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J. Differential Equations 245 (2008) 3628–3638

Journal of Differential Equations

www.elsevier.com/locate/jde

Superlinear problems without Ambrosetti and Rabinowitz growth condition

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Received 22 November 2007; revised 24 February 2008

Abstract

Superlinear elliptic boundary value problems without Ambrosetti and Rabinowitz growth condition are considered. Existence of nontrivial solution result is established by combining some arguments used by Struwe and Tarantello and Schechter and Zou (also by Wang and Wei). Firstly, by using the mountain pass theorem due to Ambrosetti and Rabinowitz is constructed a solution for almost every parameter λ by varying the parameter λ. Then, it is considered the continuation of the solutions.

MSC: 35A15; 35B38; 35J20; 35D05

Keywords: Variational methods; Critical points; Superlinear problems; Elliptic equations

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1. Introduction

Consider the following nonlinear eigenvalue Dirichlet problem

$$(P) \quad -\Delta u = \lambda f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

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[1] Supported in part by CNPq-Brazil and AGIMB—Millenium Institute MCT/Brazil.

[2] Supported in part by Fapemig-Brazil CEX 00030/07 and CNPq-Brazil Proc. 620025/2006-9.

doi:10.1016/j.jde.2008.02.035

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded smooth domain and $f(x,s)$ is a continuous function on $\overline{\Omega} \times \mathbb{R}$.

Ambrosetti and Rabinowitz in [1] established an existence of nontrivial solution result for problem (P), by assuming the following conditions:

(f_1) $$f(x,0) = 0, \qquad \lim_{s\to 0} \frac{f(x,s)}{s} = 0, \quad \text{uniformly in a.e. } x \in \Omega.$$

(f_2) There exist positive constants a and b such that

$$\left|f(x,s)\right| \leqslant a + b|s|^p, \quad 0 \leqslant p < \frac{N+2}{N-2}, \ \forall s \in \mathbb{R}, \ \forall x \in \Omega.$$

(f_3') There are constants $\theta > 2$ and $s_0 > 0$ such that

$$0 < \theta F(x,s) \leqslant sf(x,s), \quad |s| \geqslant s_0, \ \forall x \in \Omega,$$

where

$$F(x,s) = \int_0^s f(x,t)\,dt.$$

Then several researchers studied problem (P) trying to drop the above condition (f_3'), see for instance, [3,7,10]. Actually, condition (f_3') is quite natural and important not only to ensure that the Euler–Lagrange functional associated to problem (P) has a mountain pass geometry, but also to guarantee that Palais–Smale sequence of the Euler–Lagrange functional is bounded. But this condition is very restrictive eliminating many nonlinearities. We recall that (f_3') implies a weaker condition

$$F(x,s) \geqslant c|s|^\mu - d, \quad c,d > 0, \ x \in \Omega, \ s \in \mathbb{R} \text{ and } \mu > 2. \tag{1}$$

The above condition implies another much weaker condition, which is a consequence of the superlinearity of f:

(f_3) $$\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty, \quad \text{uniformly a.e. } x \in \Omega.$$

Costa and Magalhães in [2] studied problem (P) replacing condition (f_3'), among others conditions, by

$$\liminf_{s\to\infty} \frac{sf(x,s) - 2F(x,s)}{|s|^\mu} \geqslant k > 0, \quad \text{uniformly in a.e. } x \in \Omega, \tag{2}$$

where $\mu \geqslant \mu_0 > 0$. On the other hand, Willem and Zou in [9] assumed

$$H(x,s) \equiv sf(x,s) - 2F(x,s) \quad \text{is increasing in } s, \ \forall x \in \Omega,$$
$$sf(x,s) \geqslant 0, \quad \forall s \in \mathbb{R}, \qquad sf(x,s) \geqslant C_0|s|^\mu, \quad |s| \geqslant s_0 > 0, \ \forall x \in \Omega,$$

where $\mu > 2$ and $C_0 > 0$, instead of condition (f_3').

