LÁZARO SOUZA LIMA

ANALYSIS OF QUANTUM ELECTRODYNAMICS MODELS WITH APPLICATIONS TO TWO-DIMENSIONAL CONDENSED MATTER PHENOMENA

Thesis submitted to the Physics Graduate Program of the Universidade Federal de Viçosa in partial fulfillment of the requirements for the degree of *Doctor Scientiae*.

Adviser: Oswaldo M. Del Cima
Co-adviser: Daniel H. T. Franco

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Oswaldo Monteiro Del Cima
(Adviser)

To my grandmother, Ercília.

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To God, the supreme intelligence and primary cause of everything, for His providence, love, mercy, and understanding. At many moments, I felt His care for me.

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Discovering the true meaning of things is wanting to know too much...

(O Teatro Mágico)

ABSTRACT

LIMA, Lázaro Souza. D.Sc., Universidade Federal de Viçosa, November, 2023. **Analysis of quantum electrodynamics models with applications to two-dimensional condensed matter phenomena**. Adviser: Oswaldo Monteiro Del Cima. Co-adviser: Daniel Heber Theodoro Franco.

In this work, two planar quantum electrodynamics (QED_3) models with applications to condensed matter phenomena are analyzed. The first one, discussed in Chapter 2, has possible applications to the fractional quantum Hall effect. A fundamental analysis is performed in order to verify the physical consistency of the model. It is found that unitarity is jeopardized, and it is checked that the bosons of the model are non-fundamental. A diagonalization procedure of the action is performed, aiming to identify the fundamental ones and it is verified that the action written in terms of them is physically consistent. The other model analyzed is a parity-preserving QED_3, which yields theoretical results that mimic some experimental observations performed in pristine graphene. In chapter 3, it is verified that the BPHZL renormalization procedure preserves parity in this model. In chapter 4, this model is renormalized at 2-loops order, by BPHZL. In chapter 5, the algebraic renormalization scheme allows to verify that the parity preserving QED_3 is free from anomalies and it presents a quantum scale invariance, which is another feature observed in pristine graphene.

Keywords: Quantum Electrodynamics; Three Space-Time Dimensions; Algebraic Renormalization; Condensed Matter.

RESUMO

LIMA, Lázaro Souza. D.Sc., Universidade Federal de Viçosa, novembro de 2023. **Análise de modelos de eletrodinâmica quântica com aplicações a fenômenos bidimensionais da matéria condensada.**. Orientador: Oswaldo Monteiro Del Cima. Coorientador: Daniel Heber Theodoro Franco.

Neste trabalho, são analisados dois modelos de eletrodinâmica quântica (QED_3) planar com aplicações a fenômenos de matéria condensada. O primeiro, discutido no Capítulo 2, possui possíveis aplicações no efeito Hall quântico fracionário. Realiza-se uma análise fundamental para verificar a consistência física do modelo. Um comprometimento na unitariedade é identificado, e verifica-se que os bósons do modelo não são fundamentais. Um procedimento de diagonalização da ação é realizado com o objetivo de identificar os campos fundamentais, e verifica-se que a ação expressa em termos deles é fisicamente consistente. O outro modelo analisado é uma QED_3 com paridade preservada, a qual gera resultados teóricos que imitam algumas observações experimentais realizadas no grafeno puro. No capítulo 3, verifica-se que o procedimento de renormalização BPHZL preserva a paridade nesse modelo. No capítulo 4, esse modelo é renormalizado para uma ordem de 2 loops, por meio do BPHZL. No capítulo 5, o esquema de renormalização algébrica permite verificar que a QED_3 com paridade preservada está livre de anomalias e apresenta invariância quântica de escala, outra característica observada no grafeno puro.

Palavras-chave: Eletrodinâmica Quântica; Três Dimensões Espaço-Temporais; Renormalização Algébrica; Matéria Condensada.

CONTENTS

Chapter 1

Introduction

In physics, mathematics is used to describe natural phenomena. Sometimes, physicists may appropriate well-defined mathematical concepts to apply in physics; at other times, they instigate the development of them. This approach is valid as long as these concepts are treated as useful idealizations. However, often we extend their mathematical scope and attempt to attribute them a physical reality. This occurs, for instance, with the mathematical concept of dimension. The spacetime with 1+3 dimensions, meaning one temporal and three spatial dimensions, with Poincaré symmetry, is convenient for describing a wide variety of physical phenomena. Nonetheless, this does not necessarily imply that the universe inherently possesses four spacetime dimensions, even though we may be inclined to assert it. One might be tempted to reject, a priori, a model that proposes a different number of spacetime dimensions, even if it yields interesting results. This rejection could be based on the argument that it fails to depict reality and should be treated as an intriguing yet impractical fantasy. Nevertheless, there exists a significant gap between the description and the de facto reality. Therefore, a model should not be dismissed solely because it seems imaginative and departs from conventional models. Instead, the model's viability should be assessed: whether it accurately describes the intended phenomena, maintains physical consistency, and enables us to make predictions.

In what concerns the dimensions of the spacetime, some physicists had dared to propose models with spacetimes of dimensions different from the usual four. Theodor Kaluza in 1921 [1] and Oskar Klein in 1926 [2] ventured into this territory. Kaluza introduced a five-dimensional spacetime, with one dimension being temporal and the other four spatial, with the aim of unifying gravitation and electromagnetism. He successfully derived both Einstein's equations and Maxwell's equations from his proposed model, as long as an extra dimension for spacetime was accepted. Despite successfully merging gravity and electromagnetism, Kaluza's model faced two major issues: the extra coordinate was suppressed without a strong justification, and there was no explanation for the absence of observation of the extra dimension. To address these issues, Oskar Klein proposed a different approach. He suggested a five-dimensional spacetime with a circular extra dimension, making the extra coordinate periodic, compactified into small circles. This solution resolved some of the problems present in Kaluza's model, but inconsistencies remained. Despite their problems, both models played a significant role and are considered precursors to the string theory model.

Theoretical physicists were not restricted to proposing spacetimes with higher dimensions than the conventional; they also proposed those with lower dimensions. In the 1980s, some celebrated physicists, such as Alexander Polyakov [3], Fidel Schaposnik [4], and the

Nobel Prize winner Gerard 't Hooft [5], published works on quark confinement. They analyzed this highly relevant phenomenon in particle physics and physics in general, using models with two spatial dimensions. In this context, the quark confinement issue could be related to a phenomenon of lower dimensions.

Within the context of gauge theories, Jonathan Schonfeld [6] analyzed the possibility of mass generation for gauge fields in 1+2 spacetimes through a topological term added to the Lagrangian, without breaking the gauge symmetry. This marked the end of the paradigm that gauge invariance must imply massless gauge fields. Additionally, in the framework of 1+2 spacetimes, in 1982, Stanley Deser, Roman Jackiw, and Stephen Templeton [7] also analyzed massive gauge fields without breaking the gauge symmetry.

Spacetimes of lower dimensions have also been utilized in models that describe phenomena in condensed matter physics. For instance, in 1977, Yutaka Hosotani demonstrated that the equations resulting from planar quantum electrodynamics (QED_3) were identical to those governing the Josephson effect in superconducting systems [8]. Another example is the reference [9], in which Petter Minnhagen, in 1980, analyzed a model of a Coulomb gas in two spatial dimensions, considering a logarithmic potential. In this context, the considered gas was truly a planar system [1].

Still considering the subject of condensed matter, the fractional quantum Hall effect appears to be associated with a low-dimensional phenomenon. In 1983, Robert Laughlin, who later became a Nobel Prize laureate, proposed a model to describe the condensation of a two-dimensional electron gas in a high magnetic field [10]. This model aimed to provide a theoretical explanation for the quantum Hall effect. Laughlin's model successfully described the filling factors given by $\nu = \dfrac{1}{m}$, where m is an odd integer. In the same year, two other Nobel Prize winners, Philip Anderson and Duncan Haldane, also explored models within two spatial dimensions to understand the quantum Hall effect. Anderson extended the Laughlin model, allowing for fractional filling factors in the form of $\nu = \dfrac{p}{m}$, where p is an integer and m is an odd integer [11]. On the other hand, Haldane introduced a version of the Laughlin model with translational invariance, which successfully described the filling factors 2/5 and 2/7 [12]. Its must be emphasized that these models propose an electron gas existing in a true two-dimensional universe, rather than being confined to two dimensions within a three-dimensional space.

Some recent works have also introduced planar quantum electrodynamics, potentially relevant to condensed matter phenomena, as indicated in the references [13, 14]. In the former reference, a QED_3 model has been put forth, leading to the derivation of a Hall conductivity with a fractional filling factor. This suggests a potential application of the model to the fractional quantum Hall effect. Conversely, in the latter reference, a pristine-like QED_3 model has been proposed. Within this model, the Landau levels of the electron-polaron and hole-polaron quasi-particles exhibit a four-fold broken degeneracy. Additionally, there exists a Landau level with zero energy, hinting at the presence of an anomalous quantum Hall effect. Both of these models will be discussed in detail in the next chapters.

In view of the previously presented works by numerous renowned physicists, which consider spacetimes with different dimensions than the usual, exhibiting various important outcomes and describing numerous phenomena, it becomes evident that models involving

[1] Some misconceptions may arise regarding a planar system. In this case, one might think that in models of lower dimensions, the universe has three spatial dimensions and the dynamics is restricted to only two. However, that is not always the case.

lower or higher spacetime dimensions should not be dismissed as fanciful. Particularly, the models based on QED_3 merit significant attention due to their potential application in condensed matter physics. Indeed, the emergent particles, commonly referred to as quasi-particles, which arise from the collective behavior of the particles in the material, can live in a two-dimensional world, giving rise to emergent phenomena, such as quantum Hall effect and topological insulators. Therefore, describing these quasi-particles using two-dimensional models provides further evidence that there is much more physics beyond the three space dimensions.

Although a model might effectively describe a phenomenon, it must obey to fundamental physical principles. In the realm of Quantum Field Theory, it is essential to analyze the model's symmetries, spectrum, renormalizability, presence or absence of anomalies, scattering behavior, and lastly, the interaction potential. Considering these factors, the upcoming chapters will delve into the analysis of two QED_3 models. In Chapter 2 the Kaplan-Sen model, which yields a Hall conductivity featuring a fractional filling factor, is analyzed. The model's boson gauge fields are demonstrated to be non-fundamental and the fundamental fields are determined through diagonalization of the action. Moving on to Chapter 3 and subsequent ones, we investigate a parity-preserving QED_3 with two massless fermions and two massive gauge fields. In Chapter 3, we present the power-counting of the model and its super-renormalizability. This chapter also shows that the potentially divergent diagrams are up to the 2-loop order. Furthermore, we demonstrate that the BPHZL subtraction procedure preserves the parity symmetry. Chapter 4 applies the BPHZL approach to the model at the two-loops level, completing the model's renormalization process. In Chapter 5, using the algebraic renormalization procedure, we establish that the model is free from anomalies and analyze the structure of invariant counterterms. This analysis leads to the significant result that the model exhibits quantum scale invariance. Given that the analysis concerns QED_3 models, we make reference to [15] for useful results.

Prior to the aforementioned analysis, it is convenient to provide a brief discussion on the generating functional and its relationship with crucial quantities that play a substantial role in any Quantum Field Theory model.

1.1 The generating functional

The scattering matrix, often referred to as the S-matrix, is certainly one of the most important entities in Quantum Field Theory. Its components are linked to the probability of obtaining an outgoing state, given an incoming state, within a system. It enables the calculation of scattering amplitudes, decay rates, and other significant quantities. Essentially, it serves as a link between theory and experiments. We can employ the LSZ (Lehmann-Symanzik-Zimmermann) reduction formula to construct the scattering matrix, and this process involves calculating Green functions. These functions are defined as the expected values in the vacuum of temporally ordered field operators. Generally, it must to be derived

$$G_{i_1 \dots i_n}(x_1, \dots, x_n) = \langle 0 | T \Phi_{i_1}(x_1) \dots \Phi_{i_n}(x_n) | 0 \rangle \ , \tag{1.1}$$

where the index i_j is associated with all possible properties that a field may possess, such as tensor indices, flavor, color, etc., whereas T is the temporal ordering operator.

The results shown here can be found in [16] and in a number of other books on Quantum Field Theory, such as [17, 18]. The definitions may differ by constant factors, but all them are equivalent. It is possible to define the generating functional of the Green functions as follows:

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!\ \hbar^n} \int d^D x_1 ... d^D x_n J^{i_1}(x_1)...J^{i_n}(x_n) G_{i_1...i_n}(x_1, ..., x_n) \ , \tag{1.2}$$

being J^{i_k} functions in the Schwartz space, i.e., functions of rapidly decreasing as well as all their derivatives. We use the Einstein summation convention.

From the generating functional, it is possible to obtain the Green functions by

$$(-i\hbar)^n \left. \frac{\delta^n Z[J]}{\delta J^{i_1}(x_1)...\delta J^{i_n}(x_n)} \right|_{J=0} = \langle 0 | T \Phi_{i_1}(x_1)...\Phi_{i_n}(x_n) | 0 \rangle \ . \tag{1.3}$$

Thus, the problem of determining the Green functions transforms into the problem of determining the generating functional.

Is is a well established fact that the generating functional is formally given by the functional integral

$$Z[J] = \mathcal{N} \int \mathcal{D}\Phi \ e^{\frac{i}{\hbar} S[\Phi] + \frac{i}{\hbar} \int d^D x J^i(x) \Phi_i(x)} \ , \tag{1.4}$$

where D is the dimension of the considered spacetime, $\mathcal{N}$ is a normalization factor, $S[\Phi]$ is the action [2] and $\mathcal{D}\Phi$ is the integration measure. It is worth mentioning that this integration measure indicates that the integral has contributions of all the field configurations. Furthermore, it is not well defined. It is common to define the normalization condition as $Z[0] = 1$, and $\mathcal{N}$ becomes

$$\mathcal{N} = \frac{1}{\displaystyle\int \mathcal{D}\Phi \ e^{\frac{i}{\hbar} S[\Phi]}} \ . \tag{1.5}$$

Commonly, we can split the action $S[\Phi]$ into two parts, being one quadratic in the fields, called free action ($S_0[\Phi]$), whereas the other, called interaction action ($S_{\text{int}}[\Phi]$), has terms with power three or higher in the fields. It can be written mathematically as

$$S[\Phi] = S_0[\Phi] + S_{\text{int}}[\Phi] \ . \tag{1.6}$$

We define the free generating functional as

$$Z_0[J] = \mathcal{N}' \int \mathcal{D}\Phi \ e^{\frac{i}{\hbar} S_0[\Phi] + \frac{i}{\hbar} \int d^D x J^i(x) \Phi_i(x)} \ , \tag{1.7}$$

where we choose $\mathcal{N}'$ such that

$$Z_0[0] = \mathcal{N}' \int \mathcal{D}\Phi \ e^{\frac{i}{\hbar} S_0[\Phi]} = 1 \ . \tag{1.8}$$

The free Green functions are

$$\begin{aligned} \langle 0 | T \Phi_{i_1}(x_1)...\Phi_{i_n}(x_n) | 0 \rangle_0 =& (-i\hbar)^n \left. \frac{\delta^n Z_0[J]}{\delta J^{i_1}(x_1)...\delta J^{i_n}(x_n)} \right|_{J=0} \\ =& \mathcal{N}' \int \mathcal{D}\Phi \ \Phi_{i_1}(x_1)...\Phi_{i_n}(x_n) e^{\frac{i}{\hbar} \int d^D x S_0[\Phi]} \ . \end{aligned} \tag{1.9}$$

[2] Sometimes we need to add to the action other terms. For example, in abelian gauge field theories a gauge-fixing is necessary, and in Yang-Mills theories, an action of Faddeev-Popov ghosts needs to be incorporated.

Let us assume that the free action can be written as

$$S_0[\Phi] = \frac{1}{2}\int d^Dx \ \Phi_i K^{ij}(\partial)\Phi_j \ , \tag{1.10}$$

where $K^{ij}(\partial)$ is the wave operator. In (1.7), it is possible to perform a change of integration variables:

$$\Phi(x)_j \longrightarrow \Phi(x)_j - \int d^Dy \ J^l(y)K^{-1}_{jl}(x-y) \ , \tag{1.11}$$

with $K^{ij}(\partial)K^{-1}_{jl}(x-y) = \delta^i{}_l\delta^D(x-y)$,i.e., $K^{-1}_{jl}(x-y)$ is the inverse of the wave operator. Assuming that the functional integration measure does not change under translations, the free action becomes

$$\begin{aligned} Z_0[J] =& \underbrace{\mathcal{N}'\int \mathcal{D}\Phi e^{\frac{i}{\hbar}\int d^DxS_0[\Phi]}}_{Z_0[0]} e^{\frac{i}{2\hbar}\int d^Dxd^DyJ^i(x)K^{-1}_{ij}(x-y)J^j(y)} \\ =& e^{\frac{i}{2\hbar}\int d^Dxd^DyJ^i(x)K^{-1}_{ij}(x-y)J^j(y)} \ . \end{aligned} \tag{1.12}$$

Throughout the text of this work, the Feynman propagator $\Delta_{i_1i_2}(x_1-x_2)$ will play a significant role on numerous occasions. It is given by the two-point Green functions:

$$\Delta_{i_1i_2}(x_1-x_2) = \langle 0|T\Phi_{i_1}(x_1)\Phi_{i_2}(x_2)|0\rangle_0 = -i\hbar K^{-1}_{i_1i_2}(x_1-x_2) \ . \tag{1.13}$$

Is is possible to show that the free n-point Green function is

$$\langle 0|T\Phi_1(x_1)..\Phi_n(x_n)|0\rangle = \begin{cases} 0 \text{ if } n \text{ is odd;} \\ \Delta_{i_1i_2}...\Delta_{i_{1n}-i_n} + perm., \text{ if } n \text{ is even.} \end{cases} \ . \tag{1.14}$$

It is convenient to comment that in the aforementioned result, known as Wick's Theorem, when n is even, all the possible permutations of the indices have to be taken into account. In a certain way, calculating Green functions in free theories is relatively simple. The main challenge is to determine the Green functions when there exist interactions. Aiming to this purpose, we show a useful result that enables to obtain the Green functions in models with interactions through perturbation theory. Let us consider an arbitrary functional $F[\Phi]$ that has a power series expansion as

$$F[\Phi] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^Dx_1...d^Dx_n \left.\frac{\delta^{(n)}F}{\delta\Phi(x_1)...\delta\Phi(x_n)}\right|_{\Phi=0} \Phi(x_1)...\Phi(x_n) \ . \tag{1.15}$$

Therefore, the expected value of this functional in a free theory is

$$\begin{aligned} \langle 0|TF[\Phi]|0\rangle_0 =& \sum_{n=0}^{\infty}\frac{1}{n!}\int d^Dx_1...d^Dx_n \left.\frac{\delta^{(n)}F}{\delta\Phi(x_1)...\delta\Phi(x_n)}\right|_{\Phi=0} \times \\ & \times \langle 0|T\Phi(x_1)...\Phi(x_n)|0\rangle_0 \\ =& \mathcal{N}'\int \mathcal{D}\Phi \ \sum_{n=0}^{\infty}\frac{1}{n!}\int d^Dx_1...d^Dx_n \left.\frac{\delta^{(n)}F}{\delta\Phi(x_1)...\delta\Phi(x_n)}\right|_{\Phi=0} \times \\ & \times \Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)e^{\frac{i}{\hbar}\int d^DxS_0[\Phi]} \\ =& \mathcal{N}'\int \mathcal{D}\Phi \ F[\Phi]e^{\frac{i}{\hbar}\int d^DxS_0[\Phi]} \ , \end{aligned} \tag{1.16}$$

where we have used (1.9), (1.15) and the fact that the expression $\sum_{n=0}^{\infty}\frac{1}{n!}\int d^Dx_1...d^Dx_n\,\frac{\delta^{(n)}F}{\delta\Phi(x_1)...\delta\Phi(x_n)}\Big|_{\Phi=0}$ is a constant in view of the inner product as well as in view of the functional integration. The previous result was obtained considering just a single boson field, but it can be generalized to any fields.

Now we are in position to determine the Green functions considering the interaction terms. From (1.3), (1.4) and (1.5), it is possible to see that

$$\langle 0|T\Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)|0\rangle = \frac{\displaystyle\int \mathcal{D}\Phi\ \Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)e^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}e^{\frac{i}{\hbar}S_0[\Phi]}}{\displaystyle\int \mathcal{D}\Phi\ e^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}e^{\frac{i}{\hbar}S_0[\Phi]}}\,. \tag{1.17}$$

Multiplying and dividing by $\mathcal{N}'$, using (1.16) and taking into account the functionals $F_1[\Phi] = \Phi_{i_1}...\Phi_{i_n}e^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}$ e $F_2[\Phi] = e^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}$ we achieve the important expression:

$$\langle 0|T\Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)|0\rangle = \frac{\langle 0|T\Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)e^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}|0\rangle_0}{\langle 0|Te^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}|0\rangle_0}\,. \tag{1.18}$$

Thus, to calculate the Green functions, we can expand $e^{\frac{i}{\hbar}S_{\text{int}}[\Phi]}$ as a power series and calculate the terms of the Green functions order by order, considering only the free action and utilizing Wick's Theorem. Moreover, it is possible to represent these expressions diagrammatically using the well-known Feynman diagrams. These diagrams can substantially reduce calculations and enable various simplifications. It's worth mentioning that the denominator of the preceding expression serves to remove vacuum bubbles.

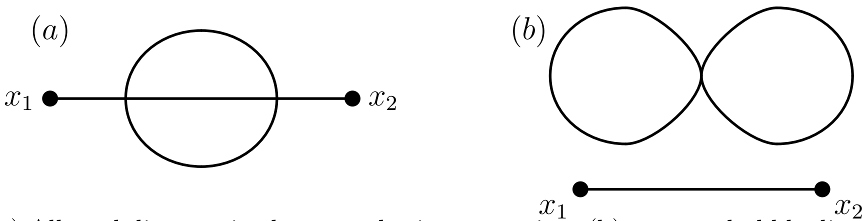


Figure 1.1: (a) Allowed diagram in the perturbative expansion; (b) vacuum bubble diagram, which is not allowed in the expansion. It is noticeable that the process that occurs between x_1 and x_2 in this diagram is not affected by the vacuum bubble.

