cdot D_v\big(D_{\gamma'}\gamma'\big) &= D_v\big(W \cdot D_{\gamma'}\gamma'\big) - D_vW \cdot \big(D_{\gamma'}\gamma'\big)^{\top} \\ &= D_v\big(W \cdot \alpha\big(\gamma',\gamma'\big)\big) + w \cdot \alpha(v,a) = w \cdot (\nabla_v\alpha)(v,v) + 3w \cdot \alpha(v,a), \end{aligned}$$

that is we get the equation $B^{\perp} = -(\nabla_v\alpha)(v,v) - 3\alpha(v,a)$. Now we need to prove that we can set the values of the $p^{i'''}(0)$ in order to satisfy this equation (see the proof of Theorem 6.2), and this is trivial.

As in 6.2, let Y denote the parallel transport of $y \in T_mM$ along γ. Then

$$\begin{aligned} Y \cdot D_{\gamma'}D_{\gamma'}u &= Y \cdot \big(D_{\gamma'}(D_{\gamma'}u)^{\top} + D_{\gamma'}(D_{\gamma'}u)^{\perp}\big) \\ &= -(D_{\gamma'}u) \cdot \alpha\big(Y,\gamma'\big) + D_{\gamma'}(Y \cdot D_{\gamma'}u) \\ &= -(D_{\gamma'}u) \cdot \alpha\big(Y,\gamma'\big) - D_{\gamma'}\big(u \cdot \alpha\big(Y,\gamma'\big)\big) \\ &= -2(D_{\gamma'}u)^{\perp} \cdot \alpha\big(Y,\gamma'\big) - u \cdot (\nabla\alpha)\big(\gamma',Y,\gamma'\big) - u \cdot \alpha\big(Y,\gamma''\big), \end{aligned}$$

where, as usual, the tensor field $\nabla\alpha$ over M is defined as $(\nabla\alpha)(x,y,z) = (\nabla_{x^{\top}}\alpha)(y,z)$, for smooth maps $x,y,z : M \to \mathbb{R}^{k+n}$.

Also, we have $a + A = a + D_vD_{\gamma'}u = 0$. Therefore

$$(D_vY) \cdot D_vD_{\gamma'}u = -(D_vY) \cdot a = -(D_vY) \cdot a^{\perp} = -\alpha(v,v) \cdot \alpha(v,y).$$

We compute separately the following, having in mind that $a + A = 0$ and $v + V = 0$:

$$\begin{aligned} \nabla_v\big((D_{\gamma'}u)^{\perp}\big) \cdot \alpha(y,v) &= D_v\big((D_{\gamma'}u)^{\perp}\big) \cdot \alpha(y,v) \\ &= A \cdot \alpha(y,v) - \alpha(y,v) \cdot \alpha(v,v) = -a \cdot \alpha(y,v) + \alpha(y,v) \cdot \alpha(v,v) = 0. \end{aligned}$$

By substitution and recalling that $\nabla_vY = 0$ and $V^{\perp} = 0$ we get

$$\begin{aligned} y \cdot D_vD_{\gamma'}D_{\gamma'}u &= D_v(Y \cdot D_{\gamma'}D_{\gamma'}u) - (D_vY) \cdot D_vD_{\gamma'}u \\ &= \nabla_v\big(-2(D_{\gamma'}u)^{\perp} \cdot \alpha\big(Y,\gamma'\big) - u \cdot (\nabla\alpha)\big(\gamma',Y,\gamma'\big) - u \cdot \alpha\big(Y,\gamma''\big)\big) + \alpha(v,v) \cdot \alpha(v,y) \\ &= -U \cdot \big(\nabla^2\alpha\big)(v,v,v,y) - U \cdot (\nabla_{a^{\top}}\alpha)(y,v) - 2U \cdot (\nabla_v\alpha)(a,y) \\ &\quad - U \cdot \alpha\big(y,\nabla_v\big(\gamma''^{\top}\big)\big) + \alpha(v,v) \cdot \alpha(v,y) \\ &= -U \cdot \big(\nabla^2\alpha\big)(v,v,v,y) - U \cdot (\nabla_{a^{\top}}\alpha)(y,v) - 2U \cdot (\nabla_v\alpha)(a,y) \\ &\quad - U \cdot \alpha(y, b - \mathcal{A}_{\alpha(v,v)}v) + \alpha(v,v) \cdot \alpha(v,y). \end{aligned}$$

Hence, from $b + B = 0$ we obtain

$$\begin{aligned} g(b,.) - U \cdot \alpha(b,.) &= -\alpha(v,v) \cdot \alpha(v,.) + U \cdot \big(\nabla^2\alpha\big)(v,v,v,.) \\ &\quad + U \cdot (\nabla_{a^{\top}}\alpha)(v,.) + 2U \cdot (\nabla_v\alpha)(a,.) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)}v,.). \end{aligned}$$

If this 1-form acts upon v we have, taking account of condition a):

d) $2U \cdot (\nabla_v\alpha)(v,a) + U \cdot \big(\nabla^2\alpha\big)(v,v,v,v) + U \cdot (\nabla_{a^{\top}}\alpha)(v,v) = 0.$

Then we may find b such that $b + B = 0$ iff there is a vector $y = y^2t_2 + \cdots + y^kt_k$ such that:

$$\begin{aligned} g(y,t_j) - U \cdot \alpha(y,t_j) &= -\alpha(v,v) \cdot \alpha(v,t_j) + U \cdot \big(\nabla^2\alpha\big)(v,v,v,t_j) \\ &\quad + U \cdot (\nabla_{a^{\top}}\alpha)(v,t_j) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)}v,t_j) + 2U \cdot (\nabla_v\alpha)\big(a^{\top},t_j\big), \end{aligned}$$

which is condition e). □

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