Recently Schechter and Zou in [5] were able to prove that under hypotheses (f_1)–(f_2) and

$$(a_3') \qquad \text{either} \quad \lim_{s\to-\infty}\frac{F(x,s)}{s^2}=+\infty \quad \text{or} \quad \lim_{s\to\infty}\frac{F(x,s)}{s^2}=+\infty$$

problem (P) has a nontrivial weak solution for almost every $\lambda > 0$. To the best of our knowledge the last assumption was originated in [5], which is weaker than (f_3), also this is a first result in this direction completely without assumption (f_3'). In the same paper [5], in order to get an existence of nontrivial solution result for all $\lambda > 0$, by substituting the condition (f_3'), Schechter and Zou assumed in addition to (a_3'), one of the conditions below

$$H(x,s) \quad \text{is convex in } s,\ \forall x \in \Omega,$$

or there are constants $C > 0$, $\mu > 2$ and $r \geqslant 0$, such that

$$\mu F(x,t) - tf(x,t) \leqslant C\bigl(1+t^2\bigr), \quad |t| \geqslant r.$$

We remark that the above second condition used in [5] is equivalent to (f_3'), which follows from

$$\liminf_{|s|\to\infty}\frac{sf(x,s)}{F(x,s)} \geqslant \theta > 2,$$

and the convexity on H is stronger than the following condition: there is $s_0 > 0$ such that

$$(f_4) \qquad \frac{f(x,s)}{s} \quad \text{is increasing in } s \geqslant s_0 \text{ and decreasing in } s \leqslant -s_0,\ \forall x \in \Omega.$$

Function $f(s) = s(2\ln s + 1)$ ($F(s) = s^2 \ln s$) satisfies condition (f_4) and it does not satisfy (f_3').

We will establish our main result, namely,

Theorem 1.1. *Under hypotheses* (f_1)–(f_4), *problem* (P) *has a nontrivial weak solution, for all* $\lambda > 0$.

Remark 1.1. In fact our result still holds if we consider a weaker condition than (f_4), namely,

$$(f_4') \qquad \text{there is } C_* > 0 \quad \text{such that} \quad H(x,t) \leqslant H(x,s) + C_*$$

for all $0 < t < s$ or $s < t < 0$, $\forall x \in \Omega$.

Indeed, the condition (f_4) is equivalent to the condition

$$H(x,s) \quad \text{is increasing in } s \geqslant s_0 \text{ and decreasing in } s \leqslant -s_0,\ \forall x \in \Omega.$$

Thus, it implies the condition (f_4'). Observe that function $H(x,s)$ is a “*quasi-monotonic*” function, and also if H is monotonic function in $s < 0$ and $s > 0$, or a convex function in $\mathbb{R}$, then it satisfies condition (f_4').

Remark 1.2. The proof is carried out by applying the mountain pass theorem due to Ambrosetti and Rabinowitz [1] getting a solution for almost all λ. In fact, this is true for all $\lambda > 0$. Whose proof is made by adapting some arguments used by Struwe and Tarantello [6] and Schechter and Zou [5] (also by Wang and Wei [8]). We recall that paper [6] treats a model that plays an important role in the Chern–Simons theory, while the paper [5] studies some class of superlinear problems by linking theory. Wang and Wei in [8] use arguments employed in [6] to obtain some existence result for an elliptic system modelling chemotaxis.

2. Preliminary results

Throughout this paper, we denote the norm of u in $H_0^1(\Omega)$ and $L^p(\Omega)$, $1 \leqslant p < +\infty$, respectively, by

$$\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2} \quad \text{and} \quad |u|_p = \left(\int_\Omega |u|^p\, dx\right)^{1/p}.$$

Our approach will be the variational techniques. Define the Euler–Lagrange functional associated to problem (P), given by

$$I_\lambda(u) = \frac{\|u\|^2}{2} - \lambda \int_\Omega F(x,u)\, dx, \quad u \in H_0^1(\Omega).$$

From the hypotheses on f, it is standard to check that I_λ is $C^1(H_0^1(\Omega), \mathbb{R})$ whose the Gateaux derivative is

$$I_\lambda(u)\cdot v = \int_\Omega \nabla u \cdot \nabla v\, dx - \lambda \int_\Omega f(x,u) v\, dx, \quad u, v \in H_0^1(\Omega).$$

Thus, the critical points of I_λ are precisely the weak solutions of problem (P).