In Feynman diagrams, as exemplified in the precedent figure, each point that represents the arguments of the Green functions is referred to as external point. On the other hand, each point corresponding to an intersection of three or more lines is called vertex and contributes for the calculation with a term from S_{int}. Besides, each internal line, i.e., each line that is not directly connected to an external point, contributes with a propagator.

1.2 The connected and 1PI generating functionals

Consider the Figure 1.2. It is possible to observe that the diagram in (a) has all external points connected, whereas in (b) the points x_1 and x_3 are not connected to x_2 and x_4. Diagrams in which all external points are connected are referred to as connected diagrams. On the other hand, a diagram that has at least one external point not connected to the others is called a disconnected diagram.

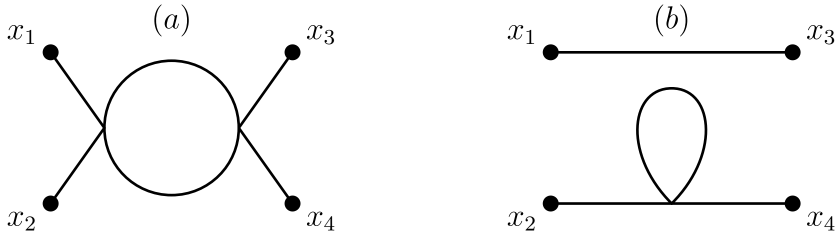


Figure 1.2: (a) Diagrama conexo; (b) diagrama desconexo.

The generating functional defined in the previous section generates all the Feynman diagrams, whether they are connected or not. However, disconnected diagrams are merely a product of connected ones. Consequently, it is useful to restrict the analysis to only the connected diagrams. To achieve this, we introduce the generating functional of the connected Green functions, denoted as $Z^c[J]$, which is defined by

$$Z[J] = e^{\frac{i}{\hbar} Z^c[J]} \Leftrightarrow Z^c[J] = -i\hbar \ln Z[J] \ . \tag{1.19}$$

Is it possible to show that $Z^c[J]$ only generates connected diagrams when functional derivatives are taken with respect to J. Thus, its natural expansion is as follows:

$$Z^c[J] = \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!\,\hbar^{n-1}} \int d^D x_1 ... d^D x_n J^{i_1}(x_1) ... J^{i_n}(x_n) \ \langle 0|T\Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)|0\rangle_{\text{con}} \ , \tag{1.20}$$

from where we conclude that

$$\langle 0|T\Phi_{i_1}(x_1)...\Phi_{i_n}(x_n)|0\rangle_{\text{con}} = (-i\hbar)^{n-1} \left. \frac{\delta^n Z^c[J]}{\delta J^{i_1}(x_1) \delta J^{i_n}(x_n)} \right|_{J=0} \ . \tag{1.21}$$

Additionally, considering solely the free action, the previous expression reads

$$\langle 0|T\Phi_{i_1}(x_1)\Phi_{i_2}(x_2)|0\rangle_{0 \text{ con}} = \langle 0|T\Phi_{i_1}(x_1)\Phi_{i_2}(x_2)|0\rangle_0 \ . \tag{1.22}$$

Therefore, at the tree-level, the two-point Green functions are necessarily connected.

It is also possible to introduce the concept of 1PI (one-particle irreducible) diagrams. These diagrams possess amputated external legs and remain connected even if an internal leg is cut, as illustrated in the following figure.

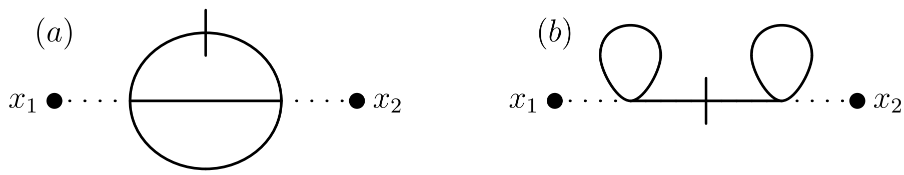


Figure 1.3: (a) 1PI diagram; (b) diagram that is not 1PI. As can be checked, the diagram (b) can be decomposed into two 1PI diagrams.

The connected diagrams can be decomposed as a product of 1PI diagrams. In this way, the analysis of Green functions can be limited to the analysis of 1PI diagrams. So, it is convenient to define the 1PI generating functional $\Gamma[\varphi]$, given by a Legendre transformation, as follows:

$$\Gamma[\varphi] = Z^c[J] - \int d^D x J^i(x) \varphi_i(x), \ \text{com} \ \varphi_i(x) = \frac{\delta Z^c[J]}{\delta J^i(x)} \ . \tag{1.23}$$

The quantity $\varphi_i(x)$ is commonly referred to as classical field. It must not be confused with the field operators. From the previous equation, we have

$$\begin{aligned}\frac{\delta\Gamma[\varphi]}{\delta\varphi_j(y)} =& \frac{\delta Z^c[J]}{\delta\varphi_j(y)} - \int d^D x \frac{\delta J^i(x)}{\delta\varphi_j(y)}\varphi_i(x) - J^j(y) \\ =& \int d^D x \frac{\delta Z^c[J]}{\delta J^i(x)}\frac{\delta J^i(x)}{\delta\varphi_j(y)} - \int d^D x \frac{\delta J^i(x)}{\delta\varphi_j(y)}\varphi_i(x) - J^j(y) \\ =& \int d^D x \varphi_i(x)\frac{\delta J^i(x)}{\delta\varphi_j(y)} - \int d^D x \frac{\delta J^i(x)}{\delta\varphi_j(y)}\varphi_i(x) - J^j(y) \\ =& - J^i(y) \ ,\end{aligned} \tag{1.24}$$

Here, the functional chain rule has been used. It is possible to show that $\Gamma[\varphi]$ is the generating functional of the 1PI diagrams, with amputated external legs. Thus, the expansion of $\Gamma[\varphi]$ is

$$\Gamma[\varphi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^D x \Gamma^{i_1 \ldots i_n}(x_1, ..., x_n)\varphi_{i_1}(x_1)...\varphi_{i_n}(x_n) \quad . \tag{1.25}$$

$\Gamma[\varphi]$ is called vertex functional and $\Gamma^{i_1 \ldots i_n}(x_1, ..., x_n)$ is the n-points 1PI Green function. Additionally, $\Gamma[\varphi]$ can be expanded formally in powers of $\hbar$ as follows:

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}[\varphi] \ , \tag{1.26}$$

being $\Gamma^{(n)}$ resulting by the sum of all 1PI diagrams with n-loops. The term of order 0 coincides with the classical action, i.e.,

$$\Gamma^{(0)}[\varphi] = S[\varphi] \ . \tag{1.27}$$

The equation (1.26) can be derived from the fact that each vertex arising from $\frac{1}{\hbar}S_{\text{int}}$ contributes with a factor $\hbar^{-1}$, whereas each propagator, given by an internal line, contributes with a factor of $\hbar$, and there exists a global factor $\hbar$ due to (1.19). So, the power of $\hbar$ for each 1PI diagram is determined by $I - V + 1$, being I the number of internal lines and V the number of vertices. The power of $\hbar$ coincides with the number of loops, as dictated by the Euler formula:

$$L = I - (V - 1) \ . \tag{1.28}$$

In this manner, the power of $\hbar$ for a 1PI diagram is identical to the number of loops in the diagram. Therefore, we can state that (1.26) is the loop-wise expansion of $\Gamma[\varphi]$. This is why, at times in the literature, Γ is referred to as the quantum action, comprised of the classical action $\Gamma^{(0)}$, along with the corrections arising from loop diagrams.

An important point to note is that, typically, diagrams are computed in momentum space, involving Fourier transforms. In this scenario, every external line corresponds to an external momentum, while each internal line is associated with a 'flowing' internal momentum. Momentum conservation holds at each vertex, and the process is similar to circuit analysis, applying Kirchhoff's laws. Furthermore, each loop corresponds to an integral over an independent internal momentum.

Chapter 2

The spectrum consistency of fractional quantum Hall effect model[1]

As mentioned earlier, QED3 serves as a potential theoretical framework for many condensed matter phenomena. While significant progress has already been made, a more thorough investigation of the fractional quantum Hall effect is still necessary. In this context, a recent relativistic model was proposed by Kaplan and Sen [13], yielding a Hall conductivity with a fractional filling factor in the low-energy limit. The proposed action of the model was constructed in three spacetime dimensions using three massive fermion families: ψ_i, χ_j, and ω_k, with flavors represented by $n\psi$, n_χ, and n_ω, respectively. The model possesses a $U(1) \times U(1)$ symmetry and two massive vector gauge fields, A_μ and Z_μ. Additionally, it has two coupling constants: e and g. The complete action, along with the gauge-fixing terms, is:

$$
\begin{aligned}
\Sigma_{\mathrm{KS}} = \int d^3x \Bigg\{ & -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4g^2} Z^{\mu\nu} Z_{\mu\nu} \\
& + \frac{\varepsilon^{\mu\nu\rho}}{4\pi} \left[(n_\psi + n_\omega) A_\mu \partial_\nu A_\rho + (n_\chi + n_\omega) Z_\mu \partial_\nu Z_\rho + 2 n_\omega A_\mu \partial_\nu Z_\rho \right] \\
& + \frac{n_\psi}{|n_\psi|} m \bar{\psi}_i \psi_i + \frac{n_\chi}{|n_\chi|} m \bar{\chi}_j \chi_j + \frac{n_\omega}{|n_\omega|} m \bar{\omega}_k \omega_k + i \bar{\psi}_i \not{D} \psi_i \\
& + i \bar{\chi}_j \not{D} \chi_j + i \bar{\omega}_k \not{D} \omega_k \underbrace{- \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 - \frac{1}{2\beta} (\partial^\mu Z_\mu)^2}_{gauge-fixing\ terms} \Bigg\} .
\end{aligned}
\tag{2.1}
$$

Here, $\not{D} \equiv \not{\partial} + i q_A e \not{A} + i q_Z g \not{Z}$, and q_A and q_Z represent the charges for the fermions, with values of 1 and 0 for ψ_i, 0 and 1 for χ_j, and 1 and 1 for ω_k, respectively. Additionally, i, j, and k range from 1 to n_ψ, n_χ, and n_ω correspondingly. The Chern-Simons terms are the topological terms responsible for the mass of the vector gauge fields. It is important to note that all fermionic masses are considered identical. Finally, $F^{\mu\nu}$ and $Z^{\mu\nu}$ represent the field strengths for the fields A^μ and Z^μ, respectively, while $\varepsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor and $\varepsilon^{012} = 1$.

[1]This chapter is based on the article submitted for publication in the journal 'Annals of Physics,' with the same title as this chapter. It can also be found on [arXiv:2204.02534 [hep-th]]. Additionally, the results were presented orally at the 'XLII Encontro Nacional de Física de Partículas e Campos' (ENFPC), which took place in Natal-RN, Brazil, from September 26th to 29th, 2022.

To derive the Hall conductivity in the low-energy limit, the authors integrated out the fermionic degrees of freedom, assuming that the fermion masses were much larger than the boson masses. Thereafter, they considered the limit $g \gg e$, in which the mass of Z^μ was much greater than the mass of A^μ. They also performed an integration over Z^μ, resulting in an effective action for A^μ referred to as the "photon" at the end of these steps. Mathematically, from the complete generating functional,

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}Z_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\chi \mathcal{D}\bar{\chi} \mathcal{D}\omega \mathcal{D}\bar{\omega} e^{i\Sigma_{KS}+\text{current terms}} , \tag{2.2}$$

they obtained

$$\mathcal{Z}_A[J,0] = \mathcal{N} \int \mathcal{D}A_\mu e^{i\Sigma_{eff}+i\int d^3x J^\mu A_\mu} , \tag{2.3}$$

which led to an effective action for the field A^μ. The achieved Hall conductivity was

$$\sigma_{xy} = \left(n_\psi + \frac{n_\chi n_\omega}{n_\chi + n_\omega} \right) \frac{e^2}{\hbar} , \tag{2.4}$$

and for certain values of n_ψ, n_χ and n_ω, it is fractional.

However, as shown throughout this chapter, A^μ and Z^μ are not fundamental fields, as evidenced by a mixed propagator whose current-current amplitude transition does not vanish. Moreover, one might inquire whether unitarity and causality are ensured, as the discussion will reveal that the masses depend on the fermionic flavor numbers. In principle, certain conditions regarding the flavor numbers could be necessary to ensure unitarity and causality. For this reason, it is worth performing a spectrum analysis of this model.

2.1 Obtaining the boson propagators of the model

We can start performing the changes $A_\mu \to eA_\mu$ and $Z_\mu \to gZ_\mu$, making the coupling constants explicit in the interaction terms. The free bosonic lagrangian, which is obtained after integrating out the fermions, reads

$$\begin{aligned} \mathcal{L}_b = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \varepsilon^{\mu\rho\nu}\left\{ \frac{E}{2}A_\mu \partial_\rho A_\nu + \frac{G}{2} Z_\mu \partial_\rho Z_\nu + \right. \\ &\left. H A_\mu \partial_\rho Z_\nu \right\} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - \frac{1}{2\beta}(\partial^\mu Z_\mu)^2 , \end{aligned} \tag{2.5}$$

where

$$E = \frac{e^2}{2\pi}(n_\psi + n_\omega), \quad G = \frac{g^2}{2\pi}(n_\chi + n_\omega), \quad H = \frac{eg}{2\pi} n_\omega . \tag{2.6}$$

The free bosonic action is $\Sigma_b = \int d^3x \mathcal{L}_b$. Integrating by parts, we obtain

$$\Sigma_b = \int d^3x \left\{ \frac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu + \frac{1}{2} Z_\nu \widetilde{\mathcal{O}}^{\mu\nu} Z_\nu + H A_\mu S^{\mu\nu} Z_\nu \right\} , \tag{2.7}$$

being the operators given by

$$\begin{aligned} &\mathcal{O}^{\mu\nu} \equiv \Box \Theta^{\mu\nu} - \frac{\Box}{\alpha}\Omega^{\mu\nu} + ES^{\mu\nu} , \quad \widetilde{\mathcal{O}}^{\mu\nu} \equiv \Box \Theta^{\mu\nu} - \frac{\Box}{\beta}\Omega^{\mu\nu} + GS^{\mu\nu} , \\ &\Theta^{\mu\nu} = \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}, \quad \Omega^{\mu\nu} \equiv \frac{\partial^\mu \partial^\nu}{\Box}, \quad S^{\mu\nu} \equiv \varepsilon^{\mu\rho\nu}\partial_\rho . \end{aligned} \tag{2.8}$$

The operator algebra is showed in the following table.

	Θ	Ω	S
Θ	Θ	0	S
Ω	0	Ω	0
S	S	0	$-\Box\Theta$

Table 2.1: Algebra of the operators Θ, Ω e S. The product must be understood as in the following example: $\Theta\Theta = \Theta$ means $\Theta^{\mu\rho}\Theta_{\rho\nu} = \Theta^{\mu}{}_{\nu}$.

The inverse of a generic operator $a\Theta + b\Omega + cS$ is given by

$$\left(\frac{a}{a^2+\Box c^2}\Theta_{\mu\nu} + \frac{1}{b}\Omega_{\mu\nu} - \frac{c}{a^2+\Box c^2}S_{\mu\nu}\right)(a\Theta^{\nu\rho} + b\Omega^{\nu\rho} + cS^{\nu\rho}) = \delta_{\mu}{}^{\rho} \ . \tag{2.9}$$

We call the last equation the inversion formula. According to (1.23), it follows that

$$\Gamma^{(0)}[A,Z] = \mathcal{Z}^c[J,j] - \int d^3x\{A_\mu J^\mu + Z_\mu j^\mu\} \ , \tag{2.10}$$

from where we identify $\Gamma^{(0)[A,Z]}$ as the classical bosonic action, whereas J^μ and j^μ are the conjugated currents to A_μ and Z_μ, respectively. Thus, the following relations read:

$$\frac{\delta\Gamma^{(0)}}{\delta A_\mu} = -J^\mu \ , \quad \frac{\delta\Gamma^{(0)}}{\delta Z_\mu} = -j^\mu \ , \quad \frac{\delta\mathcal{Z}^c}{\delta J^\mu} = A_\mu \ , \quad \frac{\delta\mathcal{Z}^c}{\delta j^\mu} = Z_\mu \ . \tag{2.11}$$

The propagator $\Delta^{AA}_{\mu\nu}(x,y)$ is obtained from $-i\dfrac{\delta^2\mathcal{Z}^c}{\delta J^\mu(x)\delta J^n(y)}$. Therefore, we need to express the conjugated currents as functions of the fields. It is worth noting that if A_μ is written in terms of J^μ and j^ν, the expression for Z_μ becomes straightforward, as the free bosonic action is symmetric under the transformation:

$$e \leftrightarrow g, \ A_\mu \leftrightarrow Z_\mu, \ J^\mu \leftrightarrow j^\mu, \ \alpha \leftrightarrow \beta. \tag{2.12}$$

Using the first and second equalities of (2.11), then

$$-J^\mu = \mathcal{O}^{\mu\nu}A_\nu + HS^{\mu\nu}Z_\nu \ , \tag{2.13a}$$

$$-j^\mu = \widetilde{\mathcal{O}}^{\mu\nu}Z_\nu + HS^{\mu\nu}A_\nu \ . \tag{2.13b}$$

where we have used the fact that, over integration, $A_\mu S^{\mu\nu}Z_\nu = Z_\mu S^{\mu\nu}A_\nu$. Consider the next expression that follows from the inversion formula:

$$\widetilde{\mathcal{O}}^{-1}_{\mu\nu} = \frac{1}{\Box+G^2}\Theta^{\mu\nu} + \frac{\beta}{\Box}\Omega_{\mu\nu} - \frac{G}{\Box(\Box+G^2)}S_{\mu\nu} \ . \tag{2.14}$$

From (2.13b), it follows that

$$Z_\nu = -\widetilde{\mathcal{O}}^{-1}_{\mu\rho}j^\rho - H\widetilde{\mathcal{O}}^{-1}_{\mu\rho}S^{\rho\kappa}A_\kappa \ . \tag{2.15}$$

Additionally, taking into account the following definition

$$D_g \equiv \Box + G^2 \ , \tag{2.16}$$

and considering the multiplicative table, it is possible to see that

$$HS^{\mu\nu}Z_\nu = -H\left(\frac{1}{D_g}S^\mu{}_\nu + \frac{G}{D_g}\Theta^\mu{}_\nu\right)j^\nu + H^2\left(\frac{\Box\Theta^{\mu\nu}}{D_g} - \frac{G}{D_g}S^{\mu\nu}\right)A_\nu \ . \tag{2.17}$$

From the previous result and (2.13a), then

$$-J^\mu = \left[\left(\Box + \frac{\Box H^2}{D_g}\right)\Theta^{\mu\nu} + \frac{\Box}{\alpha}\Omega^{\mu\nu} + \left(E - \frac{GH^2}{D_g}S^{\mu\nu}\right)\right]A_\nu - H\left(\frac{1}{D_g}S^\mu{}_\nu + \frac{G}{D_g}\Theta^\mu{}_\nu\right)j^\nu \ . \tag{2.18}$$

For the determination of A in terms of the currents, there is only one more step. Using the inversion formula, the expression for A is the following:

$$\begin{aligned} A_\rho = &-\left(\frac{(\Box + G^2 + H^2)}{\mathcal{D}}\Theta_{\rho\mu} - \frac{\alpha}{\Box}\Omega_{\rho\mu} - \frac{E(\Box + G^2) - GH^2}{\Box\mathcal{D}}S_{\rho\mu}\right)\times \\ &\times\left(J^\mu - \left[\frac{H}{\Box + G^2}S^\mu_\nu + \frac{HG}{\Box + G^2}\Theta^\mu_\nu\right]j^\mu\right) \ , \end{aligned} \tag{2.19}$$

where

$$\mathcal{D} \equiv H^4 + 2H^2(\Box - EG) + (\Box + G^2)(\Box + E^2) \ . \tag{2.20}$$

Beyond that, applying (2.12) in (2.19), the expression for Z reads:

$$\begin{aligned} Z_\rho = &-\left(\frac{(\Box + E^2 + H^2)}{\mathcal{D}}\Theta_{\rho\mu} - \frac{\beta}{\Box}\Omega_{\rho\mu} - \frac{G(\Box + E^2) - EH^2}{\Box\mathcal{D}}S_{\rho\mu}\right)\times \\ &\times\left(j^\mu - \left[\frac{H}{\Box + E^2}S^\mu_\nu + \frac{HE}{\Box + E^2}\Theta^\mu_\nu\right]J^\mu\right). \end{aligned} \tag{2.21}$$

Notwithstanding the appearance of (2.19) and (2.21), it is relatively straightforward to derive the propagators. For this, it must be recalled that

$$A_\mu(x) = \frac{\delta\mathcal{Z}^c}{\delta J^\mu(x)}, \quad Z_\mu(x) = \frac{\delta\mathcal{Z}^c}{\delta j^\mu(x)} \ , \tag{2.22}$$

so that

$$\Delta^{AA}_{\mu\nu}(x,y) = -i\left.\frac{\delta^2\mathcal{Z}^c}{\delta J^\mu(x)\delta J^\nu(y)}\right|_{J=0} = -\left. i\frac{\delta A_\nu(y)}{\delta J^\mu(x)}\right|_{J=0} \ . \tag{2.23}$$

Similar expressions for $\Delta^{ZZ}_{\mu\nu}(x,y)$, $\Delta^{AZ}_{\mu\nu}(x,y)$ and $\Delta^{ZA}_{\mu\nu}(x,y)$ can be derived. Furthermore, the last two propagators, which are referred to as mixed propagators, are identical.

It is useful to express the propagators in momentum space, as the analysis of unitarity and causality will be conducted in this realm:

$$\Delta^{ZZ}_{\mu\nu}(k) = i\left[\frac{k^2 - E^2 - H^2}{\mathcal{D}_k}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{\beta}{k^2}\frac{k_\mu k_\nu}{k^2} + \frac{1}{k^2}\frac{G(k^2 - E^2) + H^2E}{\mathcal{D}_k}i\varepsilon_{\mu\rho\nu}k^\rho\right] \ , \tag{2.24}$$

$$\Delta^{AA}_{\mu\nu}(k) = i\left[\frac{k^2 - G^2 - H^2}{\mathcal{D}_k}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{\alpha}{k^2}\frac{k_\mu k_\nu}{k^2} + \frac{1}{k^2}\frac{E(k^2 - G^2) + H^2G}{\mathcal{D}_k}i\varepsilon_{\mu\rho\nu}k^\rho\right] \ , \tag{2.25}$$

$$\Delta^{AZ}_{\mu\nu}(k) = \frac{iH}{\mathcal{D}_k}\left[\frac{i\varepsilon^{\mu\rho\nu}k_\rho}{k^2}(k^2+GE-H^2) + \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(E+G)\right] \ , \tag{2.26}$$

where $\mathcal{D}_k$ is the expressoin of $\mathcal{D}$, given by (2.20), in momentum space. In this context,

$$\mathcal{D}_k = (k^2-G^2)(k^2-E^2) - 2H^2(k^2+EG) + H^4 \ . \tag{2.27}$$

The massive excitations can be derived from the propagator poles. The boson propagators exhibit three poles:

$$k^2 = 0 \ ; \qquad k^2 = \mu^2_\pm = \frac{1}{2}\left(E^2+G^2+2H^2 \pm \sqrt{(E^2-G^2)^2+4H^2(E+G)^2}\right) \ . \tag{2.28}$$

Kaplan and Sen [13] examined the limit $g^2 >> e^2$, resulting in a significantly higher mass of Z_μ compared to the mass of A_μ. This allowed the integration of Z_μ, leading to the effective theory for A_μ. Let us reproduce this limit. At the tree level, by coupling the propagators with the external currents, we acquire the current-current transition amplitudes, defined by

$$\mathcal{A}_{\phi_1\phi_2} \equiv J^\mu_{\phi_1}\Delta^{\phi_1\phi_2}_{\mu\nu}J^\nu_{\phi_2} \ , \tag{2.29}$$

where ϕ_1 e ϕ_2 represent any two fields, whereas J_{ϕ_1} and J_{ϕ_2} are the external currents. For the boson gauge fields, they read:

$$\mathcal{A}_{AA} = J^\mu_A\Delta^{AA}_{\mu\nu}J^\nu_A = i\frac{k^2-G^2-H^2}{(k^2-\mu^2_+)(k^2-\mu^2_-)}J^\mu_A J_{A\mu} \ , \tag{2.30}$$

$$\mathcal{A}_{ZZ} = J^\mu_Z\Delta^{ZZ}_{\mu\nu}J^\nu_Z = i\frac{k^2-E^2-H^2}{(k^2-\mu^2_+)(k^2-\mu^2_-)}J^\mu_Z J_{Z\mu} \ . \tag{2.31}$$

To derive the previous equations, we used the current conservation, i.e., $k_\mu J^\mu_A = k_\mu J^\mu_Z = 0$. The poles in the limit $g^2 >> e^2$ are

$$\mu^2_\pm = \frac{1}{2}[E^2+G^2+2H^2 \pm (G^2-E^2) + \mathcal{O}(e^2/g^2)] \ . \tag{2.32}$$

Taking $g^2 \to \infty$, it is possible to see that

$$\mu^2_+ = G^2+H^2 \ , \qquad \mu^2_- = E^2+H^2 \ . \tag{2.33}$$

Therefore, in this limit, the current-current transition amplitudes for A and Z become

$$J^\mu_A\Delta^{AA}_{\mu\nu}J^\nu_A = i\frac{k^2-G^2-H^2}{(k^2-\mu^2_+)(k^2-\mu^2_-)}J^\mu_A J_{A\mu} = i\frac{1}{k^2-E^2-H^2}J^\mu_A J_{A\mu} \ . \tag{2.34}$$

$$J^\mu_Z\Delta^{ZZ}_{\mu\nu}J^\nu_Z = i\frac{k^2-E^2-H^2}{(k^2-\mu^2_+)(k^2-\mu^2_-)}J^\mu_Z J_{Z\mu} = i\frac{1}{k^2-G^2-H^2}J^\mu_Z J_{Z\mu} \ . \tag{2.35}$$

Thus, the propagator of Z_μ, in this limit, has a pole in $G^2+H^2 \approx G^2 = \dfrac{g^4}{4\pi^2}(n_\chi+n_\omega)^2$, which results in a mass of the order of $\dfrac{g^2}{2\pi}(n_\chi+n_\omega)$. On the other hand, the mass μ_-, identified as the mass of A_μ, is of order of g, much lower than the mass of Z_μ. This justify the possibility of integrating out Z_μ.

As can be observed, the masses of the vector gauge fields depend on the fermion flavors. In this context, some natural questions arise. For example, whether causality and unitarity are ensured for all the fermion flavors. This is a crucial point, because the model does not present a fractional filling factor for all the flavor numbers, and certain filling factors might, in principle, be not allowed. Furthermore, the presence of the mixed propagators requires a more thorough examination.

2.2 Spectrum analysis of the model

Owing to the fact that poles in the Feynman propagators are related to the particle masses, it is essential to investigate the former concerning causality and unitarity.

Causality constitutes a fundamental aspect of a physically consistent model, as it signifies that effects do not precede their causes. Additionally, the information cannot be transmitted instantaneously. In the context of Minkowski space, two events can be causally connected solely if they are separated by a temporal or light interval, indicating that the effect resides within the light cone of the cause. A model is called causal if there are no tachyons in its spectrum – particles characterized by a negative squared mass when the metric $\eta_{\mu\nu} = \mathrm{diag}(+--)$ is considered.

Unitarity is also a fundamental feature of a consistent model, as it is associated with probability conservation. In the realm of quantum field theory, the probabilities are linked to the S-matrix. A natural assumption is its unitarity, driven precisely by the need for probability conservation. By the optical theorem [17,19], a consequence of the unitarity of the S-matrix is that at the tree-level the current-current transition amplitudes, defined in the previous section by (2.29), must be non-negative. Therefore, a necessary condition for a model's unitarity is that at the tree-level the current-current transition amplitudes of all the particles in the model's spectrum must be non-negative. Particles exhibiting negative values in this regard are called ghosts in this context, and they have negative norm states. For a unitary theory, if there are ghosts in the spectrum, they have to mutually cancel each other or they decouple from the model.

The Kaplan-Sen model has both fermions and bosons in its spectrum. The spectrum analysis will be conducted separately for each sector of the model. Let us start from the fermionic sector. The fermion propagators are

$$\Delta^{\phi}(k) = i\frac{\not{k} - n_\phi/|n_\phi|m}{k^2 - m^2} \ , \tag{2.36}$$

where ϕ represents ψ, χ e ω. As can be seen, there is only one pole, $k^2 = m^2$, and the causality is ensured regarding this sector. Let us consider a external fermionic current as $J_\phi = (\theta_1, \theta_2)^\intercal$ and $\bar{J}_\phi = J_\phi^\dagger \gamma_0 = (\theta_1^*, -\theta_2^*)$. Due to the fermions be massive, it is allowed to choose the rest frame, where the fermion momentum is $k^\mu = (m, 0, 0)$. Therefore, the fermionic current-current transition amplitudes are given by

$$\begin{aligned}\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{\phi\phi}|_{k^2=m^2}\}\} =& \mathcal{I}m\left\{\lim_{k^2\to m^2}(k^2-m^2)i\bar{J}_\phi^\dagger\frac{\not{k} - n_\phi/|n_\phi|m}{k^2-m^2}J_\phi\right\} \qquad (2.37)\\ =& m(|\theta_1|^2+|\theta_2|^2) - \frac{n_\phi}{|n_\phi|}m(|\theta_1|^2-|\theta_2|^2) \ . \qquad (2.38)\end{aligned}$$

It is worth noting that $n_\phi/|n_\phi| = \pm 1$. Thus, the aforementioned transition amplitude is $2m|\theta_1|^2$ or $2m|\theta_2|^2$, always being non-negative. Therefore, the necessary unitarity condition is ensured for the fermions of the model at the tree-level.

In the following, we analyze the boson spectrum. In (2.28), the radicand is non-negative, which means that $\mu_\pm^2$ is real. Additionally, it can be straightforwardly checked that μ_+^2 is positive. Furthermore, taking into account that

$$\begin{aligned}&(H^2 - EG)^2 \geq 0 \Rightarrow 4H^4 + 4E^2G^2 - 8H^2EG \geq 0\\ &\Rightarrow 4H^4 + 2E^2G^2 \geq 8H^2EG - 2E^2G^2 \ ,\end{aligned} \tag{2.39}$$

Adding to both sides $E^4 + G^4 + 4H^2G^2 + 4H^2G^2$, we are led to

$$(E^2 + G^2 + 2H^2)^2 \geq (E^2 - G^2)^2 + 4H^2(E+G)^2 \ . \tag{2.40}$$

This allows for the conclusion that μ_-^2 is always greater than or equal to zero, according to (2.28). Therefore, there are no issues with the causality of the model regarding the bosonic spectrum as well. So, the causality is ensured for the whole model and no conditions on fermion flavors or coupling constants regarding this issue are necessary

Let us now consider, in the bosonic sector, the vector external currents J_A^μ and J_Z^μ, which couple to the fields A_μ and Z_μ, respectively. They can be decomposed in a basis of the 1+2 Minkowski spacetime given by the set of vectors $k^\mu, \widetilde{k}^\mu, \varepsilon^\mu$, where $k^\mu = (k^0, k^1, k^2)$, $\widetilde{k}^\mu = (k^0, -k^1, -k^2)$, and $\varepsilon^\mu = (0, \varepsilon^1, \varepsilon^2)$, which satisfy the following covariant constraints:

$$k^\mu \varepsilon_\mu = \tilde{k}^\mu \varepsilon_\mu = 0 \ ; \qquad k^\mu k_\mu = \tilde{k}^\mu \tilde{k}_\mu = \mu^2 \ ; \qquad \varepsilon^\mu \varepsilon_\mu = -1 \ . \tag{2.41}$$

Here, μ is the mass parameter of the gauge field taken into consideration, and it can be zero. Beyond that, since the currents are conserved, it follows that

$$k_\mu J_A^\mu = k_\mu J_Z^\mu = 0. \tag{2.42}$$

Taking into consideration $\mu_\pm$, it is possible to use the rest frame, so that $k^\mu = \widetilde{k}^\mu = (\mu_\pm, 0, 0)$. Thus, considering firstly μ_+, it follows that

$$J_A^\mu = A_A k^\mu + B_A \widetilde{k}^\mu + C_A \varepsilon^\mu \ , \tag{2.43a}$$

$$J_Z^\mu = A_Z k^\mu + B_Z \widetilde{k}^\mu + C_Z \varepsilon^\mu \ . \tag{2.43b}$$

The application of the covariant constrains, along with (2.42), led to

$$k_\mu J_A^\mu = (A_A + B_A)\mu_+^2 = 0 \Rightarrow A_A = -B_A \ . \tag{2.44}$$

Similarly, it can be observed that A_Z equals $-B_Z$. This observation allows for the consideration of the massive pole μ_+^2:

$$J_A^\mu = C_A \varepsilon^\mu \ , \qquad J_Z^\mu = C_Z \varepsilon^\mu \ . \tag{2.45}$$

In a similar way, for μ_-^2 it can be seen that

$$J_A^\mu = C'_A \varepsilon^\mu \ , \qquad J_Z^\mu = C'_Z \varepsilon^\mu \ . \tag{2.46}$$

Finally, the unitarity of the bosonic sector can be analyzed. We take into consideration solely the terms that contribute to the imaginary part of the residue. It can be observed that

$$\begin{aligned}
\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{AZ}|_{k^2=\mu_+^2}\}\} =& \mathcal{I}m\left\{\lim_{k^2\to\mu_+^2}\left[(k^2-\mu_+^2)i\frac{H(E+G)}{(k^2-\mu_+^2)(k^2-\mu_-^2)}\times\right.\right. \\
&\left.\left.\times\ J_Z^\mu\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)J_A^\nu\right]\right\} \\
=& -C_A C_Z \frac{H(E+G)}{\mu_+^2-\mu_-^2} \ .
\end{aligned} \tag{2.47}$$

Similarly, for the pole μ_-^2, it follows that

$$\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{AZ}|_{k^2=\mu_-^2}\}\} = C'_A C'_Z \frac{H(E+G)}{\mu_+^2 - \mu_-^2} \ . \tag{2.48}$$

It is necessary to emphasize that the imaginary part of the transition amplitude $\mathcal{A}_{AZ}$ is not necessarily zero at both poles, which shows that the fields A^μ and Z^μ are not fundamental fields[2]. Additionally, it is possible that these quantities are, in fact, negative, depending on the constants C_A, C_Z, C'_A, and C'_Z, which threatens the unitarity of the model. To address these issues, a diagonalization procedure will be performed.

2.3 The fundamental vector gauge fields of the model

We start rewriting the bosonic free action as

$$\widetilde{\Sigma}_{AZ} = \int d^3x \ V_\mu^\intercal \mathcal{O}^{\mu\nu} V_\nu \ , \tag{2.49}$$

where

$$V_\mu \equiv \begin{bmatrix} A_\mu \\ Z_\mu \end{bmatrix} \ , \quad \mathcal{O}^{\mu\nu} \equiv \begin{bmatrix} \frac{1}{2}(\Box\Theta^{\mu\nu} + ES^{\mu\nu}) & \frac{1}{2}HS^{\mu\nu} \\ \frac{1}{2}HS^{\mu\nu} & \frac{1}{2}(\Box\Theta^{\mu\nu} + GS^{\mu\nu}) \end{bmatrix} \ . \tag{2.50}$$

It must to be stressed that the gauge-breaking terms are not present at this moment. If each of the operators $\mathcal{O}^{\mu\nu}$ has the same set of eigenvectors, then we can diagonalize[3] all of them simultaneously, thus finding the fundamental fields of the model.

Let us solve the following equation

$$\begin{vmatrix} \frac{1}{2}(\Box\Theta^{\mu\nu} + ES^{\mu\nu} - \Lambda^{\mu\nu}) & \frac{1}{2}HS^{\mu\nu} \\ \frac{1}{2}HS^{\mu\nu} & \frac{1}{2}(\Box\Theta^{\mu\nu} + GS^{\mu\nu} - \Lambda^{\mu\nu}) \end{vmatrix} = 0 \ , \tag{2.51}$$

where the eigenvalues are written as $\frac{1}{2}\Lambda^{\mu\nu}$. It leads to

$$(\Box\Theta^{\mu\nu} + ES^{\mu\nu} - \Lambda^{\mu\nu})(\Box\Theta^{\mu\nu} + GS^{\mu\nu} - \Lambda^{\mu\nu}) = H^2 S^{\mu\nu} S^{\mu\nu} \ . \tag{2.52}$$

The previous expression represents a set of distinct equations obtained by varying the pairs μ, ν. When computing the determinant of the operator $\mathcal{O}_{\mu\nu}$, one must refrain from performing a summation over repeated indices. Since $\Theta^{\mu\nu}$ e $S^{\mu\nu}$ are independent of each other, it can be seen that the only possible eigenvalue $\Lambda^{\mu\nu}$ is

$$\Lambda^{\mu\nu} = \Box\Theta^{\mu\nu} + \lambda S^{\mu\nu} \ , \tag{2.53}$$

where λ is a constant to be determined. Therefore, it follows that

$$(E-\lambda)(G-\lambda)S^{\mu\nu}S^{\mu\nu} = H^2 S^{\mu\nu} S^{\mu\nu} \ , \tag{2.54}$$

[2] In fact, during propagation, even at the tree level, the field $A\mu$, as it propagates, may transform into the field Z_μ, indicating that A_μ and Z_μ can be expressed in terms of the true fundamental fields.

[3] Although $\mathcal{O}^{\mu\nu}$ is a matrix of operators, there is no issue in applying the usual diagonalization methods since, when performing Fourier transforms, derivatives become momenta, represented by vector components, and are treated as numbers.

which leads to the equation

$$(E-\lambda)(G-\lambda)=H^2 \ . \tag{2.55}$$

Thus, the two eigenvalues of the matrix $\mathcal{O}^{\mu\nu}$ are:

$$\begin{aligned}
\Lambda_1^{\mu\nu} &= \Box\Theta^{\mu\nu}+\frac{1}{2}\left(G+E+\sqrt{(G-E)^2+4H^2}\right)S^{\mu\nu} \ , \\
\Lambda_2^{\mu\nu} &= \Box\Theta^{\mu\nu}+\frac{1}{2}\left(G+E-\sqrt{(G-E)^2+4H^2}\right)S^{\mu\nu} \ ,
\end{aligned} \tag{2.56}$$

So, the orthonormal eigenvectors v_{λ_1} and v_{λ_2} are given by

$$v_{\Lambda_1}=(-\xi,\zeta) \ \ ; \qquad v_{\Lambda_2}=(\zeta,\xi) \ , \tag{2.57}$$

where

$$\xi=\frac{2H}{\sqrt{4H^2+\left[(E-G)-\sqrt{(E-G)^2+4H^2}\right]^2}} \ , \tag{2.58a}$$

$$\zeta=\frac{(E-G)-\sqrt{(E-G)^2+4H^2}}{\sqrt{4H^2+\left[(E-G)-\sqrt{(E-G)^2+4H^2}\right]^2}} \tag{2.58b}$$

Finally, we are able to determine the fundamental gauge fields of the model, which are denoted by W_+^μ and W_-^μ:

$$\begin{bmatrix} W_+^\mu \\ W_-^\mu \end{bmatrix}=\begin{bmatrix} -\xi & \zeta \\ \zeta & \xi \end{bmatrix}\begin{bmatrix} A^\mu \\ Z^\mu \end{bmatrix} \ . \tag{2.59}$$

Consequently, the free bosonic action, which is diagonal in terms of the fundamental vector gauge fields, along with gauge-fixing terms, is given by

$$\begin{aligned}
\Sigma_{W_\pm} &= \int d^3x\bigg\{\frac{1}{2}W_+^\mu\left[\Box\left(\Theta_{\mu\nu}+\frac{1}{\alpha_+}\Omega_{\mu\nu}\right)+M_+S_{\mu\nu}\right]W_+^\nu \\
&\quad +\frac{1}{2}W_-^\mu\left[\Box\left(\Theta_{\mu\nu}+\frac{1}{\alpha_-}\Omega_{\mu\nu}\right)+M_-S_{\mu\nu}\right]W_-^\nu\bigg\} \ , \\
&= \int d^3x\bigg\{-\frac{1}{4}F_+^{\mu\nu}F_{\mu\nu}^+-\frac{1}{4}F_-^{\mu\nu}F_{\mu\nu}^-+\frac{1}{2}M_+\varepsilon_{\mu\rho\nu}W_+^\mu\partial^\rho W_+^\nu \\
&\quad +\frac{1}{2}M_-\varepsilon_{\mu\rho\nu}W_-^\mu\partial^\rho W_-^\nu-\frac{1}{2\alpha_+}(\partial_\mu W_+^\mu)^2-\frac{1}{2\alpha_-}(\partial_\mu W_-^\mu)^2\bigg\} \ ,
\end{aligned} \tag{2.60}$$

In addition, $F_+^{\mu\nu}$ and $F_-^{\mu\nu}$ are the field strengths for the bosons W_+^μ e W_-^μ, whereas

$$M_+=\xi^2E+\zeta^2G-2\xi\zeta H \ ; \qquad M_-=\zeta^2E+\xi^2G+2\xi\zeta H \ . \tag{2.61}$$

As will be seen in the next section, each bosonic propagator will now possess only one pole, not two as was the case with A_μ and Z_μ. Furthermore, as there is no longer a Chern-Simons term involving W_+^μ and W_-^μ, there is no mixed propagator involving these fields, which provides us with another indication that these are the fundamental fields of the model.

Let us analyze the regime where $g^2 >> e^2$. In this case, $\xi \approx H/G << 1$, while $\zeta \approx -1$, which yields

$$M_+ \approx G \; , \qquad M_- \approx E - \frac{H^2}{G} \; . \tag{2.62}$$

In this limit, $M_+ \gg M_-$, and this fact, along with (5.4), allows to identify W_+^μ with Z^μ and W_-^μ with A^μ. An additional observation is that the squared mass M_-^2 approaches to a constant value. On the other hand, μ_-^2, previously identified as the squared mass for A_μ (see eq. (2.33)), grows in proportion to g^2. Hence, when considering the photon as W_- in the limit $g \gg e$, it possesses a constant and slight mass, of order of e^2. This is not the case when A_μ is considered as the photon, as seen in the previous section, since the photon in this context would have a mass that grows with g, although lower then the mass of Z_μ. It is due to the fact that A_μ still has a contribution from W_+, however small it may be.