First of all, notice that I_λ verifies the mountain pass geometry, in a uniform way on compact sets:

Lemma 2.1.

(a) *I_λ is unbounded from below.*
(b) *$u = 0$ is a strict local minimum for I_λ.*

Proof of (a). From (f_3) follows that, for all $M > 0$ there exists $C_M > 0$, such that

$$F(x,s) \geqslant Ms^2 - C_M, \quad \forall x \in \Omega,\ \forall s > 0 \tag{3}$$

(we are supposing f is superlinear on $+\infty$).

Take $\phi \in H_0^1(\Omega)$ with $\phi > 0$, from (3) we obtain

$$I_\lambda(t\phi) \leqslant t^2\left(\frac{\|\phi\|^2}{2} - \lambda M\right) \int_\Omega \phi^2\, dx + C|\Omega|,$$

where $C > 0$ is a constant and $|\Omega|$ denotes the Lebesgue measure of Ω.

Thus,

$$\lim_{t\to\infty} I_\lambda(t\phi) = -\infty.$$

This proves (a). □

Proof of (b). Firstly, from (f_1) and (f_2) it follows that, for all given $\epsilon > 0$ there exists $C_\epsilon > 0$, such that

$$F(x,s) \leqslant \frac{\epsilon}{2} s^2 + C_\epsilon |s|^{p+1}, \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}. \tag{4}$$

Then

$$I_\lambda(t\phi) \geqslant \frac{\|u\|^2}{2} - \frac{\lambda\epsilon}{2} \int\limits_\Omega u^2\,dx - C_\epsilon |u|^{p+1}, \quad \forall u \in H_0^1(\Omega) \tag{5}$$

$$\geqslant \left(\frac{1}{2} - \frac{\lambda\epsilon}{2\lambda_1}\right)\|u\|^2 - C_\epsilon |u|^{p+1}, \quad \forall u \in H_0^1(\Omega), \tag{6}$$

where λ_1 is the first eigenvalue of the Dirichlet problem for $(-\Delta, H_0^1(\Omega))$.

So, $u = 0$ is local minimum of I_λ. □

Fix $0 < \lambda_0 < \mu_0$. Now, we can see that geometry on I_λ works uniformly on $[\lambda_0, \mu_0]$. By choosing $\epsilon > 0$ such that $\frac{1}{2} - \frac{\mu_0\epsilon}{2\lambda_1} \geqslant \frac{1}{4}$, we obtain

$$I_\lambda(u) \geqslant \frac{1}{4}\|u\|^2 - C_\epsilon \|u\|^{p+1}, \quad \forall u \in H_0^1(\Omega),\ 0 < \lambda \leqslant \mu_0,\ C_\epsilon > 0.$$

That is, there exist $\rho > 0$ and $R > 0$, such that

$$I_\lambda(u) \geqslant R, \qquad \|u\| = \rho, \quad \forall \lambda \leqslant \mu_0. \tag{7}$$

By choosing $e \in H_0^1(\Omega)$ such that $I_{\lambda_0}(e) < 0$, we infer that

$$\frac{I_\lambda(e)}{\lambda} < \frac{I_{\lambda_0}(e)}{\lambda_0} < 0, \quad \lambda_0 \leqslant \lambda \leqslant \mu_0.$$

Also we have

$$\frac{I_\lambda(u)}{\lambda} \leqslant \frac{I_\mu(u)}{\mu}, \quad \forall u \in H_0^1(\Omega),\ \mu < \lambda. \tag{8}$$

Define

$$P = \bigl\{\gamma : [0,1] \to H_0^1(\Omega)\colon \gamma \text{ is continuous and } \gamma(0) = 0 \text{ and } \gamma(1) = e\bigr\},$$

and for $\lambda_0 \leqslant \lambda \leqslant \mu_0$, let

$$c_\lambda = \inf_{\gamma \in P} \max_{t \in [0,1]} I_\lambda\big(\gamma(t)\big).$$

We recall that the map $c : [\lambda_0, \mu_0] \to \mathbb{R}_+$, given by $c(\lambda) = c_\lambda$, is such that c_λ/λ is decreasing, left semi-continuous and bounded from below by $c_{\mu_0} > 0$.