To conclude this section, we would like to demonstrate how one can obtain the Hall conductivity using the fundamental fields in the limit of $g^2 \gg e^2$. Since W_+^μ and W_-^μ are decoupled, we can easily integrate out the degrees of freedom with masses much greater than M_-, which in this case includes all fermionic degrees of freedom and the degrees of freedom of W_+^μ, thereby obtaining an effective theory for W_-^μ. First, we revert the constants E, G, and H to their values expressed in the model using (2.6). With this consideration, we have

$$M_- = (n_\psi + n_\omega)\frac{e^2}{2\pi} - \frac{\dfrac{n_\omega^2 e^2 g^2}{(2\pi)^2}}{(n_\chi + n_\omega)\dfrac{g^2}{2\pi}} = \frac{e^2}{2\pi}\left(n_\psi + \frac{n_\omega n_\chi}{n_\chi + n_\omega}\right) \; . \tag{2.63}$$

As a consequence, the effective action for W_-^μ, which is identified with the photon, is

$$\Sigma_{\text{photon}} = \int d^3x \Big\{ -\frac{1}{4} F_-^{\mu\nu} F^-_{\mu\nu} + \nu \frac{e^2}{4\pi} \varepsilon_{\mu\rho\nu} W_-^\mu \partial^\rho W_-^\nu - J_-^\mu W_{-_\mu} \Big\} + \cdots \; , \tag{2.64}$$

where $\nu = \left(n_\psi + \frac{n_\chi n_\omega}{n_\chi + n_\omega}\right)$. Taking functional derivatives with respect to W_-^μ, the motion equation and the classical current are obtained in the low energies limit:

$$J_-^\alpha = \partial_\mu F_-^{\mu\alpha} + \nu \frac{e^2}{4\pi} \varepsilon_{\alpha\mu\nu} F_-^{\mu\nu} \; . \tag{2.65}$$

$F_-^{\mu\nu}$ can be identified as the field strength to the photon. In three spacetime dimensions, it is given by [20]

$$F_-^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y \\ E_x & 0 & -B \\ E_y & B & 0 \end{bmatrix} \; . \tag{2.66}$$

The current along the x-direction, i.e., the current considering $\alpha = 1$ in (2.65) is

$$J_{-x} = -\partial_t E_x + \partial_y B + \underbrace{\nu \frac{e^2}{2\pi}}_{\sigma_{xy}} E_y \; . \tag{2.67}$$

It must be stressed that there is an electric current in the x direction induced by an electric field in the y direction. In other words, we obtain an electric current perpendicular to the applied field, which is associated with the Hall effect. Upon restoring the Planck constant, we arrive at the same Hall conductivity with a fractional filling factor as obtained by Kaplan and Sen [13]:

$$\sigma_{xy} = \nu \frac{e^2}{2\pi}, \qquad \text{with } \nu = \left(n_\psi + \frac{\nu_\chi n_\omega}{n_\chi + n_\omega} \right) . \tag{2.68}$$

2.4 Analysis of unitarity

The form of the action $\Sigma_{W_\pm}$ suggests that there will be no issue with unitarity, at least at the tree level, since it exhibits two decoupled Chern-Simons terms. Nonetheless, it is worthwhile to verify whether the necessary condition for unitarity is met. Firstly, let us specify the bosonic propagators of the fundamental fields.

$$\Delta^{++}_{\mu\nu}(k) = -\frac{i}{k^2 - M_+^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - i\frac{\alpha_+}{k^2}\frac{k_\mu k_\nu}{k^2} + \frac{M_+}{k^2(k^2 - M_+^2)}\varepsilon_{\mu\rho\nu}k^\rho , \tag{2.69}$$

$$\Delta^{--}_{\mu\nu}(k) = -\frac{i}{k^2 - M_-^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - i\frac{\alpha_-}{k^2}\frac{k_\mu k_\nu}{k^2} + \frac{M_-}{k^2(k^2 - M_-^2)}\varepsilon_{\mu\rho\nu}k^\rho . \tag{2.70}$$

Here, Δ^{++} is the momentum space propagator for the field W_+, while Δ^{--} is the propagator for W_-. Additionally, Δ^{+-}, which would be the mixed propagator, is zero, simplifying our analysis. As expected, $\Delta^{++}_{\mu\nu}$ has a pole at M_+^2, while $\Delta^{--}_{\mu\nu}$ has a pole at M_-^2, indicating that M_+ is the mass of the W_+^μ field, and M_- is the mass of the W_-^μ field. There are no issues with causality in this case, as the poles are always positive, regardless of the flavors and coupling constants of the model.

For the unitarity analysis, we must once again calculate the current-current transition amplitudes using $k^\mu, \widetilde{k}^\mu, \varepsilon^\mu$. In this situation, considering the massive poles, it is possible to see that $J_+^\mu = C_+\varepsilon^\mu$ and $J_-^\mu = C_-\varepsilon^\mu$. Consequently,

$$\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{++}|_{k^2=M_+^2}\}\} = -J_+^\mu \eta_{\mu\nu} J_+^\nu = C_+^2 > 0 , \tag{2.71}$$

$$\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{--}|_{k^2=M_-^2}\}\} = -J_-^\mu \eta_{\mu\nu} J_-^\nu = C_-^2 > 0 . \tag{2.72}$$

For the massless poles, it is not possible to assume that $k^\mu = (\mu, 0, 0)$, since there is no rest frame. In this context, we assume that $k^\mu = (\mu, 0, \mu)$, yielding the currents decomposed as $J_+^\mu = (\mu A_+, C_+, \mu A_+)$ and $J_-^\mu = (\mu A_-, C_-, \mu A_-)$. Thus, it can be checked that

$$\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{++}|_{k^2=0}\}\} = 0 , \quad \mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{--}|_{k^2=0}\}\} = 0 . \tag{2.73}$$

The previous equation indicates that the massless degrees of freedom of the boson gauge fields do not propagate. It was already expected, because these degrees of freedom were introduced by the gauge-fixing term and are associated to the longitudinal modes.

The Kaplan-Sen model, rewritten in terms of the fundamental gauge fields, fulfills the necessary condition for unitarity, at the tree level, as verified in (2.71) and (2.73). The complete verification of unitarity of the model demands the Froissart-Martin limit [21–23], a procedure that is not made here, but is a perspective of future work.

To conclude this section, it is worth emphasizing that spectrum consistency could only be ensured after the diagonalization procedure. In principle, one might argue that all the previous analyses presented along this chapter would be unnecessary, given that the Kaplan-Sen action includes standard terms, and causality and unitarity should be expected to straightforwardly be guaranteed. However, (2.47) and (2.48) indicate the opposite, as they show the jeopardizing of the unitarity. Furthermore, as demonstrated, there is a difference in the mass of the photon when considering it as W^-_μ instead of A_μ in the limit $g \gg e$, which justifies the work conducted here. Additionally, one's expectations do not necessarily align with reality.

2.5 Transformation of the gauge fields

We conclude the discussion on the model presented in [13] by showing the transformations of the fields W^μ_+ and W^ν_- from the perspective of the $U(1)\times U(1)$ symmetry. The complete action after the diagonalization procedure is

$$\begin{aligned}\Sigma = \Sigma_{W_\pm} + \int d^3x \left\{ \frac{n_\psi}{|n_\psi|} m\bar{\psi}_i\psi_i + \frac{n_\chi}{|n_\chi|} m\bar{\chi}_j\chi_j + \frac{n_\omega}{|n_\omega|} m\bar{\omega}_k\omega_k + i\bar{\psi}_i \not{D}_\psi \psi_i + i\bar{\chi}_j \not{D}_\chi \chi_j \right.\\ \left. + i\bar{\omega}_k \not{D}_\omega \omega_k \right\} ,\end{aligned} \tag{2.74}$$

and the covariant derivatives are

$$\begin{aligned}&\not{D}_\psi = (\not{\partial} - ie\xi \not{W}_+ + ie\zeta \not{W}_-) \ , \quad \not{D}_\chi = (\not{\partial} + ig\zeta \not{W}_+ + ig\xi \not{W}_-) \ ,\\ &\not{D}_\omega = [\not{\partial} + (-ie\xi + ig\zeta)\not{W}_+ + (ie\zeta + ig\xi)\not{W}_-] \ .\end{aligned} \tag{2.75}$$

Here, ξ and ζ are given by (5.64).

Let us suppose that the fermions transform under $U(1)\times U(1)$ as

$$\delta_\pm \psi_i = -iA^\psi_\pm \phi_\pm(x)\psi_i \ , \quad \delta_\pm \chi_i = -iA^\chi_\pm \phi_\pm(x)\chi_i \ , \quad \delta_\pm \omega_i = -iA^\omega_\pm \phi_\pm(x)\omega_i \ , \tag{2.76}$$

where the subscripts + and - refer to the $U(1)$ symmetry involving W^μ_+ and W^μ_-, respectively. The transformation parameters are given by $\phi_\pm(x)$, and the $A^j_\pm$ are the eigenvalues of the generators of $U(1)\times U(1)$.

Our objective is to determine how the boson fields should transform starting from the fermion transformations. Let us begin by determining how W^μ_+ transforms. To do this, we set $\phi_-(x) = 0$ and impose that the covariant derivatives transform in the same way as the fermions. In this way, we have

$$\not{D}'_\psi \psi'_i = e^{-iA^\psi_+\phi_+(x)} \not{D}_\psi \psi_i \ , \tag{2.77}$$

which implies that

$$(\not{\partial} - ie\xi \not{W}'_+) e^{-iA^\psi_+\phi_+(x)} \psi_i = e^{-iA^\psi_+\phi_+(x)} (\not{\partial} - ie\xi \not{W}_+)\psi_i \ . \tag{2.78}$$

The previous equation leads to

$$\not{W}'_+ = \not{W}_+ - \frac{A^\psi_+}{e\xi} \not{\partial}\phi_+(x) \ . \tag{2.79}$$

Repeating this procedure for the fields χ_i and ω_i, the W^μ_+ transformation can be written as follows:

$$\not{W}'_+ = \not{W}_+ + \frac{A^\chi_+}{g\zeta} \not{\partial}\phi_+(x) \ , \qquad \not{W}'_+ = \not{W}_+ + \frac{A^\omega_+}{g\zeta - e\xi} \not{\partial}\phi_+(x) \ . \tag{2.80}$$

Thus, considering the transformation of W_+^μ under $U(1) \times U(1)$ given by

$$W'_{+\mu} = W_{+\mu} + \frac{1}{g}\partial_\mu \phi_+(x) \ , \tag{2.81}$$

and the eigenvalues are

$$A_+^\psi = -\frac{e}{g}\xi \ , \qquad A_+^\chi = \zeta \ , \qquad A_+^\omega = \zeta - \frac{e}{g}\xi \ . \tag{2.82}$$

In a similar way, we can define the transformation of W_- under $U(1) \times U(1)$ as:

$$W'_{-\mu} = W_{-\mu} + \frac{1}{e}\partial_\mu \phi_-(x) \ , \tag{2.83}$$

so that the remaining eigenvalues are

$$A_-^\psi = \zeta \ , \qquad A_-^\chi = \frac{g}{e}\xi \ , \qquad A_-^\omega = \zeta + \frac{g}{e}\xi \ . \tag{2.84}$$

In this way, we have completed the tree-level analysis of the model proposed in [13] in terms of the fundamental fields. It still needs to be studied from the perspective of algebraic renormalization to verify if the classical symmetries can be extended at the quantum level.

Chapter 3

Quantum Parity Conservation in Planar Quantum Electrodynamics[1]

The application of a renormalization procedure to a model can break a classical symmetry, leading to a potential misconception if one interprets this breaking as an anomaly. An anomaly refers to the impossibility of extending a classical symmetry when the model is quantized, and it is independent of the renormalization procedure. Therefore, a breaking due to a renormalization method does not necessarily indicate an anomaly. In general, the algebraic renormalization procedure [16] serves as a useful and powerful tool for verifying whether the observed symmetry breakings result from non-invariant counterterms or genuine anomalies.

In three spacetime dimensions, since the work of reference [25], there was a belief that in order to maintain gauge invariance in a QED_3 model with an odd number of fermions, even perturbatively, parity should necessarily be broken. However, this belief was proven to be unfounded in the references [26, 27], where a massless QED_3 was analyzed using both BPHZL (Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein) renormalization procedure and algebraic renormalization. The BPHZL procedure introduced a parity-violating term, but it turned out to be a non-invariant counterterm rather than an anomaly. Indeed, the symmetry breaking was due to the renormalization method.

When analyzing a QED_3 model through the BPHZL procedure, a natural question that arises, motivated by [26, 27], is whether it breaks parity. We will discuss this issue by applying the BPHZL to the model proposed in reference [14], which consists of a parity-preserving massless pristine-like graphene QED_3, with $U(1) \times U(1)$ symmetry. We will refer to this model as parity-preserving QED_3. It yields interesting results, such as the four-fold broken degeneracy of the Landau levels of the quasiparticles electron-polaron and hole-polaron, a phenomenon observed experimentally in pristine graphene in reference [28]. Furthermore, the model presents a Landau level with zero energy, which suggests the possibility of anomalous quantum Hall effect [14]. The zero-energy Landau level has already been observed in [29].

[1]This chapter is based on the article published in the International Journal of Theoretical Physics, Volume 60, Issue 8, pages 3063–3075, in 2021, with the authors O. M. Del Cima, L. S. Lima, and E. S. Miranda. (Ref. [24])

3.1 The Parity-Preserving QED$_3$

In this section we present the main features of the parity-preserving QED$_3$ proposed in [14]. It consists of a Maxwell-Chern-Simons action, which is invariant under parity, with a $U(1) \times U(1)$. We have already incorporated, preserving parity, the Lowenstein-Zimmermann mass terms, which are required for renormalizing massless theories by BPHZL [30, 31]. A detailed discussion about this renormalizaion procedure will take place in the next sections. It is worth noting that the version of the action used here is the same as that presented in [24], and the original one is recovered when $s \to 1$.

$$\begin{aligned}\Sigma^{(s-1)} = \int d^3x \Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\alpha\nu}A_\mu\partial_\alpha a_\nu + i\overline{\psi}_+ \not{D}\psi_+ \\ & + i\overline{\psi}_- \not{D}\psi_- \underbrace{- m(s-1)\overline{\psi}_+\psi_+ + m(s-1)\overline{\psi}_-\psi_-}_{\text{Lowenstein-Zimmermann mass term}} \\ & + b\partial^\mu A_\mu + \frac{\alpha}{2}b^2 + \overline{c}\Box c + \pi\partial^\mu a_\mu + \frac{\beta}{2}\pi^2 + \overline{\xi}\Box\xi \Big\} \ .\end{aligned} \tag{3.1}$$

The covariant derivatives are $\not{D}\psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm$, whereas m and μ are the mass parameters with mass dimension equal to 1. Additionally, the the coupling constants are e (electric charge) and g (pseudochiral charge[2]), with mass dimension equal to 1/2. The field strengths are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and they are related to the electromagnetic field (A_μ) and the pseudochiral gauge field (a_μ), respectively. Furthermore, the Dirac spinors ψ_+ and ψ_- are two kinds of fermions and the subscript $\pm$ is associated to the sign of the pseudospin [14, 15]. Moreover, the fields c and ξ are two kinds of ghosts[3], whereas $\overline{c}$ and $\overline{\xi}$ are two anti-ghosts. The fields b and π are two fields of Lautrup-Nakanishi [32], which play a role of Lagrange multipliers, fixing the gauge. Finally, the Lowenstein-Zimmermann parameter s take values from 0 to 1 and plays the same role of the external momentum in the BPHZL renormalization procedure as a subtraction variable.

The representation used for the γ matrices in three-dimensional spacetime is $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$, and below are some of their properties that will be useful:

$$\begin{aligned}&\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbb{I} + i\varepsilon^{\mu\nu\alpha}\gamma_\alpha \ , \quad \mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 2\eta^{\mu\nu} \ , \quad \mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\alpha\} = 2i\varepsilon^{\mu\nu\alpha} \ , \\ &\mathrm{Tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_n}\} = \eta^{\mu_{n-1}\mu_n}\mathrm{Tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_{n-2}}\} + i\varepsilon^{\mu_{n-1}\mu_n\alpha}\mathrm{Tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_{n-2}}\gamma_\alpha\} \ .\end{aligned} \tag{3.2}$$

It is interesting to point that the trace (Tr) of the product of an even number of γ matrices does not exhibit the Levi-Civita symbol. On the other hand, an odd number does.

[2] For pseudochiral symmetry, it is possible to express the two spinors, ψ_+ and ψ_-, each comprising two components and forming part of a four-component spinor. The transformation of these fields, when written in this manner, shares the same form as the chiral transformation in four spacetime dimensions, involving the γ_5 matrix. Furthermore, in pristine graphene near the Dirac points, the quasiparticles are massless, and sublattices A and B can be regarded as mirror images of each other.

[3] Despite appearing in the action due to BRS symmetry, neither the ghost fields (c and ξ) nor the antighost fields ($\overline{c}$ and $\overline{\xi}$) will appear in the vacuum polarization diagrams (2-point Green functions for A^μ and a^μ), self-energies (2-point Green functions for fermions), or vertex diagrams, in any perturbative order, since they are free quantum fields, decoupling from the model. They were intentionally added for further algebraic renormalization of the model.

3.2 Classical symmetries: BRS and parity

The action $\Sigma^{(s-1)}$ (3.1) is invariant under the Becchi-Rouet-Stora (BRS) symmetry:

$$
\begin{gathered}
s\psi_+ = i(c+\xi)\psi_+ \ , \quad s\overline{\psi}_+ = -i(c+\xi)\overline{\psi}_+ \ ; \\
s\psi_- = i(c-\xi)\psi_- \ , \quad s\overline{\psi}_- = -i(c-\xi)\overline{\psi}_- \ ; \\
sA_\mu = -\frac{1}{e}\partial_\mu c \ , \quad sc = 0 \ ; \quad sa_\mu = -\frac{1}{g}\partial_\mu \xi \ , \quad s\xi = 0 \ ; \\
s\overline{c} = \frac{b}{e} \ , \quad sb = 0 \ ; \quad s\overline{\xi} = \frac{\pi}{g} \ , \quad s\pi = 0 \ ;
\end{gathered} \tag{3.3}
$$

For a detailed discussion of the BRS symmetry, please refer to the following references: [16, 33, 34]. The action is also invariant under the parity symmetry:

$$
\begin{aligned}
&\psi_+ \xrightarrow{P} \psi_+^P = -i\gamma^1\psi_- \ , \quad \psi_- \xrightarrow{P} \psi_-^P = -i\gamma^1\psi_+ \ , \quad \overline{\psi}_+ \xrightarrow{P} \overline{\psi}_+^P = i\overline{\psi}_-\gamma^1 \ , \\
&\overline{\psi}_- \xrightarrow{P} \overline{\psi}_-^P = i\overline{\psi}_+\gamma^1 \ ; \\
&A_\mu \xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ ; \quad \phi \xrightarrow{P} \phi^P = \phi \ , \quad \phi = \{b, c, \overline{c}\} \ ; \\
&a_\mu \xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ ; \quad \chi \xrightarrow{P} \chi^P = -\chi \ , \quad \chi = \{\pi, \xi, \overline{\xi}\} \ .
\end{aligned} \tag{3.4}
$$

The propagators in momentum space are

$$
\Delta^{\mu\nu}_{AA}(k) = -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\alpha}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \ , \tag{3.5a}
$$

$$
\Delta^{\mu\nu}_{aa}(k) = -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\beta}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \ , \tag{3.5b}
$$

$$
\Delta^{\mu\nu}_{Aa}(k) = \frac{\mu}{k^2(k^2-\mu^2)}\varepsilon^{\mu\alpha\nu}k_\alpha \ , \quad \Delta^{\mu}_{Ab}(k) = \Delta^{\mu}_{a\pi}(k) = \frac{k^\mu}{k^2} \ , \tag{3.5c}
$$

$$
\Delta_{bb}(k) = \Delta_{\pi\pi}(k) = 0 \ , \quad \Delta_{\overline{c}c}(k) = \Delta_{\overline{\xi}\xi}(k) = -\frac{i}{k^2} \ , \tag{3.5d}
$$

$$
\Delta_{++}(k) = i\frac{\not{k} - m(s-1)}{k^2 - m^2(s-1)^2} \ , \quad \Delta_{--}(k) = i\frac{\not{k} + m(s-1)}{k^2 - m^2(s-1)^2} \ . \tag{3.5e}
$$

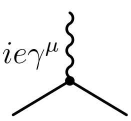
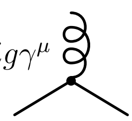
It should be noted that, as observed in Chapter 2, a mixed propagator ($\Delta^{\mu\nu}_{Aa}$) for the two vector gauge fields exists. However, the imaginary part of the current-current transition amplitudes vanishes, making the diagonalization of the action unnecessary.

From now on, all the calculations will be performed considering the Landau gauge, where $\alpha = \beta = 0$.

The diagramatic convensions for the propagators are:

$\Delta^{\mu\nu}_{AA} \equiv$  , $\Delta^{\mu\nu}_{aa} \equiv$  , $\Delta^{\mu\nu}_{Aa} \equiv$  , $\Delta_{\pm\pm} \equiv$ ——— . (3.6)

Additionally, the Feynman rules for the interaction vertices are:

$V_{\pm A^\mu \pm} \equiv$ $ie\gamma^\mu$  , $V_{\pm a^\mu \pm} \equiv$ $\pm ig\gamma^\mu$  . (3.7)

3.3 The Power-Counting, the Subtraction Operator, the Vacuum Polarization and Self-Energy

In order to renormalize the divergences, if they exist, the superficial degrees of divergence of the diagrams have to be determined. For each propagator Δ_{XY}, we can assign an ultraviolet (UV) and an infrared (IR) dimensions d_{XY} and r_{XY}, respectively, given by the propagator behavior on the limits UV ($k,\ s \to \infty$) and IR ($k,\ (s-1) \to 0$). We may define d_{XY} such as $\Delta_{XY}(k) \sim k^{d_{XY}}$ when $k,\ s \to \infty$. On the other hand, we have r_{XY} such as $\Delta_{XY}(k) \sim k^{r_{XY}}$ when $k,\ (s-1) \to 0$. The UV and IR dimensions of the fields, denoted by d and r respectively, are conditioned by the following inequalities [30, 31]:

$$d_X + d_Y \geq 3 + d_{XY} \ , \qquad r_X + r_Y \leq 3 + r_{XY} \ , \tag{3.8}$$

For instance, the propagator (3.5a) in the ultraviolet limit, when $k,\ (s-1) \to 0$, behaves asymptotically as $\Delta^{\mu\nu}_{AA}(k) \sim k^{-2}$, while in the infrared limit, when $k,\ (s-1) \to 0$, it behaves as $\Delta^{\mu\nu}_{AA}(k) \sim k^{0}$. So, we can assign the values $d_{AA} = -2$ and $r_{AA} = 0$ to the UV and IR dimensions of that propagator, respectively. Thus, applying the inequalities (3.8) yields the following relations:

$$2d_A \geq 1 \ , \qquad 2r_A \leq 3 \ . \tag{3.9}$$

The previous procedure can be repeated, allowing us to derive all the inequalities involving the UV and IR dimensions of all fields. The following table summarizes these dimensions.