In fact, from (8) follows the monotonicity. While the estimate (7) implies that $c_\lambda \geqslant R > 0$.

Now, we are going to check the left semi-continuity of c_λ/λ. Fix $\mu \in [\lambda_0, \mu_0]$ and $\epsilon > 0$. Then fix $\gamma \in P$ such that

$$c(\mu) \leqslant \max_{t \in [0,1]} I_\mu\big(\gamma(t)\big) \leqslant c(\mu) + \frac{\epsilon\mu}{4}.$$

Let $R_0 = \max_{t \in [0,1]} \int_\Omega F(x, \gamma(t))\, dx$. Then, for $\lambda > \mu/2$ and such that $1/\lambda < 1/\mu + \epsilon/2\mu$

$$\begin{aligned} I_\lambda\big(\gamma(t)\big) &= \big(I_\lambda\big(\gamma(t)\big) - I_\mu\big(\gamma(t)\big)\big) + I_\mu\big(\gamma(t)\big) \\ &\leqslant I_\mu\big(\gamma(t)\big) + (\mu - \lambda) \int_\Omega F\big(x, \gamma(t)\big)\, dx \\ &\leqslant R_0|\lambda - \mu| + c_\mu + \frac{\epsilon\mu}{4}, \quad \forall t \in [0,1], \end{aligned}$$

that is,

$$c_\lambda \leqslant c_\mu + \frac{\epsilon\mu}{2}, \quad \text{if } |\mu - \lambda| < \frac{\epsilon\mu}{4R_0}.$$

Hence, if $\mu > \lambda$, it follows that

$$\frac{c_\mu}{\mu} - \epsilon < \frac{c_\mu}{\mu} \leqslant \frac{c_\lambda}{\lambda} \leqslant \frac{c_\mu}{\lambda} + \frac{\epsilon}{2} \leqslant \frac{c_\mu}{\mu} + \epsilon.$$

This proves the left semi-continuity of c_λ/λ and c_λ. □

The next lemma estimates the dependence of the parameter λ of the derivative I'_λ in the $H^{-1,2}(\Omega)$-norm $\|\,.\,\|_*$.

Lemma 2.2. *There exists $C > 0$, such that*

$$\big\|I'_\mu(u) - I'_\lambda(u)\big\|_* \leqslant C\big(1 + \|u\|\big)|\mu - \lambda|, \quad \forall \lambda, \mu > 0.$$

Proof. Notice that (f_2) implies an inequality like

$$\big|f(x,u)\big|^{2N/(N+2)} \leqslant C_1 + C_2|s|^{2Np/(N+2)}, \quad \forall x \in \Omega,\ \forall s \in \mathbb{R},$$

for some constants $C_1, C_2 > 0$, and then

$$\int_\Omega \big|f(x,u)\big|^{2N/(N+2)}\, dx \leqslant C_1|\Omega| + C_2\bigg(\int_\Omega |u|^{2Np/(N-2)}\, dx\bigg).$$

Therefore, there exist positive constants d_i $(i = 1, 2, 3)$, such that

$$\int_\Omega \left|f(x,u)\right|^{2N/(N-2)}\,dx \leqslant d_1 + d_2\|u\|^{2N/(N+2)}, \quad \text{for all } u \in H_0^1(\Omega).$$

Now, for all $v \in H_0^1(\Omega)$ with $\|v\| \leqslant 1$, we have

$$I'_\mu(u)v - I'_\lambda(u)v = (\lambda - \mu)\int_\Omega f(x,u)v\,dx.$$

Therefore

$$\left|I'_\mu(u).v - I'_\lambda(u).v\right| \leqslant |\lambda - \mu|\left[\int_\Omega |f|^{2N/(N+2)}\,dx\right]^{(N+2)/2N}\left[\int_\Omega |v|^{2N/(N-2)}\,dx\right]^{(N-2)/2N}.$$