	ψ_+	ψ_-	A_μ	a_μ	b	π	c	$\bar{c}$	ξ	$\bar{\xi}$	s	$s-1$
d	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	1	0	1	1	1
r	1	1	1	1	1	1	0	1	0	1	0	1

Table 3.1: UV (d) and IR (r) dimensions of the fields.

Taking into account all the previous results, we can determine the power-counting of the model, establishing the superficial degrees of divergence of an arbitrary 1PI diagram γ:

$$\begin{pmatrix} d(\gamma) \\ r(\gamma) \end{pmatrix} = 3 - \sum_f \begin{pmatrix} d_f \\ r_f \end{pmatrix} N_f - \sum_b \begin{pmatrix} d_b \\ \frac{3}{2} r_b \end{pmatrix} N_b + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_e + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_g - N_{Aa} \ . \tag{3.10}$$

In the power-counting formula, N_f and N_b are the numbers of external fermion and boson lines, respectively, while N_{Aa} is the number of internal lines associated with the mixed propagator[4] $\Delta^{\mu\nu}_{Aa}$. Furthermore, N_e and N_g are respectively the power of the coupling constants e and g that appear in the integral corresponding to a diagram. It is interesting to note that the model is super-renormalizable since, by increasing the number of vertices,

[4] Although the current-current transition amplitude is zero, the propagator can appear when considering all possible virtual processes, that is, they must be taken into account in the Feynman diagrams. It is important to note that $\Delta^{\mu\nu}_{Aa} \sim k^{-3}$ in the ultraviolet and $\Delta^{\mu\nu}_{Aa} \sim k^{-1}$. In both cases, the power is one unit lower than in the propagators $\Delta^{\mu\nu}_{AA}$ and $\Delta^{\mu\nu}_{aa}$, which means that diagrams containing internal lines with mixed propagators necessarily have lower superficial degrees of divergence.

we naturally increase the number of coupling constants, which reduces the superficial UV degree of divergence of the diagrams as we increase the loop orders.

The diagrams, at 1-loop, for vacuum polarizations, self-energies, and vertex functions are presented in Figure 3.1. Additionally, using the power-counting formula (3.10), we have shown in Table 3.2 the superficial degrees of UV and IR divergence for each of these diagrams. It should be mentioned that for any diagram $\gamma_{i\pm}$, the subscript $\pm$ refers to the lines of internal or external legs with ψ_+ or ψ_-.

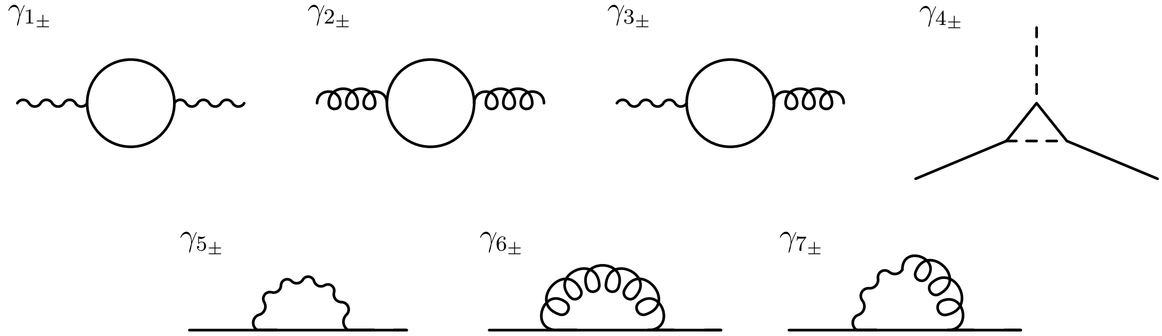


Figure 3.1: The 1-loop diagrams $\gamma_{1\pm}$, $\gamma_{2\pm}$, and $\gamma_{3\pm}$ represent the vacuum polarization tensors, $\gamma_{4\pm}$ represents the interaction vertices, and $\gamma_{5\pm}$, $\gamma_{6\pm}$, and $\gamma_{7\pm}$ are the self-energies. The solid lines represent external legs or the propagators of ψ_+ or ψ_-, while the dashed lines in $\gamma_{4\pm}$ denote propagators of A_μ, a_μ, mixed propagator, or external legs of A_μ or a_μ.

	$\gamma_{1\pm}$	$\gamma_{2\pm}$	$\gamma_{3\pm}$	$\gamma_{4\pm}^{(a)}$	$\gamma_{4\pm}^{(b)}$	$\gamma_{5\pm}$	$\gamma_{6\pm}$	$\gamma_{7\pm}$
d	1	1	1	-1	-2	0	0	-1
r	1	1	1	1	0	2	2	1

Table 3.2: Superficial degrees of dibergence of the diagrams in Fig. 3.1. $\gamma_{4\pm}^{(b)}$ is obtained when considering the internal dashed line as a mixed propagator, whereas $\gamma_{4\pm}^{(a)}$ is obtained when considering Δ_{AA} or Δ_{aa}.

According to the Power-Counting Theorem[5], $\gamma_{4\pm}$ and $\gamma_{7\pm}$ are absolutely convergent, since they possess d lower than zero. On the other hand, the diagrams $\gamma_{1\pm}$, $\gamma_{2\pm}$, $\gamma_{3\pm}$, $\gamma_{5\pm}$ and $\gamma_{6\pm}$ are superficially divergent in the UV limit, and a renormalization procedure must be applied to them. For this purpose, the BPHZL subtraction scheme will be used and it will be discussed in the following section.

3.4 The BPHZL Subtraction Scheme

As mentioned earlier, when a model involves massless fields, an appropriate renormalization procedure is the BPHZL subtraction scheme. This is because of the need to handle the IR divergences that arise due to the renormalization of UV divergences. This is the reason for choosing BPHZL over, for example, the BPHZ [36] (Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein) subtraction scheme. Before presenting the BPHZL method, some definitions need to be introduced. For further details, we refer to the Refs. [30, 31]

[5]The Power-Counting theorem establishes that a diagram without any subdivergences converges absolutely if its superficial UV degree of divergence is negative and its superficial IR degree of divergence is positive. Some details can be found at [35].

Definition 1. *Let Λ be a Feynman diagram. A **subdiagram** λ_i of Λ is a subset of vertices within Λ along with the lines that connect them. A subdiagram is called a **trivial subdiagram** if it either consists of the empty diagram or represents the entire diagram itself. Conversely, a subdiagram is called a **proper diagram** if it is not trivial.*

From now on, a subdiagram has to be intended as a 1PI subdiagram.

Definition 2. *Consider two subdiagrams λ_i and λ_j. Also, consider the following conditions:*

$$\lambda_i \subseteq \lambda_j \ ; \quad \lambda_j \subseteq \lambda_i \ ; \quad \lambda_i \cap \lambda_j = \varnothing \ . \tag{3.11}$$

*If none of these conditions is satisfied, we say that λ_i and λ_j **overlap**. If $\lambda_i \cap \lambda_j = \varnothing$, we say that λ_i and λ_j are **disjoint**.*

Definition 3. *Let $\{\lambda_1 \cdots \lambda_n\}$ be a set of mutually disjoint proper subdiagrams. A **reduced** subdiagram $\Lambda/\lambda_1 \cdots \lambda_n$ is obtained from Λ by contracting all λ_i, $i = 1, \cdots, n$, to points (reduced vertices). Under these conditions, it follows that*

$$d(\Lambda) = d(\Lambda/\lambda_1 \cdots \lambda_n) + \sum_{i=1}^{n} d(\lambda_i) \ , \tag{3.12}$$

$$r(\Lambda) = r(\Lambda/\lambda_1 \cdots \lambda_n) + \sum_{i=1}^{n} r(\lambda_i) \ . \tag{3.13}$$

Definition 4. *Let $U(\Lambda)$ be a set of subdiagrams, proper or trivial, of a diagram Λ. Let $\lambda_i, \lambda_j \subseteq \Lambda$ be two arbitrary elements of $U(\Lambda)$. If λ_i, λ_j do not overlap, for all $\lambda_i, \lambda_j \in U(\Lambda)$, then we say that $U(\Lambda)$ is a Λ-**forest**.*

Definition 5. *Given a subdiagram λ of a diagram Λ, consider the UV and IR subtraction degrees, denoted by $\delta(\lambda)$ and $\rho(\lambda)$, respectively. Let $\{\lambda_1, \cdots, \lambda_n\}$ be an arbitrary set of subdiagrams of λ, and $\lambda/\lambda_1 \cdots \lambda_n$ be an arbitrary reduced diagram. Under these conditions, the following inequalities hold:*

$$\delta(\lambda) \geq d(\lambda) + b(\lambda) \ , \tag{3.14a}$$

$$\rho(\lambda) \leq r(\lambda) - c(\lambda) \ , \tag{3.14b}$$

where $b(\lambda)$ and $c(\lambda)$ are positive integers satisfying the following constraints:

$$\delta(\lambda) \geq d(\lambda/\lambda_1 \cdots \lambda_n) + \sum_{i=1}^{n} \delta(\lambda_i) \ , \tag{3.15a}$$

$$\rho(\lambda) \leq r(\lambda/\lambda_1 \cdots \lambda_n) + \sum_{i=1}^{n} \rho(\lambda_i) \ , \tag{3.15b}$$

$$\rho(\lambda) \leq \delta(\lambda) + 1 \ . \tag{3.15c}$$

Now we are in a position to define how to subtract the UV divergences of a divergent diagram using the BPHZL procedure. Consider an arbitrary n-loop diagram Λ, divergent in the ultraviolet, with the integral given by

$$\int d^3k_1 \cdots d^3k_n I_\Lambda(p, k, s) \ , \tag{3.16}$$

where k_i represent internal momenta, p represents external momenta, and s is the Lowenstein-Zimmermann parameter. The renormalized integrand $R_\Lambda(p,k,s)$, is given by the Zimmermann's forest formula:

$$R_\Lambda(p,k,s) = S_\Lambda \sum_{U\in\mathcal{F}_\Lambda} \prod_{\lambda\in U} (-\tau_\lambda S_\lambda) I_\Lambda(p,k,s) \ . \tag{3.17}$$

Here, S_γ is the substitution operator, which alters the p, k, and s, in the lines corresponding to the subdiagram γ, to p^γ, k^γ, and s^γ, while $\mathcal{F}_\Lambda$ is the family of all the forests of Λ. Additionally, τ_λ is the operator that acts in the terms corresponding to γ, defined by

$$1 - \tau_\lambda = \left(1 - t^{\rho(\lambda)-1}_{p^\lambda,(s^\lambda-1)}\right)\left(1 - t^{\delta(\lambda)}_{p^\lambda,s^\lambda}\right) \ , \tag{3.18}$$

where

$$t^d_{x,y} = \begin{cases} \text{Taylor polynomial of degree } d \text{ around } x = y = 0 \text{ if } d \geq 0 \\ 0 \text{ se } d < 0 \end{cases} \ . \tag{3.19}$$

Furthermore, we can define $-\tau_\lambda = 1$ if $\lambda = \varnothing$. Another important point to mention is that if a forest contains λ_i, λ_j, such that $\lambda_i \subset \lambda_j$, then, in the product in (3.17), we should start from the smaller to the larger, that is, $(-\tau_{\lambda_i} S_{\lambda_i})$ should act first on $I_\Lambda(p,k,s)$ than $(-\tau_{\lambda_j} S_{\lambda_j})$. We emphasize that $\varnothing$ is also considered a forest. Finally, it is important to note that forests containing at least one convergent subdiagram do not contribute to (3.17). Hence, we take into consideration forests that contain only divergent subdiagrams, which are called the renormalization parts of Λ. Once the subtraction procedure is performed, the limit $s = 1$ has to be taken, recovering the massless case.

An important fact to note is that 1-loop diagrams have no proper subdiagrams. In this case, the renormalized integrand is just

$$R_\gamma(p,k,s) = \left(1 - t^{\rho(\gamma)-1}_{p,s-1}\right)\left(1 - t^{\delta(\gamma)}_{p,s}\right) I_\gamma(p,k,s) \ , \tag{3.20}$$

and the constraints (3.15) become

$$\begin{gathered} \delta(\gamma) = d(\gamma) + b(\gamma) \ , \quad \rho(\gamma) = r(\gamma) - c(\gamma) \ ; \\ \delta(\gamma) \geq d(\gamma) \ , \quad \rho(\gamma) \leq r(\gamma) \ , \quad \rho(\gamma) \leq \delta(\gamma) + 1 \ . \end{gathered} \tag{3.21}$$

A last comment regarding the BPHZL subtraction scheme is that local invariant (by the symmetries) counterterms have to be considered in the effective action and they must be fixed by suitable normalization conditions, similar to what was discussed in [16] in the context of the BPHZ subtraction scheme.

3.5 BPHZL at 1-loop in the Model

3.5.1 The vacuum polarization tensors

The renormalization procedures for $\gamma_{1\pm}$, $\gamma_{2\pm}$, and $\gamma_{3\pm}$ are similar, as all these diagrams have the same loop structure, and their integrands are the same, except for factors dependent on the constants $\pm e^2$, $\pm g^2$, and $\mp eg$, respectively. Let us analyze first $\gamma_{1\pm}$. The vacuum polarization tensors, $\Pi^{\mu\nu}_{\gamma_{1\pm}}(p,s)$, in this situation, are

$$\Pi^{\mu\nu}_{\gamma_{1\pm}}(p,s) = \int \frac{d^3k}{(2\pi)^3} \underbrace{\left\{-e^2\mathrm{Tr}\left[\gamma^\mu \frac{\not{k} \mp m(s-1)}{k^2 - m^2(s-1)^2}\gamma^\nu \frac{\not{k} - \not{p} \mp m(s-1)}{(k-p)^2 - m^2(s-1)^2}\right]\right\}}_{I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)} . \quad (3.22)$$

The negative sign before e^2 arises due to the fermion loop.

Taking into account the conditions in (3.21), with $b(\gamma_{1\pm}) = c(\gamma_{1\pm}) = 0$, the subtraction degrees are $\delta(\gamma_{1\pm}) = \rho(\gamma_{1\pm}) = 1$. Consequently, the subtracted (renormalized) 1-loop integrands by BPHZL, expressed in terms of the original integrand $I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)$, are:

$$\begin{aligned} R^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) &= (1 - t^0_{p,s-1})(1 - t^1_{p,s}) I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) \\ &= (1 - t^0_{p,s-1} - t^1_{p,s} + t^0_{p,s-1}t^1_{p,s}) I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) . \end{aligned} \quad (3.23)$$

The terms on the right-hand side of the previous equation are given by:

$$t^0_{p,s-1} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) = I(0,k,1) , \quad (3.24)$$

$$t^1_{p,s} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) = I^{\mu\nu}_{\gamma_{1\pm}}(0,k,0) + p^\rho \frac{\partial}{\partial p^\rho} I^{\mu\nu}_{\gamma_{1\pm}}\bigg|_{\substack{p=0\\s=0}} + s\frac{\partial}{\partial s} I^{\mu\nu}_{\gamma_{1\pm}}\bigg|_{\substack{p=0\\s=0}} , \quad (3.25)$$

$$t^0_{p,s-1} t^1_{p,s} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) = I^{\mu\nu}_{\gamma_{1\pm}}(0,k,0) + \frac{\partial}{\partial s} I^{\mu\nu}_{\gamma_{1\pm}}\bigg|_{\substack{p=0\\s=0}} . \quad (3.26)$$

As already mentioned previously, it is necessary to take $s = 1$ after writing the renormalized integrand to recover the massless case. Hence, using (4.18), (3.25) and (3.26), the renormalized integrand $R^{\mu\nu}_{\gamma_{1\pm}}(p,k,1)$ reads

$$R^{\mu\nu}_{\gamma_{1\pm}}(p,k,1) = \underbrace{I^{\mu\nu}_{\gamma_{1\pm}}(p,k,1)}_{\text{paridade par}} - \underbrace{I^{\mu\nu}_{\gamma_{1\pm}}(0,k,1)}_{\text{paridade par}} - \underbrace{p^\rho \frac{\partial}{\partial p^\rho} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)}_{\text{paridade ímpar}}\Bigg|_{p=s=0} , \quad (3.27)$$

where

$$I^{\mu\nu}_{\gamma_{1\pm}}(p,k,1) = -e^2\mathrm{Tr}\left\{\gamma^\mu \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k} - \not{p}}{(k-p)^2}\right\} , \quad (3.28)$$

$$I^{\mu\nu}_{\gamma_{1\pm}}(0,k,1) = -e^2\mathrm{Tr}\left\{\gamma^\mu \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k}}{k^2}\right\} , \quad (3.29)$$

$$p^\rho \frac{\partial}{\partial p^\rho} I^{\mu\nu}_{\gamma_{1\pm}}\bigg|_{p=s=0} = -e^2\mathrm{Tr}\left\{\gamma^\mu \frac{\not{k} \mp m}{k^2 - m^2}\gamma^\nu \left[\frac{-\not{p}}{k^2 - m^2} + \frac{2p\cdot k(\not{k} \mp m)}{(k^2 - m^2)^2}\right]\right\} . \quad (3.30)$$

Additionally, the vacuum polarization tensor is given by

$$\Pi^{(R)\mu\nu}_{\gamma_{1\pm}}(p,s) = \int \frac{d^3k}{(2\pi)^3} R^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) . \quad (3.31)$$

After performing the calculations, using the results of Appendix B, it is possible to see that

$$\Pi^{(R)\mu\nu}_{\gamma_{1\pm}}(p,1) = -\frac{e^2}{16}\frac{\eta^{\mu\nu}p^2 - p^\mu p^\nu}{\sqrt{p^2}} \mp \frac{e^2 m}{4\pi|m|}\varepsilon^{\mu\alpha\nu}p_\alpha . \quad (3.32)$$

The term with the Levi-Civita in the equation above was responsible for the parity breaking by the BPHZL method in [26]. However, it could be reabsorbed into the quantum action as a non-invariant counterterm [26, 27], thus not being an anomaly; it was the BPHZL method that was responsible for breaking parity in that case. However, here, a slightly different process occurs. If we sum the contributions from $\Pi^{(R)\mu\nu}_{\gamma 1+}$ and $\Pi^{(R)\mu\nu}_{\gamma 1-}$, we see that the terms with the Levi-Civita cancel each other mutually.

Following the same procedure for the diagrams $\gamma_{2\pm}$ and $\gamma_{3\pm}$, it can be concluded that

$$\Pi^{(R)\mu\nu}_{\gamma_{2\pm}}(p,1) = -\frac{g^2}{16}\frac{\eta^{\mu\nu}p^2 - p^\mu p^\nu}{\sqrt{p^2}} \mp \frac{g^2 m}{4\pi|m|}\varepsilon^{\mu\alpha\nu}p_\alpha \; , \tag{3.33}$$

$$\Pi^{(R)\mu\nu}_{\gamma_{3\pm}}(p,1) = \frac{eg}{16}\frac{\eta^{\mu\nu}p^2 - p^\mu p^\nu}{\sqrt{p^2}} \pm \frac{egm}{4\pi|m|}\varepsilon^{\mu\alpha\nu}p_\alpha \; . \tag{3.34}$$

Again, considering $\Pi^{(R)\mu\nu}_{\gamma 2+} + \Pi^{(R)\mu\nu}_{\gamma 2-}$ and $\Pi^{(R)\mu\nu}_{\gamma 3+} + \Pi^{(R)\mu\nu}_{\gamma 3-}$, it is possible to see that the terms containing the Levi-Civita tensor cancel mutually. Therefore, there is no Levi-Civita tensor in the total vacuum polarization at 1-loop, when the contribution of all vacuum polarization diagrams is considered, and BPHZL does not break parity in this case, differently from what happened in [26]. It is still necessary to analyze the self-energies diagrams at 1-loop and the vacuum polarization diagrams at 2-loops, as will be seen.

3.5.2 The Self-Energies

In this section, we analyze the BPHZL subtraction method applied to the diagrams $\gamma_{5\pm}$ and $\gamma_{6\pm}$, which correspond to self-energy diagrams with a superficial degree of UV divergence equal to zero. Some details will be omitted since they are quite similar to those in the previous section. The analyzes for $\gamma_{5\pm}$ and $\gamma_{6\pm}$ are entirely analogous, as the corresponding integrals differ only by the constants involving e^2 and g^2.