So that, there exists $C > 0$ such that

$$\left\|I'_\mu(u) - I'_\lambda(u)\right\|_* \leqslant C\left(1 + \|u\|\right)|\mu - \lambda|, \quad \forall\lambda, \mu > 0. \qquad \square$$

Remark 2.1. We recall that the map $b : [\lambda_0, \mu_0] \to \mathbb{R}_+$, given by $b(\lambda) = c_\lambda/\lambda$, is monotone decreasing. Thus, b_λ and c_λ are differentiable at almost all values $\lambda \in (\lambda_0, \mu_0)$.

The proof of the next lemma is done by adapting some arguments employed in the proof of Lemma 3.3 in [6] and Lemma 2.5 in [8].

Lemma 2.3. *Suppose the map* $c : [\lambda_0, \mu_0] \to \mathbb{R}_+$, *given by* $c(\lambda) = c_\lambda$, *is differentiable in* μ, *then there exists a sequence* $(u_n) \in H_0^1(\Omega)$ *such that*

$$I_\mu(u_n) \to c_\mu, \quad I'_\mu(u_n) \to 0, \quad \text{and} \quad \|u_n\|^2 \leqslant C_1,$$

as $n \to \infty$ *and actually* $C_1 = 2c_\mu + 2\mu(2 - c'(\mu)) + 1$.

Proof. Assume, by contradiction, that the lemma were false. Then, for that C_1 there exists $\delta > 0$ such that

$$\left\|I'_\mu(u)\right\| \geqslant 2\delta, \quad \text{for all } u \in N_\delta = \left\{v \in H_0^1(\Omega)\colon\ \|v\|^2 \leqslant C_1,\ \left|I_\mu(v) - c_\mu\right| \leqslant \delta\right\}.$$

Let C_2 be such that

$$\left|\int_\Omega F(x,u)\,dx\right| = \frac{1}{2\mu}\left|2I_\mu(u) - \|u\|^2\right| \leqslant C_2, \quad \forall u \in N_\delta. \tag{9}$$

Let $V : N_\delta \to H_0^1(\Omega)$ be the pseudo-gradient vector field for I_μ in N_δ, that is, V is locally Lipschitz, $\|V\| \leqslant 1$ and

$$I'_\mu(u).\left(V(u)\right) \leqslant -\delta, \quad \text{for all } u \in N_\delta \quad \text{(see [4])}.$$

Now, fix $\{\lambda_n\}$ a sequence in (λ_0, μ_0) such that $\mu < \lambda_{n+1} < \lambda_n$, converging to μ, $|\lambda_n - \mu| \leqslant \delta/4$ and $|c_\mu - c_{\lambda_n}| \leqslant \delta/4$. For each n, let $\gamma_n \in P$ be such that

$$\max_{t\in[0,1]} I_\mu\bigl(\gamma_n(t)\bigr) \leqslant c_\mu + (\lambda_n - \mu). \tag{10}$$

Consider the open set

$$A_n = \bigl\{t \in [0,1]:\ I_{\lambda_n}\bigl(\gamma_n(t)\bigr) > c_{\lambda_n} - (\lambda_n - \mu)\bigr\}.$$

By definition of c_{λ_n}, A_n is nonempty set. If $v \in \gamma_n(A_n)$, from (10) we have

$$\int\limits_\Omega F(x,v)\,dx = \frac{I_\mu(v) - I_{\lambda_n}(v)}{\lambda_n - \mu} \leqslant \frac{c_\mu - c_{\lambda_n}}{\lambda_n - \mu} + 2 = -c'(\mu) + 2 + o_n(1),$$

where we have used $c_\mu - c_{\lambda_n} = (c'(\mu) + o_n(1))(\mu - \lambda_n)$.