Let us analyze $\gamma_{5\pm}$. The corresponding integrands are

$$I_{\gamma_{5\pm}} = \left\{-e^2\gamma^\mu\left[\frac{1}{k^2-\mu^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\right]\left[\frac{(\not{k} - \not{p}) \mp m(s-1)}{(k-p)^2 - m^2(s-1)^2}\right]\gamma^\nu\right\} \; , \tag{3.35}$$

and the self-energies are

$$\Sigma(\gamma_{5\pm}) = \int \frac{d^3k}{(2\pi)^3} I_{\gamma_{5\pm}} \; . \tag{3.36}$$

Again, it is needed to derive the subtraction degrees, respecting the consistence conditions expressed in (3.21). They are $\delta(\gamma_{5\pm}) = 0$ and $\rho(\gamma_{5\pm}) = 1$. The application of the BPHZL procedure yields

$$\begin{aligned} R_{\gamma_{5\pm}}(p,k,s) &= (1 - t^0_{p,s-1})(1 - t^0_{p,s}) I_{\gamma_{5\pm}}(p,k,s) \\ &= (1 - t^0_{p,s-1} - t^0_{p,s} + t^0_{p,s-1}t^0_{p,s}) I_{\gamma_{5\pm}}(p,k,s) \; . \end{aligned} \tag{3.37}$$

Once more, it is necessary to take $s = 1$, obtaining

$$R_{\gamma_{5\pm}}(p,k,1) = I_{\gamma_{5\pm}}(p,k,1) - I_{\gamma_{5\pm}}(0,k,1) \; , \tag{3.38}$$

where

$$I_{\gamma_{5\pm}}(p,k,1) = 2e^2\left\{\frac{1}{k^2-\mu^2}\frac{1}{(k-p)^2}\left[\not{k} - \frac{\not{k}(k\cdot p)}{k^2}\right]\right\} \ , \tag{3.39}$$

$$I_{\gamma_{5\pm}}(0,k,1) = 2e^2\frac{\not{k}}{k^2(k^2-\mu^2)} \ . \tag{3.40}$$

Furthermore, the expressions for the contributions to the renormalized self-energies $\Sigma^{(R)}_{\gamma_{5\pm}}(p,s)$ are defined by

$$\Sigma^{(R)}_{\gamma_{5\pm}}(p,s) = \int \frac{d^3k}{(2\pi)^3}\ R_{\gamma_{5\pm}}(p,k,s), \tag{3.41}$$

Using the results of Appendic B again, we have

$$\begin{aligned}\Sigma^{(R)}_{\gamma_{5\pm}} = -\frac{ie^2\not{p}}{4\pi}\Bigg[&\frac{1}{4\sqrt{p^2}}\left(\frac{p^2}{\mu^2}+\frac{3\mu^2}{p^2}+2\right)\ln\left(\frac{\mu^2-p^2}{(\sqrt{\mu^2}-\sqrt{p^2})^2}\right)\\ &-\frac{|\mu|}{2}\left(\frac{1}{\mu^2}+\frac{3}{p^2}\right)+i\pi\frac{p^2}{4\mu^2\sqrt{p^2}}\Bigg] \ ,\end{aligned} \tag{3.42}$$

with $\Sigma^{(R)}_{\gamma_{5\pm}} \equiv \Sigma^{(R)}_{\gamma_{5\pm}}(p,1)$.

Similarly, the renormalized self-energies for $\gamma_{6\pm}$, denoted by $\Sigma^{(R)}_{\gamma_{6\pm}}$, are

$$\begin{aligned}\Sigma^{(R)}_{\gamma_{6\pm}} = -\frac{ig^2\not{p}}{4\pi}\Bigg[&\frac{1}{4\sqrt{p^2}}\left(\frac{p^2}{\mu^2}+\frac{3\mu^2}{p^2}+2\right)\ln\left(\frac{\mu^2-p^2}{(\sqrt{\mu^2}-\sqrt{p^2})^2}\right)\\ &-\frac{|\mu|}{2}\left(\frac{1}{\mu^2}+\frac{3}{p^2}\right)+i\pi\frac{p^2}{4\mu^2\sqrt{p^2}}\Bigg] \ .\end{aligned} \tag{3.43}$$

Thus, the self-energy for ψ_+ at 1-loop is $\Sigma^{(R)}_{+} = \Sigma^{(R)}_{\gamma_{5+}} + \Sigma^{(R)}_{\gamma_{6+}}$, whereas for ψ_- it is $\Sigma^{(R)}_{-} = \Sigma^{(R)}_{\gamma_{5-}} + \Sigma^{(R)}_{\gamma_{6-}}$. Therefore, it follows that

$$\Sigma^{(R)}_{+} = \Sigma^{(R)}_{-} = \frac{(e^2+g^2)}{4\pi}\not{p}\ \mathcal{O}(p^2,\mu) \ , \tag{3.44}$$

where

$$\begin{aligned}\mathcal{O}(p^2,\mu) = -i\Bigg[&\frac{1}{4\sqrt{p^2}}\left(\frac{p^2}{\mu^2}+\frac{3\mu^2}{p^2}+2\right)\ln\left(\frac{\mu^2-p^2}{(\sqrt{\mu^2}-\sqrt{p^2})^2}\right)\\ &-\frac{|\mu|}{2}\left(\frac{1}{\mu^2}+\frac{3}{p^2}\right)+i\pi\frac{p^2}{4\mu^2\sqrt{p^2}}\Bigg] \ .\end{aligned} \tag{3.45}$$

The terms in the quantum action due to the self-energies are $\overline{\psi}_+\Sigma^{(R)}_{+}\psi_+$ and $\overline{\psi}_-\Sigma^{(R)}_{-}\psi_-$. It is needed to analyze them under parity transformations. Evidently, $\mathcal{O}(p^2,\mu)$ does not break the parity symmetry. Hence, it is necessary to analyze only the transformation of the terms $\overline{\psi}_+\not{p}\psi_+$ and $\overline{\psi}_-\not{p}\psi_-$. Using (3.2) and (3.4), we have

$$\begin{aligned}\overline{\psi}_+\not{p}\psi_+ \overset{P}{\longmapsto}\ & i\overline{\psi}_-\gamma^1(\gamma^0p_0-\gamma^1p_1+\gamma^2p_2)(-i\gamma^1\psi_-)\\ =\ &\overline{\psi}_-(\gamma^1\gamma^0\gamma^1p_0-\gamma^1\gamma^1\gamma^1p_1+\gamma^1\gamma^2\gamma^1p_2)\psi_-\\ =\ &\overline{\psi}_-(-\gamma^1\gamma^1\gamma^0p_0+\gamma^1p_1-\gamma^1\gamma^1\gamma^2p_2)\psi_-\\ =\ &\overline{\psi}_-(\gamma^0p_0+\gamma^1p_1+\gamma^2p_2)\psi_- = \overline{\psi}_-\not{p}\psi_- \ .\end{aligned} \tag{3.46}$$

similarly,

$$\overline{\psi}_-\not{p}\psi_- \stackrel{P}{\longmapsto} \overline{\psi}_+\not{p}\psi_+ \ . \tag{3.47}$$

Therefore, considering all the 1-loop self-energies, it follows that

$$\overline{\psi}_+\Sigma_+^{(R)}\psi_+ + \overline{\psi}_-\Sigma_-^{(R)}\psi_- \stackrel{P}{\longmapsto} \overline{\psi}_-\Sigma_-^{(R)}\psi_- + \overline{\psi}_+\Sigma_+^{(R)}\psi_+ \ . \tag{3.48}$$

Thus, we conclude that BPHZL does not break parity when applied to the self-energies in this model. Since it also does not break parity for the vacuum polarization, it preserves the parity symmetry in this model at 1-loop. It remains to verify only the 2-loop diagrams.

3.6 Parity Analysis at 2-loops

To complete the proof that BPHZL does not break parity in parity-preserving QED$_3$ [14], it is still necessary to investigate the potentially divergent diagrams at the 2-loops level. There are no divergent diagrams at superior orders and the model is super-renormalizable.

Let us assume that a term Λ containing $\varepsilon^{\mu\alpha\nu}A_\mu p_\alpha A_\nu$ or $\varepsilon^{\mu\alpha\nu}a_\mu p_\alpha a_\nu$ might appear in the quantum action after applying BPHZL at 2-loops [6]. Λ could be written as

$$\Lambda = A_\mu \Lambda^{\mu\nu} A_\nu \ , \tag{3.49}$$

where $\Lambda^{\mu\nu}$ is the 1PI Feynman diagram, obtained from $\dfrac{\delta^2\Gamma}{\delta A_\mu \delta A_\nu}$, except by a constant factor, being Γ the quantum action. The superficial and the actual degrees of divergence of $\Lambda^{\mu\nu}$ are d and d', respectively, and we have $d' \leq d$. By the power-counting formula, it follows that

$$d(\Lambda^{\mu\nu}) = 3 - N_f d_f - d_b N_b - \frac{1}{2}N_e - \frac{1}{2}N_g - N_{Aa} \ . \tag{3.50}$$

According to Table 3.2 and the power-counting formula, a diagram of 2-loops with bosonic external legs has $d = 0$, since there are four coupling constants in this case, namely e^4, e^2g^2, or g^4. So, it follows that $d' \leq 0$. Hence, if $A_\mu\Lambda^{\mu\nu}A_\nu$ yields a term that contains $\varepsilon^{\mu\alpha\nu}A_\mu p_\alpha A_\nu$ or $\varepsilon^{\mu\alpha\nu}a_\mu p_\alpha a_\nu$, it has to be at least as $\varepsilon^{\mu\alpha\nu}A_\mu \frac{p_\alpha}{\sqrt{p^2}} A_\nu$ or $\varepsilon^{\mu\alpha\nu}a_\mu \frac{p_\alpha}{\sqrt{p^2}} a_\nu$, to respect the degree of divergence of $\Lambda^{\mu\nu}$. Therefore, if there is a term that breaks parity, it has to be non-local[7], which would not renormalize any parameters of the model. Thus, through the analysis of UV dimensions, we conjecture that BPHZL does not break parity.

Complementing the analysis involving the UV dimensions that we conducted, it is opportune to investigate the tensorial structure of the vacuum polarization diagrams, represented in the Figure 3.2, at 2-loops. There are 36 types of vacuum polarization diagrams, with 12 of them being convergent, namely, those represented by $\gamma_{10_\pm}$ and $\gamma_{13_\pm}$, whose superficial UV degrees of divergence are $d(\gamma_{10_\pm}) = d(\gamma_{13_\pm}) = -1$. In contrast, $\gamma_{8_\pm}$, $\gamma_{9_\pm}$, $\gamma_{11_\pm}$, and $\gamma_{12_\pm}$ are potentially logarithmically divergent since $d(\gamma_{8_\pm}) = d(\gamma_{9_\pm}) = d(\gamma_{11_\pm}) = d(\gamma_{12_\pm}) = 0$ and they need to be renormalized.

---

[6] The term $\varepsilon^{\mu\alpha\nu}A_\mu p_\alpha a_\nu$ can be discarded as it is even under parity

[7] A local operator, when evaluated at a point x in spacetime, depends solely on that point and its neighborhood. Non-local operators, on the other hand, when evaluated at x, have contributions from other points that are beyond the vicinity of x. For example, consider $\frac{1}{p^2}$, which corresponds to the operator $\frac{1}{\Box}$ in configuration space. Since $\Box_x \frac{1}{\Box_x} = 1$, $\Box_x G(x-y) = \delta(x-y) \Rightarrow \Box_x \int dy, G(x-y) = 1$, thus $\frac{1}{\Box_x} = \int dy G(x-y)$, meaning that $\frac{1}{\Box_x}$ depends on all points in spacetime due to the integration over y and is therefore non-local.

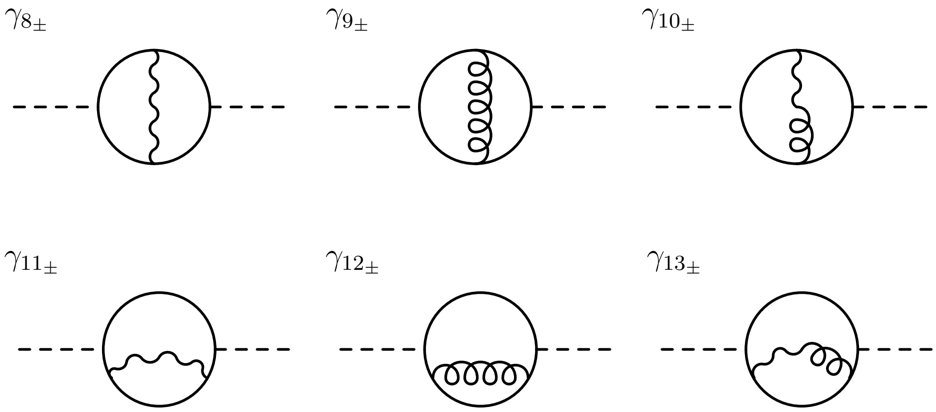


Figure 3.2: Vacumm polarizations at 2-loops, whose continuous lines represent propagators of ψ_+ ou de ψ_-, while dashed lines represent external legs of the fields A_μ or a_μ.

In order to verify whether there can be a parity violation, it is convenient to write the integrals corresponding to these diagrams:

$$\Pi^{\mu\nu}_{\gamma_{8\pm}}(p,s) = \lambda_8^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\; e^2\; \widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ , \tag{3.51}$$

$$\Pi^{\mu\nu}_{\gamma_{9\pm}}(p,s) = \lambda_9^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\; g^2\; \widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ ; \tag{3.52}$$

and

$$\Pi^{\mu\nu}_{\gamma_{11\pm}}(p,s) = \lambda_{11}^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\; e^2\; \widetilde{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ , \tag{3.53}$$

$$\Pi^{\mu\nu}_{\gamma_{12\pm}}(p,s) = \lambda_{12}^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\; g^2\; \widetilde{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ ; \tag{3.54}$$

where $\lambda_i = e$ $(i = 8, 9, 11, 12)$ if the two external legs are A_μ and, if they are then a_μ, $\lambda_i = g$. Besides,

$$\begin{aligned}\widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s) = -\mathrm{Tr}\Bigg\{ &\gamma^\mu \left[i\frac{\not{k}_1 \mp m(s-1)}{k_1^2 - m^2(s-1)^2}\right] \gamma_\alpha \left[-i\frac{1}{(k_1-k_2)^2-\mu^2} \times \right.\\ &\times \left. \left(\eta^{\alpha\beta} - \frac{(k_1^\alpha - k_2^\alpha)(k_1^\beta - k_2^\beta)}{(k_1-k_2)^2}\right)\right] \left[i\frac{\not{k}_2 \mp m(s-1)}{k_2^2 - m^2(s-1)^2}\right] \gamma^\nu \times \\ &\times \left[i\frac{(\not{k}_2 - \not{p}) \mp m(s-1)}{(k_2-p)^2 - m^2(s-1)^2}\right] \gamma_\beta \left[i\frac{(\not{k}_1 - \not{p}) \mp m(s-1)}{(k_1-p)^2 - m^2(s-1)^2}\right] \Bigg\}\ ,\end{aligned} \tag{3.55}$$

whereas

$$
\begin{aligned}
\widetilde{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s) = -\mathrm{Tr}\Big\{ \gamma^{\mu}\left[i\frac{\not{k}_1 \mp m(s-1)}{k_1^2 - m^2(s-1)^2}\right] \gamma^{\nu} \left[i\frac{(\not{k}_1 - \not{p}) \mp m(s-1)}{(k_1-p)^2 - m^2(s-1)^2}\right] \times \\
\times \gamma_{\alpha}\left[-i\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^{\alpha}k_2^{\beta}}{k_2^2}\right)\right]\left[i\frac{(\not{k}_1 - \not{k}_2 - \not{p}) \mp m(s-1)}{(k_1-k_2-p)^2 - m^2(s-1)^2}\right] \times \\
\times \gamma_{\beta}\left[i\frac{(\not{k}_1 - \not{p}) \mp m(s-1)}{(k_1-p)^2 - m^2(s-1)^2}\right]\Big\} \ . \qquad (3.56)
\end{aligned}
$$

Here, p is the external momentum, and the subscripts $+$ and $-$ refer to the internal lines ψ_+ and ψ_-, respectively

Observing the integrands above, it is possible to see that traces of four to eight γ matrices will be generated. However, only traces of five and seven γ matrices produce Levi-Civita symbols. Furthermore, each of the terms in the integrand corresponding to the fermionic propagators can contribute with a maximum of one γ matrix. Taking, for example, the integrand $\widetilde{I}^{\mu\nu}_{\pm}$, one of the parts containing the trace of five γ matrices is:

$$
\begin{aligned}
\mathcal{Z}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = -\mathrm{Tr}\{\gamma^{\mu}\ [\mp\ im(s-1)]\ \gamma^{\nu}\ [\mp im(s-1)]\ \gamma_{\alpha}\ [\ \Delta^{\alpha\beta}(k_1,k_2)\ ] \\
\times\ [\ \mp im(s-1)\ ]\ \gamma_{\beta}\ [\ i\ (\not{k}_1 - \not{p})\ ]\} \ ,
\end{aligned} \qquad (3.57)
$$

which can be written as

$$
\mathcal{Z}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = \pm\varepsilon^{\mu\nu\rho}\mathcal{X}_{5\rho}(k_1,k_2,p,s) + \mathcal{Y}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) \ . \qquad (3.58)
$$

Here, the first term has odd parity while the second one has even parity. Employing the same strategy for all terms that can yield a trace of five matrices γ, we obtain:

$$
\widehat{I}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = \pm\varepsilon^{\mu\nu\rho}\widehat{\mathcal{A}}_{5\rho}(k_1,k_2,p,s) + \widehat{\mathcal{S}}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) \ , \qquad (3.59)
$$

$$
\widetilde{I}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = \pm\varepsilon^{\mu\nu\rho}\widetilde{\mathcal{A}}_{5\rho}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) \ , \qquad (3.60)
$$

where $\widehat{\mathcal{S}}^{\mu\nu}_{5\pm}$ and $\widetilde{\mathcal{S}}^{\mu\nu}_{5\pm}$ are parity-even tensors, and the subscripts $+$ and $-$ refer to the internal lines associated with ψ_+ and ψ_-, respectively. The subscript with the numeral 5 is used to remind us that we are dealing only with the part related to the trace of 5 matrices γ. Consequently, the total parts of the integrands concerning the trace of 5 matrices γ, given by $\widehat{I}^{\mu\nu}_{5} = \widehat{I}^{\mu\nu}_{5+} + \widehat{I}^{\mu\nu}_{5-}$ and $\widetilde{I}^{\mu\nu}_{5} = \widetilde{I}^{\mu\nu}_{5+} + \widetilde{I}^{\mu\nu}_{5-}$, are

$$
\widehat{I}^{\mu\nu}_{5}(k_1,k_2,p,s) = \widehat{\mathcal{S}}^{\mu\nu}_{5+}(k_1,k_2,p,s) + \widehat{\mathcal{S}}^{\mu\nu}_{5-}(k_1,k_2,p,s) \ , \qquad (3.61)
$$

$$
\widetilde{I}^{\mu\nu}_{5}(k_1,k_2,p,s) = \widetilde{\mathcal{S}}^{\mu\nu}_{5+}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}^{\mu\nu}_{5-}(k_1,k_2,p,s) \ , \qquad (3.62)
$$

Consequently, none of the terms in the integrand with the structure of five γ contain a Levi-Civita symbol. We can repeat the previous approach for terms containing the trace of seven γ matrices, obtaining:

$$
\widehat{I}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) = \pm\varepsilon^{\mu\nu\rho}\widehat{\mathcal{A}}_{7\rho}(k_1,k_2,p,s) + \widehat{\mathcal{S}}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) \ , \qquad (3.63)
$$

$$
\widetilde{I}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) = \pm\varepsilon^{\mu\nu\rho}\widetilde{\mathcal{A}}_{7\rho}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) \ , \qquad (3.64)
$$

Once again, we conclude that there is no generation of terms involving Levi-Civita, which allows us to ensure that parity is not violated[8], even at 2-loops, using the BPHZL method. Thus, we conclude that the BPHZL method does not break parity in parity-preserving QED_3, as further investigation into higher loop orders is not required. The preservation of the parity by BPHZL yields also as a consequence that, when analyzing the parity preserving QED_3 in the realm of algebraic renormalization, only parity even terms can be generated in the quantum action.

[8]If there was a violation, it would be due to the introduction of Lowenstein-Zimmermann terms.

Chapter 4

2-loops Renormalization of a Parity-Preserving Quantum Planar Electrodynamics by BPHZL[1]

In the previous chapter, we discussed the preservation of the parity symmetry by BPHZL when applied to the parity-preserving QED_3. In this chapter, we will apply the BPHZL subtraction scheme to the divergent diagrams at 2-loops. Many results used here have been established in the Appendices, especially in Appendix B. The detailed description of the BPHZL method was provided in the previous chapter.

4.1 The Zimmermann's Forests

As discussed in the previous chapter, the diagrams shown in Figure 3.2 that require renormalization are $\gamma_{8_\pm}$, $\gamma_{9_\pm}$, $\gamma_{11_\pm}$, and $\gamma_{12_\pm}$. The first step in applying the BPHZL subtraction scheme at the 2-loop level is to identify the subdiagrams and determine whether they are divergent, indicating the presence of subdivergences within the complete diagram. It is also necessary to check whether the subdiagrams overlap to construct the Zimmermann's forests. Notably, $\gamma_{8_\pm}$ and $\gamma_{9_\pm}$ are topologically equivalent, and in the Landau gauge, their differences stem solely from distinct coupling constants derived from the Feynman rules. Similar conclusions can be drawn for $\gamma_{11_\pm}$ and $\gamma_{12_\pm}$.

Let us start analyzing the subdiagrams originated from $\gamma_{8_\pm}$. In the following figure, all the proper subdiagrams of $\gamma_{8_\pm}$ are presented.

---

[1] This chapter is based on an article currently in the writing phase, titled "The BPHZL Renormalization of Parity-Preserving Massless Planar Quantum Electrodynamics," authored by D.O.R. Azevedo, O. M. Del Cima, L. S. Lima, and E. S. Miranda.

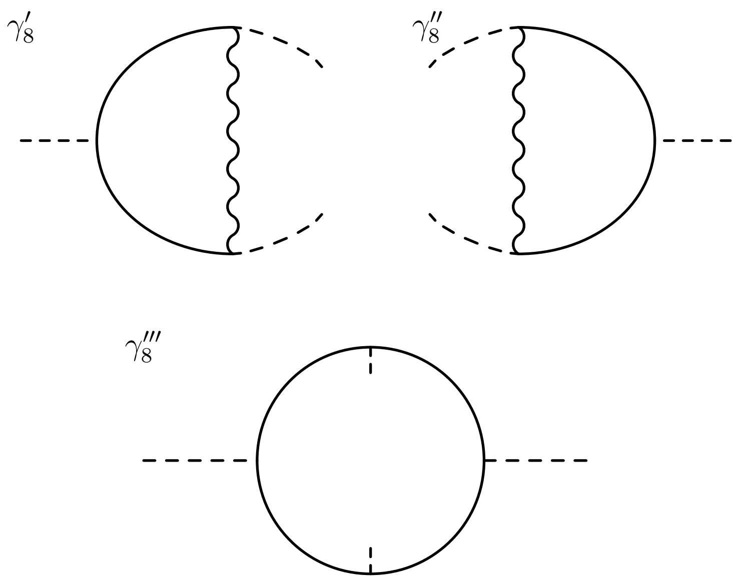


Figure 4.1: 1PI subdiagrams originated from $\gamma_{8_\pm}$. Dashed lines must be treated as external lines for these subdiagrams, even though some them are internal lines for the complete diagram.

As can be seen, γ_8' and γ_8'' are 1-loop vertex-type diagrams, and they are convergent. Additionally, γ_8''' is also convergent. Thus, there are no subdivergences for γ_8, and the BPHZL subtraction will be entirely analogous to that performed at the 1-loop. This becomes evident when we determine the family of forests for γ_8:

$$\mathcal{F}(\gamma_{8_\pm}) = \{\varnothing, \{\gamma_{8_\pm}\}\} \ , \tag{4.1}$$

Besides, as $\gamma_{9_\pm}$ is topologically equivalent to $\gamma_{8_\pm}$, it follows that the family of forests for this diagram is

$$\mathcal{F}(\gamma_{9_\pm}) = \{\varnothing, \{\gamma_{9_\pm}\}\} \ , \tag{4.2}$$

Let us repeat the previous approach, but now considering $\gamma_{11\pm}$. In this context, the subdiagrams are represented in the following figure.

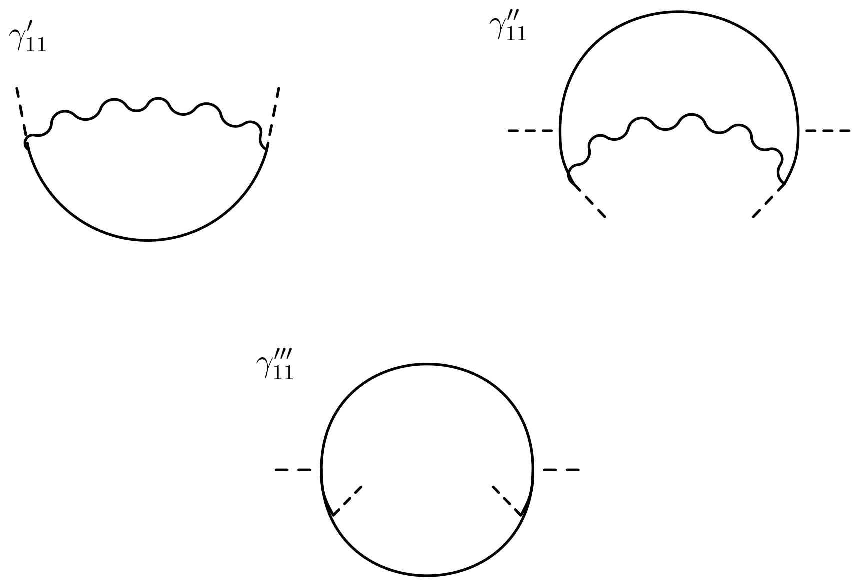


Figure 4.2: 1PI subdiagrams related to the diagrams $\gamma_{11_\pm}$. Dashed lines must be treated as external lines in these subdiagrams, even though some of them are internal lines when considering the complete diagram.

Now the situation is slightly different. The subdiagram γ'_{11} is a self-energy diagram at 1-loop and is divergent, as verified in the previous chapter. On the other hand, the subdiagrams γ''_{11} and γ'''_{11} are similar to γ'''_{8} and are convergent. Therefore, the family of forests for the diagram $\gamma_{11_\pm}$ is

$$\mathcal{F}(\gamma_{11_\pm}) = \{\varnothing, \{\gamma'_{11_\pm}\}, \{\gamma_{11_\pm}\}, \{\gamma'_{11_\pm}, \gamma_{11_\pm}\}\} \ . \tag{4.3}$$

Considering $\gamma_{12_\pm}$ and denoting the self-energy-like subdiagram as $\gamma'_{12_\pm}$, the family of forests in this case is given by

$$\mathcal{F}(\gamma_{12_\pm}) = \{\varnothing, \{\gamma'_{12_\pm}\}, \{\gamma_{12_\pm}\}, \{\gamma'_{12_\pm}, \gamma_{12_\pm}\}\} \ . \tag{4.4}$$

4.2 Renormalization of the diagrams $\gamma_{11_\pm}$ and $\gamma_{12_\pm}$

Let us start from the renormalization of $\gamma_{11_\pm}$ and $\gamma_{12_\pm}$. Despite having subdivergences, their renormalization is simpler than the renormalization of $\gamma_{8_\pm}$ and $\gamma_{9_\pm}$. Taking into account the Zimmermann's forest formula (3.17), the renormalized integrand for $\gamma_{11_\pm}$ reads:

$$R_{\gamma_{11_\pm}}(p, k_1, k_2, s) = \lambda_{11} e^2 S_{\gamma_{11_\pm}} \sum_{U \in \mathcal{F}_{\gamma_{11_\pm}}} \prod_{\lambda \in U} (-\tau_\lambda S_\lambda) \tilde{I}^{\mu\nu}_{\pm}(k_1, k_2, p, s) \ , \tag{4.5}$$

where we have used (3.53) and (3.56). As already discussed, τ_λ is related to the Taylor operator that acts on p^λ and s^λ, being p^λ the external momentum of the subdiagram whereas s^λ is its Lowenstein-Zimmermann parameter. Considering $\gamma'_{11_\pm}$, let us denote by $p^{\gamma'_{11\pm}}$ its external momentum, $k^{\gamma'_{11\pm}}$ its internal momentum and $s^{\gamma'_{11\pm}}$ as its Lowenstein-Zimmermann parameter. In this case, it is necessary to know the results of applying the substitution operator $S_{\gamma'_{11_\pm}}$ in $\tilde{I}^{\mu\nu}_{\pm}(k_1, k_2, p, s)$, which means basically to determine $p^{\gamma'_{11\pm}}$, $k^{\gamma'_{11\pm}}$ and $s^{\gamma'_{11\pm}}$ as functions of p, k_1, k_2 and s, allowing to change the latter variables by the variables of the subdiagram in the terms that correspond to it. It is possible to determine $p^{\gamma'_{11\pm}}$ and $k^{\gamma'_{11\pm}}$ following a similar approach as in Ref. [37]. The next figure shows how to relate the quantities of the diagram and its subdiagram:

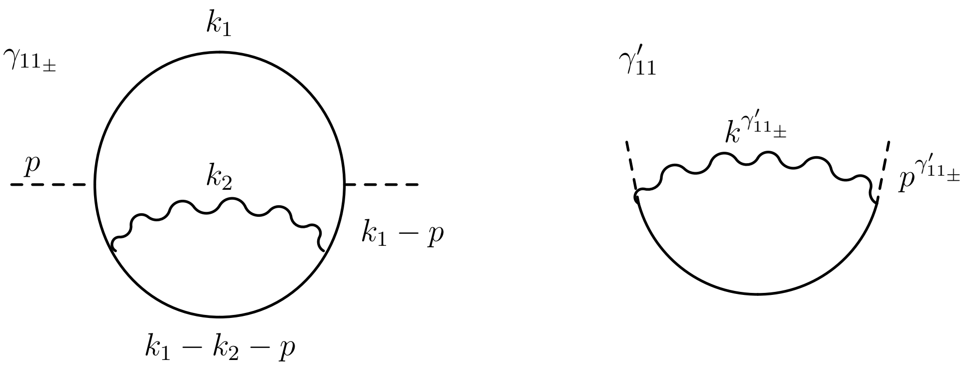


Figure 4.3: Complete diagram $\gamma_{11_\pm}$ and its subdivergent diagram, with the momenta explicit.

Comparing both, it is possible to see that

$$k^{\gamma'_{11\pm}} = k_2 \ ; \qquad p^{\gamma'_{11\pm}} = k_1 - p \ . \tag{4.6}$$

Additionally, the parameters s that are in the expression of $\gamma_{11_\pm}$ but result from the structure of $\gamma'_{11_\pm}$ must be replaced by $s^{\gamma'_{11\pm}}$.

The application of the BPHZL procedure also requires the determination of the subtraction degrees δ (UV) and ρ (IR), obeying the constraints (3.14a), (3.14b) and (3.15). For the divergent subdiagram, it is possible to utilize those obtained for the self-energy at 1-loop. Therefore,

$$\delta(\gamma'_{11_\pm}) = 0 \; ; \qquad \rho(\gamma'_{11_\pm}) = 1 \; . \tag{4.7}$$

For the complete diagram $\gamma_{11_\pm}$, there are the following constraints:

$$\delta\left(\gamma_{11_\pm}\right) = d\left(\gamma_{11_\pm}\right) + b\left(\gamma_{11_\pm}\right) \;\; , \tag{4.8}$$

$$\delta\left(\gamma_{11_\pm}\right) \geq d\left(\gamma_{11_\pm}/\gamma'_{11_\pm}\right) + \delta\left(\gamma'_{11_\pm}\right) \;\; , \tag{4.9}$$

$$\rho\left(\gamma_{11_\pm}\right) = r\left(\gamma_{11_\pm}\right) - c\left(\gamma_{11_\pm}\right) \;\; , \tag{4.10}$$

$$\rho\left(\gamma_{11_\pm}\right) \leq r\left(\gamma_{11_\pm}/\gamma'_{11_\pm}\right) + \rho\left(\gamma'_{11_\pm}\right) \;\; , \tag{4.11}$$

$$\rho\left(\gamma_{11_\pm}\right) \leq \delta\left(\gamma_{11_\pm}\right) + 1 \; . \tag{4.12}$$

The reduced diagram, when $\gamma'_{11_\pm}$ is contracted to a point, is shown in the following figure:

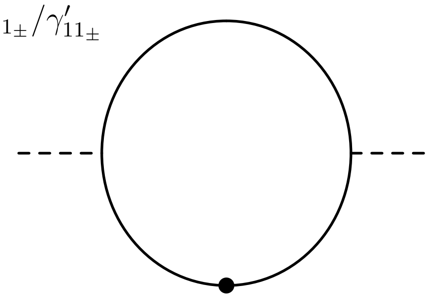


Figure 4.4: Subdiagrama reduzido.

To derive $d(\gamma_{11_\pm}/\gamma'_{11_\pm})$ and $r(\gamma_{11_\pm}/\gamma'_{11_\pm})$, it is still possible to use the power-counting formula (3.10), bearing in mind that there is the addition of one fermionic propagator, due to the contraction of the divergent subdiagram, which reduces one unit of d and r. Thus, it is possible to see that

$$d(\gamma_{11_\pm}/\gamma'_{11_\pm}) = 0 \; , \qquad r(\gamma_{11_\pm}/\gamma'_{11_\pm}) = 0 \tag{4.13}$$

So, the constraints become

$$\begin{aligned}
&\delta\left(\gamma_{11_\pm}\right) = d\left(\gamma_{11_\pm}\right) + b\left(\gamma_{11_\pm}\right) \;\; , \\
&\delta\left(\gamma_{11_\pm}\right) \geq +\delta\left(\gamma'_{11_\pm}\right) \;\; , \\
&\rho\left(\gamma_{11_\pm}\right) = r\left(\gamma_{11_\pm}\right) - c\left(\gamma_{11_\pm}\right) \;\; , \\
&\rho\left(\gamma_{11_\pm}\right) \leq \rho\left(\gamma'_{11_\pm}\right) \;\; , \\
&\rho\left(\gamma_{11_\pm}\right) \leq \delta\left(\gamma_{11_\pm}\right) + 1 \; .
\end{aligned} \tag{4.14}$$

So, taking $c(\gamma_{11_\pm}) = b(\gamma_{11_\pm}) = 0$, it follows that

$$\delta(\gamma_{11_\pm}) = 0 \; , \qquad \rho(\gamma_{11_\pm}) = 1 \; . \tag{4.15}$$

The next step is to determine the Taylor operators $\tau_{\gamma_{11_\pm}}$ and $\tau_{\gamma'_{11_\pm}}$. Since the subtraction degrees are the same for both the complete diagram and the subdiagram, determining $\tau_{\gamma_{11_\pm}}$ automatically determines $\tau_{\gamma'_{11_\pm}}$. For $\gamma_{11\pm}$, the Taylor operator reads

$$\begin{aligned} 1-\tau_{\gamma_{11}} =& \left(1-t_{p,(s-1)}^{\rho(\gamma_{11\pm})-1}\right)\left(1-t_{p,s}^{\delta(\gamma_{11\pm})}\right) \\ =& 1-t^0_{p,(s-1)}-t^0_{p,s}+t^0_{p,(s-1)}t^0_{p,s} \\ =& 1-t^0_{p,(s-1)} \; . \end{aligned} \tag{4.16}$$

The aforementioned expression was obtained using the fact that $t^0_{p,s}$ annihilates p and s, such that $t^0_{p,(s-1)}t^0_{p,s}=t^0_{p,s}$, because $t^0_{p,(s-1)}$ acts on a expression that has neither p nor s, effectively becoming the identity operator. It can be concluded that:

$$\tau_{\gamma_{11}} = t^0_{p,(s-1)} \; , \qquad \tau_{\gamma'_{11}} = t^0_{p^{\gamma'_{11}},(s^{\gamma'_{11}}-1)} \; . \tag{4.17}$$

Now we are in position of determining the renormalized integrand for γ_{11}. Expanding the expression (4.5), we get

$$\frac{R^{\gamma_{11}}_{\mu\nu}(p,k_1,k_2,s)}{\lambda_{11}e^2} = [1-\tau_{\gamma_{11}}-S_{\gamma_{11}}\tau_{\gamma'_{11}}S_{\gamma'_{11}}-\tau_{\gamma_{11}}S_{\gamma_{11}}\tau_{\gamma'_{11}}S_{\gamma'_{11}}]\tilde{I}_{\mu\nu}(p,k_1,k_2,s) \; . \tag{4.18}$$

It remains to determine the terms of $\widetilde{I}^{\mu\nu}(p,k_1,k_2,s)$ which correspond to the subdiagram γ'_{11}. Observing the expression in (3.56), it is possible to see that it is situated between γ_α and γ_β. Using the trace properties, together with (4.6), it is possible to obtain the following equality:

$$S_{\gamma'_{11}}\widetilde{I}^{\mu\nu}(p,k_1,k_2,s) = -\mathrm{Tr}\{I^{\mu\nu}_{\gamma_{11}/\gamma'_{11}}(p,k_1,s)\, I^{\mu\nu}_{\gamma'_{11}}(p^{\gamma_{11}},k^{\gamma_{11}},s^{\gamma_{11}})\} \; , \tag{4.19}$$

where

$$\begin{aligned} I^{\mu\nu}_{\gamma_{11}/\gamma'_{11}}(p,k_1,s) = & \left[i\frac{(\not{k}_1-\not{p})\mp m(s-1)}{(k_1-p)^2-m^2(s-1)^2}\right]\gamma^\mu\left[i\frac{\not{k}_1\mp m(s-1)}{k_1^2-m^2(s-1)^2}\right]\times \\ & \times\gamma^\nu\left[i\frac{(\not{k}_1-\not{p})\mp m(s-1)}{(k_1-p)^2-m^2(s-1)^2}\right] \; , \end{aligned} \tag{4.20}$$

while

$$\begin{aligned} I^{\mu\nu}_{\gamma'_{11}}(p^{\gamma_{11}},k^{\gamma_{11}},s^{\gamma_{11}}) =& \gamma_\alpha\left[-i\frac{1}{(k^{\gamma'_{11}})^2-\mu^2}\left(\eta^{\alpha\beta}-\frac{(k^{\gamma'_{11}})^\alpha(k^{\gamma'_{11}})^\beta}{(k^{\gamma'_{11}})^2}\right)\right]\times \\ & \times\left[i\frac{(\not{p}^{\gamma'_{11}})-(\not{k}^{\gamma'_{11}})\mp m(s-1)}{((p^{\gamma'_{11}})-(k^{\gamma'_{11}}))^2-m^2(s-1)^2}\right]\gamma_\beta \; . \end{aligned} \tag{4.21}$$

Now it is possible to expand the non-trivial terms of (4.18):

$$\tau_{\gamma_{11}}\widetilde{I}_{\gamma_{11}} = i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{k_1^\theta k_1^\omega k_1^\xi(k_1^\chi-k_2^\chi)}{k_1^2k_1^2k_1^2(k_1-k_2)^2}\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta}-\frac{k_2^\alpha k_2^\beta}{k_2^2}\right) \; , \tag{4.22}$$

$$
\begin{aligned}
S_{\gamma_{11}}(\tau_{\gamma'_{11}}S_{\gamma'_{11}}\widetilde{I}_{\gamma_{11}}) = {} & i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{(k_1^\theta - p^\theta)(k_1^\xi - p^\xi)}{((k_1-p)^2 - m^2(s-1)^2)^2}\times \\
& \times\frac{k_1^\omega}{k_1^2 - m^2(s-1)^2}\,\frac{k_2^\chi}{k_2^2}\,\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right) + \\
& + 2i\,\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{(k_1^\xi - p^\xi)}{((k_1-p)^2 - m^2(s-1)^2)^2}\,\frac{m^2(s-1)^2}{k_1^2 - m^2(s-1)^2}\times \\
& \times\frac{k_2^\chi}{k_2^2}\,\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right) + \\
& + i\,\mathrm{Tr}[\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{k_1^\omega}{((k_1-p)^2 - m^2(s-1)^2)^2}\,\frac{m^2(s-1)^2}{k_1^2 - m^2(s-1)^2}\times \\
& \times\frac{k_2^\chi}{k_2^2}\,\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right) .
\end{aligned}
\tag{4.23}
$$

$$
\tau_{\gamma_{11}}S_{\gamma_{11}}(\tau_{\gamma'_{11}}S_{\gamma'_{11}}\widetilde{I}_{\gamma_{11}}) = i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{k_1^\theta k_1^\omega k_1^\xi}{k_1^2 k_1^2 k_1^2}\,\frac{k_2^\chi}{k_2^2}\,\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right) .
\tag{4.24}
$$

It is still necessary to take the limit $s \to 1$ to recover the massless regime. The terms in (4.18) depending on s become

$$
\begin{aligned}
I_{\pm}^{\mu\nu}(p, k_1, k_2, s=1) = {} & -\mathrm{Tr}\left\{\left(i\frac{(\not{k}_1 - \not{p})}{(k_1-p)^2}\right)\gamma_\mu\left(i\frac{\not{k}_1}{k_1^2}\right)\gamma_\nu\left(i\frac{(\not{k}_1 - \not{p})}{(k_1-p)^2}\right)\times\right. \\
& \left.\times\,\gamma_\alpha\left[-i\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right)\right]\left(i\frac{(\not{k}_1 - \not{k}_2 - \not{p})}{(k_1-k_2-p)^2}\right)\gamma_\beta\right\} \\
= {} & i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{(k_1^\theta - p^\theta)}{(k_1-p)^2}\,\frac{k_1^\omega}{k_1^2}\,\frac{(k_1^\xi - p^\xi)}{(k_1-p)^2}\,\frac{(k_1^\chi - k_2^\chi - p^\chi)}{(k_1-k_2-p)^2}\times \\
& \times\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right) ,
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
S_{\gamma_{11}}(\tau_{\gamma'_{11}}S_{\gamma'_{11}}\widetilde{I}_{\gamma_{11}})|_{s=1} = {} & i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\frac{(k_1^\theta - p^\theta)(k_1^\xi - p^\xi)}{((k_1-p)^2)^2}\,\frac{k_1^\omega}{k_1^2}\,\frac{k_2^\chi}{k_2^2} \\
& \frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right) .
\end{aligned}
\tag{4.26}
$$

The renormalization of $\gamma_{11_\pm}$ by BPHZL now is summarized in considering the expressions (4.25), (4.22), (4.24) and (4.26). Several results of Appendix B will be used to solve the integrals, such as the Feynman parametrization and the J_r integrals. It is worth emphasizing that these expressions do not break parity, because there are not traces of five or seven γ matrices, which agrees with the conclusions of the previous chapter.

It must be recalled that the integrands are integrated with respect to the momentum k_1 and k_2. Regarding the subtraction terms originated from the divergent subdiagram, namely (4.24) and (4.26), it is possible to see that (4.24) has a power three in the momentum k_1 in the numerator, and after the Feynman parametrization and considering the integration over the internal momentum, it can be written as

$$\int \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} \frac{k_1^\theta k_1^\omega k_1^\xi f(k_2)}{(k_1^2 - c)^\alpha} \ , \tag{4.27}$$

where $f(k_2)$ encompasses all the other terms, such as the integration in the Feynman parameters and c has no momentum k_1. Therefore, the integration with respect to k_1 vanishes, because there ir a odd number of momentum k_1 in the numerator and there is no term of the form $k_1 \cdot p$ in the denominator. By a similar argument, considering the integration over k_2, it follows that the contribution of (4.26) vanishes. Thus, the subtraction terms due to the divergent subdiagram vanish.