Since $\|v\|^2 = 2I_\mu(v) + 2\mu\int_\Omega F(x,v)\,dx$, we obtain for $v \in \gamma_n(A_n)$

$$\|v\|^2 \leqslant 2c_\mu + 2(\lambda_n - \mu) + 2\mu\bigl(2 - c'(\mu) + o_n(1)\bigr) \leqslant C_1$$

for n large.

It is easy to see that inequality (9) is satisfied for $v \in \gamma_n(A_n)$. Thus $\gamma_n(A_n) \subset N_\delta$, because

$$c_{\lambda_n} - (\lambda_n - \mu) \leqslant I_{\lambda_n}(v), \qquad I_\mu(v) \leqslant c_\mu + (\lambda_n - \mu),$$
$$\bigl|I_{\lambda_n}(v) - I_\mu(v)\bigr| = (\lambda_n - \mu)\left|\int\limits_\Omega F(x,v)\,dx\right| \leqslant C_2|\lambda_n - \mu| \tag{11}$$

and for n large,

$$c_\mu - \delta < I_\mu(v) < c_\mu + \delta, \quad \forall v \in \gamma_n(A_n).$$

From Lemma 2.4, we have $I'_{\lambda_n}(u).(V(u)) < -\delta/2$, for all $u \in N_\delta$. Now consider a Lipschitz continuous cut-off function η such that $0 \leqslant \eta \leqslant 1$, $\eta(u) = 0$ in $u \notin N_\delta$, and $\eta(u) = 1$ for $u \in N_{\delta/2}$. Let ϕ be the flow generated by ηV, that is,

$$\frac{\partial\phi}{\partial r}(u,r) = \eta\bigl(\phi(u,r)\bigr)V\bigl(\phi(u,r)\bigr), \quad \text{for all } r, \quad \text{and} \quad \phi(u,0) = u.$$

From the ODE uniqueness result we have

$$\text{if } u \notin N_\delta \quad \text{then } \phi(u,r) = u, \quad \forall r \geqslant 0,$$
$$\text{if } u \in N_\delta \quad \text{then } \phi(u,r) \in N_\delta, \quad \forall r \geqslant 0,$$
$$\text{if } u \in H_0^1(\Omega) \quad \text{then } I'_{\lambda_n}\bigl(\phi(u,r)\bigr)\left(\frac{\partial\phi}{\partial r}(u,r)\right) \leqslant 0, \quad \forall r \geqslant 0,$$
$$\text{if } \phi(u,r) \in N_{\delta/2}, \quad \forall r \in [0,r_0] \quad \text{then } I_{\lambda_n}\bigl(\phi(u,r)\bigr) \leqslant I_{\lambda_n}(u) - \frac{\delta r_0}{2}.$$

It is easy to see that if $u \in N_{\delta/2}$ then $I_{\lambda_n}(\phi(u,1)) \leqslant I_{\lambda_n}(u) - \delta/2$.

Since $e \notin N_\delta$, we have $\phi(e,r) = e$ and $\phi(0,r) = 0$, for all $r \geqslant 0$, and then $\phi(\gamma, r) \in P$, for all r real and $\gamma \in P$.

This implies that $h_n(t) = \phi(\gamma_n(t), 1)$ is a continuous path in P such that

$$I_{\lambda_n}\bigl(h_n(t)\bigr) \leqslant I_{\lambda_n}\bigl(\gamma_n(t)\bigr), \quad \forall t,$$

and then for its maximum point $s_n \in [0,1]$, we should have $s_n \in A_n$, and

$$c_\mu - o_n(1) = c_{\lambda_n} \leqslant \max_{t\in[0,1]} I_{\lambda_n}\bigl(h_n(t)\bigr) = I_{\lambda_n}\bigl(h_n(s_n)\bigr) \leqslant I_{\lambda_n}\bigl(\gamma_n(s_n)\bigr) - \delta/2.$$

On the other hand, from (10) and (11) we have

$$I_{\lambda_n}\bigl(\gamma_n(s_n)\bigr) \leqslant I_\mu\bigl(\gamma_n(s_n)\bigr) + C_2|\lambda_n - \mu| \leqslant c_\mu + (1 + C_2)|\lambda_n - \mu|,$$

which is a contradiction. □

The next lemma follows directly from Lemma 2.3.

Lemma 2.4. *For almost all $\lambda > 0$, c_λ is a critical value for I_λ.*

3. Proof of main theorem

As c_λ is left semi-continuous, from Lemma 2.4, for each $\mu > 0$ we can fix sequences $\{u_n\}$ in $H_0^1(\Omega)$ and $\{\lambda_n\} \in \mathbb{R}$ such that $\lambda_n \to \mu$, $c_{\lambda_n} \to c_\mu$, as $n \to \infty$,

$$I_{\lambda_n}(u_n) = c_{\lambda_n} \quad \text{and} \quad I'_{\lambda_n}(u_n) = 0.$$

We claim that u_n is bounded. If it is not bounded we define $\omega_n = u_n/\|u_n\|$. As in [5] we will show that ω_n converges to 0 in $L^{p+1}(\Omega)$, $n \to \infty$. Without loss of generality we suppose that there are $\omega \in H_0^1(\Omega)$ and $h \in L^{p+1}(\Omega)$ such that

$$\begin{aligned} \omega_n(x) &\to \omega(x), \quad \text{a.e. in } \Omega,\ n \to \infty,\\ \bigl|\omega_n(x)\bigr| &\leqslant h(x), \quad \text{a.e. in } \Omega,\ \text{for all } n,\\ \omega_n &\to \omega, \quad \text{in } L^{p+1}(\Omega),\ n \to \infty,\\ \omega_n &\to \omega, \quad \text{in } L^2(\Omega),\ n \to \infty. \end{aligned}$$

Let $\Omega_{\neq} = \{x \in \Omega\colon\ \omega(x) \neq 0\}$. If $x \in \Omega_{\neq}$ then

$$\lim_n \frac{F(x, u_n(x))}{u_n(x)^2}\omega_n(x)^2 = \infty.$$

Applying the Fatou Lemma and the limit

$$\lim_n \int_\Omega \frac{F(x,u_n)}{u_n^2}\omega_n^2 = \frac{1}{2\mu},$$

we conclude that $\Omega_{\neq}$ has zero measure and $\omega = 0$ a.e. in Ω.

Let $t_n \in [0,1]$ such that

$$I_{\lambda_n}(t_n u_n) = \max_{t\in[0,1]} I_{\lambda_n}(t u_n).$$

Since $I'_{\lambda_n}(t_n u_n)(t_n u_n) = 0$, from (f'_4), we have

$$\begin{aligned}
2I_{\lambda_n}(t u_n) &\leqslant 2I_{\lambda_n}(t_n u_n) - I'_{\lambda_n}(t_n u_n)(t_n u_n) \\
&= \lambda_n \int\limits_{\Omega} \bigl[t_n u_n f(x, t_n u_n) - 2F(x, t_n u_n)\bigr]\,dx \\
&\leqslant \lambda_n \int\limits_{\Omega} \bigl[u_n f(x, u_n) - 2F(x, u_n) + C_*\bigr]\,dx = \lambda_n C_* |\Omega| + 2c_{\lambda_n}
\end{aligned}$$

for all $t \in [0,1]$.

On the other hand, for all $R_0 > 0$,

$$2I_{\lambda_n}(R_0\omega_n) = R_0^2 - 2\lambda_n \int\limits_{\Omega} F(x, R_0\omega_n)\,dx = R_0^2 + o_n(1),$$

which contradicts $2I_{\lambda_n}(R_0\omega_n) \leqslant \lambda_n C_* |\Omega| + 2c_{\lambda_n}$, for n large.

Now we have a bounded sequence $\{u_n\}$ such that

$$I_\mu(u_n) \to c_\mu \quad \text{and} \quad I'_\mu(u_n) \to 0, \quad \text{as } n \to \infty.$$

The proof is done.

Acknowledgment

The authors would like to thank the anonymous referee for valuable suggestions.

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