It remains the following expressions, that effectively contribute to the renormalized integrand:

$$\tau_{\gamma_{11}} \widetilde{I}_{\gamma_{11}} = i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta] \frac{k_1^\theta k_1^\omega k_1^\xi (k_1^\chi - k_2^\chi)}{k_1^2 k_1^2 k_1^2 (k_1 - k_2)^2} \frac{1}{k_2^2 - \mu^2} \left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2} \right) \ , \tag{4.28}$$

$$\begin{aligned} \widetilde{I}_{\gamma_{11}} = i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta] \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{(k_1^\theta - p^\theta)}{(k_1 - p)^2} \frac{k_1^\omega}{k_1^2} \times \\ \times \frac{(k_1^\xi - p^\xi)}{(k_1 - p)^2} \frac{(k_1^\chi - k_2^\chi - p^\chi)}{(k_1 - k_2 - p)^2} \frac{1}{k_2^2 - \mu^2} \left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2} \right) \ . \end{aligned} \tag{4.29}$$

It is possible to perform a change of variables in the integral $\widetilde{I}_{\gamma_{11}}$, making $k_1 - p \to k_1$, obtaining

$$\begin{aligned} \widetilde{I}_{\gamma_{11}} = i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta] \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{k_1^\theta}{k_1^2} \frac{(k_1^\omega - p^\omega)}{(k_1 - p)^2} \times \\ \times \frac{k_1^\xi}{k_1^2} \frac{(k_1^\chi - k_2^\chi)}{(k_1 - k_2)^2} \frac{1}{k_2^2 - \mu^2} \left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2} \right) \ . \end{aligned} \tag{4.30}$$

In this context, the renormalized diagram becomes

$$\begin{aligned} \frac{\gamma_{11}^{(R)}}{\lambda_{11} e^2} = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{R_{\mu\nu}^{\gamma_{11}}}{\lambda_{11} e^2} &= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \left(\widetilde{I}_{\mu\nu}(p, k_1, k_2, 1) - \tau_{\gamma_{11}} \widetilde{I}_{\mu\nu}(p, k_1, k_2, 1) \right) \\ &= i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta] \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \times \\ &\times \frac{k_1^\theta k_1^\xi (k_1^\chi - k_2^\chi)(\eta^{\alpha\beta} k_2^2 - k_2^\alpha k_2^\beta)(2k_1^\omega k_1^\sigma p_\sigma - k_1^\omega p^2 - p^\omega k_1^2)}{(k_1^2)^3 (k_1 - p)^2 k_2^2 (k_1 - k_2)^2 (k_2^2 - \mu^2)} \ . \end{aligned} \tag{4.31}$$

As can be checked, the previous integral is convergent, since it behaves as $1/k$ at large momenta.

The following Feynman parametrization (see Eq. (B.17)) is necessary:

$$\begin{aligned} \frac{1}{(k_1^2)^3 (k_1 - p)^2 k_2^2 (k_1 - k_2)^2 (k_2^2 - \mu^2)} = \frac{\Gamma(7)}{\Gamma(3)} \int dx\, dy\, dv\, dw\, dz \times \\ \times \frac{\delta(1 - x - y - v - w - z) x^2}{(x k_1^2 + y((k_1 - p)^2) + v k_2^2 + w(k_1 - k_2)^2 + z(k_2^2 - \mu^2))^7} . \end{aligned} \tag{4.32}$$

Here, the Feynman parameters are x , y, v, w and z and they range from 0 to 1. Using the previous result, along with (4.31), and integrating over the parameter v, we obtain the following expression for the renormalized diagram:

$$\frac{\gamma_{11}^{(R)}}{\lambda_{11}e^2} = \int d\Theta \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{k_1^\theta k_1^\xi(k_1^\chi - k_2^\chi)(\eta^{\alpha\beta}k_2^2 - k_2^\alpha k_2^\beta)(2k_1^\omega k_1^\sigma p_\sigma - k_1^\omega p^2 - p^\omega k_1^2)}{(k_2^2 + 2p'\cdot k_2 - c)^7} . \tag{4.33}$$

where

$$d\Theta = dx\, dy\, dw\, dz \frac{x^2}{(1-x-y)^7} i\,\mathrm{Tr}[\gamma_\theta\gamma_\mu\gamma_\omega\gamma_\nu\gamma_\xi\gamma_\alpha\gamma_\chi\gamma_\beta]\,\frac{\Gamma(7)}{\Gamma(3)} , \tag{4.34}$$

whereas

$$p' = -\frac{w}{1-x-y}k_1 = \qquad c' = \frac{1}{1-x-y}(\mu^2 z - (w+x+y)k_1^2 - 2yk_1\cdot p - yp^2) . \tag{4.35}$$

It is also possible to define

$$a = -\frac{w}{1-x-y} . \tag{4.36}$$

From the results of the Appendix B, integrating first over k_2, it follows that

$$\frac{\gamma_{11}^{(R)}}{\lambda_{11}e^2} = \int d\Theta \frac{d^3k_1}{(2\pi)^3}\frac{i(-1)^7\pi^{3/2}}{\Gamma(7)(2\pi)^3}k_1^\theta k_1^\xi \eta_{\rho\sigma}(-k_1^\rho k_1^\sigma p^\omega + 2k_1^\rho k_1^\omega p^\sigma - k_1^\omega p^\rho p^\sigma)[I_{k_2}^{\alpha\beta\chi}(J_2) + I_{k_2}^{\alpha\beta\chi}(J_3)] , \tag{4.37}$$

where

$$I_{k_2}^{\alpha\beta\chi}(J_2) = \left[(c' + p'^2)^{-\frac{11}{2}}\Gamma\left(\frac{11}{2}\right)(p'^2\eta^{\alpha\beta} - p'^\alpha p'^\beta) - (c' + p'^2)^{-\frac{9}{2}}\Gamma\left(\frac{9}{2}\right)\eta^{\alpha\beta}\right]k_1^\chi , \tag{4.38}$$

$$\begin{aligned} I_{k_2}^{\alpha\beta\chi}(J_3) = -&\left[-(c' + p'^2)^{-\frac{11}{2}}\Gamma\left(\frac{11}{2}\right)p'^\chi(p'^2\eta^{\alpha\beta} - p'^\alpha p'^\beta) + \right. \\ &\left. + \frac{1}{2}(c' + p'^2)^{-\frac{9}{2}}\Gamma\left(\frac{9}{2}\right)(4\eta^{\alpha\beta}p'^\chi - \eta^{\chi\alpha}p'^\beta - \eta^{\beta\chi}p'^\alpha)\right] . \end{aligned} \tag{4.39}$$

It is worth noting that $\eta_{\rho\sigma}$ appeared in (4.37) to write all the momenta with contravariant components. This will also be used in the following, making $p'^2 = \eta_{\kappa\gamma}p'^\kappa p'^\gamma$.

For simplification, the change $k_1 \to k$ is made. Substituting c' and p', we have

$$c' + p'^2 = u\left(k^2 + 2k\cdot p'' - c''\right) , \tag{4.40}$$

with u, p'' and c'' given by:

$$\begin{gathered} u = \left(\frac{w^2 - (1-x-y)(w+x+y)}{(1-x-y)^2}\right) \qquad p'' = -\frac{2y(1-x-y)}{w^2 - (1-x-y)(w+x+y)}p \\ c'' = \frac{(1-x-y)(yp^2 - z\mu^2)}{w^2 - (1-x-y)(w+x+y)} \end{gathered} \tag{4.41}$$

The form of (4.40) allows to integrate over k, using the J_r integrals once more. In the end of the day, the renormalized diagram is

$$
\begin{aligned}
\frac{\gamma_{11}^{(R)}}{\lambda_{11}e^2} = \int d\Theta' \eta_{\rho\sigma} \Bigg\{ \Gamma\left(\frac{11}{2}\right) \frac{a^2(1+a)}{u^{\frac{11}{2}}} \Big[&- \Big(\eta^{\alpha\beta}\eta_{\kappa\gamma} J_7^{\theta\xi\rho\sigma\kappa\gamma\chi} + \\
&- J_7^{\theta\xi\rho\sigma\alpha\beta\chi} \Big) p^\omega + 2\left(\eta^{\alpha\beta}\eta_{\kappa\gamma} J_7^{\theta\xi\rho\omega\kappa\gamma\chi} + \right. \\
&- \left. J_7^{\theta\xi\rho\omega\alpha\beta\chi} \right) p^\sigma - \left(\eta^{\alpha\beta}\eta_{\kappa\gamma} J_6^{\theta\xi\omega\kappa\gamma\chi} - J_6^{\theta\xi\omega\alpha\beta\chi} \right) p^\rho p^\sigma \Big] + \\
&- \Gamma\left(\frac{9}{2}\right) \frac{1}{2u^{\frac{9}{2}}} \Big[- \Big(2(1+2a)\eta^{\alpha\beta} J_5^{\theta\xi\rho\sigma\chi} - a\eta^{\chi\alpha} J_5^{\theta\xi\rho\sigma\beta} + \\
&- a\eta^{\beta\chi} J_5^{\theta\xi\rho\sigma\alpha} \Big) p^\omega + 2\left(2(1+2a)\eta^{\alpha\beta} J_5^{\theta\xi\rho\omega\chi} - a\eta^{\chi\alpha} J_5^{\theta\xi\rho\omega\beta} + \right. \\
&- \left. a\eta^{\beta\chi} J_5^{\theta\xi\rho\omega\alpha} \right) p^\sigma - \left(2(1+2a)\eta^{\alpha\beta} J_4^{\theta\xi\omega\chi} - a\eta^{\chi\alpha} J_4^{\theta\xi\omega\beta} + \right. \\
&- \left. a\eta^{\beta\chi} J_4^{\theta\xi\omega\alpha} \right) p^\rho p^\sigma \Big] \Bigg\}
\end{aligned} \tag{4.42}
$$

with J_7 and J_6 being functions of $p'', c'', \alpha = 11/2$, while J_5 and J_4 are functions of $p'', c'', \alpha = 9/2$. Furthermore, $d\Theta'$ is $d\Theta$ with the constant factors absorbed, encompassing all the integration elements of the Feynman parameters x, y, w and z.

We prove that all the terms resulting from the renormalized diagram $\gamma_{11_\pm}$ are non-local. From (B.7), the terms that contain a J_7 are given by ($\lfloor \cdot \rfloor$ denotes the floor function):

$$
\begin{aligned}
J_7^{\mu_1 \cdots \mu_r}(11/2, p'', c'') = i(-1)^{25/2} \frac{\pi^{3/2}}{\Gamma(\alpha)(2\pi)^3} \sum_{j=0}^{\lfloor \frac{7}{2} \rfloor} \left(-\frac{1}{2}\right)^j \Gamma(11/2 - 3/2 - j) \times \\
\times (c'' + p''^2)^{3/2 - 11/2 + j} \left(\Omega(p)_{7-2j}^{\mu_1 \cdots \mu_{7-2j}} \zeta_{2j}^{\mu_{7-2j+1} \cdots \mu_7} + \text{dist. perm.} \right) .
\end{aligned} \tag{4.43}
$$

The highest j is $\lfloor 7/2 \rfloor = 3$. Therefore, the term $(c'' + p''^2)$ has the maximum power given by -1, which results in a denominator containing terms with p^2, where p is the external momentum. This yields non-local terms in configuration space. By a similar argument, it is also possible to check that the terms containing J_6, J_5 and J_4 yield non-local terms. Thus, the renormalized diagram $\gamma_{11_\pm}$ does not contribute to the renormalization of any parameters of the model. It is worth to mention, that invariant counterterms by the symmetries could be added to the action, and they must be fixed by suitable normalization conditions. However, as will be shown in Chapter 5, using the algebraic renormalization procedure, these invariant counterterms are zero.

The BPHZL renormalization of $\gamma_{12_\pm}$ is entirely analogous, and allows us to conclude that this diagram will also yield only non-local terms.

4.3 Renormalization of the diagrams $\gamma_{8_\pm}$ and $\gamma_{9_\pm}$

Although the expressions of the forests corresponding to $\gamma_{8_\pm}$ and $\gamma_{9_\pm}$ are simpler than those for $\gamma_{10_\pm}$ and $\gamma_{11_\pm}$, their renormalization procedures are more challenging due to the greater number of terms that are generated. It is worth noting that the expressions of $\gamma_{8_\pm}$ and $\gamma_{9_\pm}$ are the same, except for a constant factor. Hence, it is sufficient to renormalize only one of them. We choose to perform the renormalization of $\gamma_{8_\pm}$.

Using the Zimmermann's forest formula and the same subtraction degrees in (4.7), we obtain

$$\begin{aligned} R^{\mu\nu}_{\gamma_{8\pm}}(p,k_1,k_2,s) =& \lambda_8 e^2 S_{\gamma_{8\pm}} \sum_{U\in\mathcal{F}_{\gamma_{8\pm}}} \prod_{\lambda\in U} (-\tau_\lambda S_\lambda) \widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s) \\ =& \lambda_8 e^2 (1 - t^0_{p,(s-1)}) \widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s) \ . \end{aligned} \tag{4.44}$$

where $\widehat{I}_{\mu\nu}$ is given by (3.55). After performing the calculations and taking $s \to 1$ to recover the massless case, the previous expression reads

$$\begin{aligned} R^{\mu\nu}_{\gamma_{8\pm}}(p,k_1,k_2,1) =& \, i\lambda_8 e^2 \mathrm{Tr}[\gamma^\mu \gamma_\theta \gamma_\alpha \gamma_\chi \gamma^\nu \gamma_\xi \gamma_\beta \gamma_\omega] \times \\ & \times \frac{k_1^\theta k_2^\chi [(k_1-k_2)^2 \eta^{\alpha\beta} - (k_1^\alpha - k_2^\alpha)(k_1^\beta - k_2^\beta)]}{k_1^2((k_1-k_2)^2 - \mu^2)(k_1-k_2)^2 k_2^2} \times \\ & \times \frac{-k_1^2 k_2^2 (k_1^\omega p^\xi + k_2^\xi p^\omega - p^\omega p^\xi) - k_1^\omega k_2^\xi \left(k_1^2(-2k_2\cdot p + p^2) + (-2k_1\cdot p + p^2)(k_2-p)^2\right)}{k_1^2 k_2^2 (k_1-p)^2 (k_2-p)^2} \end{aligned} \tag{4.45}$$

Considering the highest power of k in the numerator, and taking into account $d^3k_1 d^3k_2$, and considering the power of k in the denominator, it is possible to check that the previous renormalized integrand will give a convergent integral, because $d^3k_1 d^3k_2 R^{\mu\nu}_{\gamma_{8\pm}} \sim 1/k$.

We intend to integrate the renormalized integrand primarily with respect to the momentum k_1, using the results in Appendix B. Therefore, the denominator must take the form $(k_1^2 + 2k\cdot p' - c')$, for some p'^μ and c' that depend on k_2^μ, p^μ, μ and numerical factors. Using the generalized Feynman parametrization (B.17), we obtain

$$\begin{aligned} & \frac{1}{(k_1)^2(k_2)^2[(k_1-k_2)^2-\mu^2](k_1-k_2)^2(k_1-p)^2(k_2-p)^2} = \\ & = \int \frac{dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \ \ \Gamma(8) \ \delta(1-x_1 ... - x_6) \ x_1 x_2}{[k_1^2 x_1 + k_2^2 x_2 + [(k_1-k_2)^2 - \mu^2] x_3 + (k_1-k_2)^2 x_4 + (k_1-p)^2 x_5 + (k_2-p)^2 x_6]^8} \\ & = \Gamma(8) \int dx_1 dx_2 dx_4 dx_5 dx_6 \frac{x_1 x_2}{(1-x_2)^8} \ \frac{1}{(k_1^2 + 2k_1\cdot p' - c')^8} \ , \end{aligned} \tag{4.46}$$

where the Feynman parameters are denoted as $x_1, \ ..., x_6$, and we have chosen to eliminate the parameter x_3 by integrating the distribution δ. Additionally, we have $\alpha = 8$ and the quantities p'^μ and c' are

$$\begin{aligned} p'^\mu &= \frac{[(x_1+x_2+x_5+x_6-1)k_2^\mu - x_5 p^\mu]}{1-x_2} \\ c' &= \frac{(x_1+x_5-1) \ k_2^2 + 2x_6 \ k_2\cdot p + (1-x_1-x_2-x_4-x_5-x_6)\mu^2 - (x_5+x_6)p^2}{1-x_2} \end{aligned} \tag{4.47}$$

Next, we turn our attention to the numerator of (4.45). We will expand it in terms of powers of the momentum k_1. Each of these terms, after integrating with respect to this internal momentum, will correspond to a term containing a $J_r(8,p',c')$, being r the power of k_1. The number of terms after the expansion is large, so it is convenient to introduce a new notation to make the expressions more readable. We define

$$f_n^{\mu_1...\mu_n} = k_1^{\mu_1}...k_1^{\mu_n} \quad ; \qquad g_n^{\mu_1...\mu_n} = k_2^{\mu_1}...k_2^{\mu_n} \quad . \tag{4.48}$$

Additionally, if the subscript is greater than the number of indices, there is a scalar product implicit. For instance,

$$f_3^{\mu} = k_1^{\mu} k_1^2 \ . \tag{4.49}$$

Furthermore, the following properties hold:

$$f_n^{\mu_1...\mu_n} f_m^{\mu_n...\mu_{m+n}} = f_{m+n}^{\mu_1...\mu_{m+n}} \quad ; \qquad g_n^{\mu_1...\mu_n} g_m^{\mu_n...\mu_{m+n}} = g_{m+n}^{\mu_1...\mu_{m+n}} \tag{4.50}$$

With all these considerations, the numerator in (4.45) can be expanded as $\sum_{l=0}^{6} X_l$, where

$$\begin{aligned} X_0 = \Big[& - f_6^{\theta\omega} g_3^{\chi} p^{\xi} - f_5^{\theta} g_4^{\xi\chi} p^{\omega} + f_5^{\theta} g_3^{\chi} p^{\omega} p^{\xi} + 2 f_6^{\theta\omega} g_3^{\chi\xi\delta} p_\delta - f_6^{\theta\omega} g_2^{\chi\xi} p^2 + \\ & 2 f_5^{\theta\omega\delta} \left(g_4^{\chi\xi} - 2 g_3^{\chi\xi\delta} p_\delta + g_2^{\chi\xi} p^2 \right) p_\delta - f_4^{\theta\omega} \left(g_4^{\chi\xi} - 2 g_3^{\chi\xi\delta} p_\delta + g_2^{\chi\xi} p^2 \right) p^2 \Big] \eta^{\alpha\beta} \ ; \\ X_1 = & -\frac{f_1^{\delta'} g_{1\delta'}}{f_2} X_0 \ ; \quad X_2 = \frac{g_2}{f_2} X_0 \ , \quad X_3 = -\frac{f_2^{\alpha\beta}}{f_2 \eta^{\alpha\beta}} X_0 \ ; \quad X_4 = \frac{f_1^{\alpha} g_1^{\beta}}{f_2 \eta^{\alpha\beta}} X_0 \ ; \\ X_5 = & \frac{f_1^{\beta} g_1^{\alpha}}{f_2 \eta^{\alpha\beta}} X_0 \ ; \quad X_6 = -\frac{g_2^{\alpha\beta}}{f_2 \eta^{\alpha\beta}} X_0 \quad . \end{aligned} \tag{4.51}$$

It is important to mention that the tensor indices of X_l were omitted, but each of these terms has the indices θ ,ω ,χ ,ξ ,α ,β, which must be contracted with the γ-matrices. Thus, the expression for the renormalized integrand (4.45) can be rewritten as

$$R^{\mu\nu}_{\gamma_{8\pm}} = \sum_{l=0}^{6} \int d\Omega \frac{X_l}{(k_1^2 + 2k_1 \cdot p' - c')^8} \quad , \tag{4.52}$$

where

$$d\Omega = \lambda_8 e^2 \mathrm{Tr}\left(\gamma^{\mu}\gamma_{\theta}\gamma_{\alpha}\gamma_{\chi}\gamma^{\nu}\gamma_{\xi}\gamma_{\beta}\gamma_{\omega}\right) \Gamma(8) dx_1 dx_2 dx_4 dx_5 dx_6 \frac{x_1 x_2}{(1-x_2)^8} \ . \tag{4.53}$$

and we have omitted its tensor indices. For $\gamma_{9\pm}$, $d\Omega$ is the same, apart from $\lambda_8 e^2$, which must be changed to $\lambda_9 g^2$.

Let us compute the contributions of $\gamma_{8\pm}$ after the BPHZL renormalization procedure. For this purpose, it is convenient to introduce another notation:

$$\{p_i k_{n-i}\}^{\alpha_1...\alpha_n} = p^{\alpha_1} \cdots p^{\alpha_i} k^{\alpha_{i+1}} ... k^{\alpha_n} + \text{dist. perm} \ , \tag{4.54}$$

$$(Ap^{\alpha_1} + Bk^{\alpha_1}) \cdots (Ap^{\alpha_n} + Bk^{\alpha_n}) = \sum_{i=0}^{n} A^i B^{n-i} \{p_i k_{n-i}\}^{\alpha_1...\alpha_n} \ , \tag{4.55}$$

where A, B are numbers, and by "dist. perm." we mean all the distinct permutations, which are obtained by exchanging the tensor indices between k and p. We refer to $\{\cdot\}^{\cdot}$ as **expansion operator**. Furthermore,

$$\begin{aligned}\{p_i k_{n-i}\}^{\alpha_1\ldots\alpha_n\bar{\beta}_k} =& k^\beta (p^{\alpha_1}\cdots p^{\alpha_i}k^{\alpha_{i+1}}\ldots k^{\alpha_n} + \text{dist. perm}) \\ & k^\beta \{p_i k_{n-i-1}\}^{\alpha_1\ldots\alpha_n} \ ,\end{aligned} \tag{4.56}$$

that is, an index with an upper bar is not exchanged. In addition, if the sum of the subscripts is greater than the number of indices, there is a contraction omitted in the quantity whose subscript is responsible for the asymmetry. For instance, $\{p_1 k_3\}^{\alpha\beta} = p^\alpha k^\beta k^2 + \text{dist. perm.}$ The generalization of the expansion operator for more than two quantities is straightforward. Furthermore, the expansion operator can be used for the J_r integrals discussed in Appendix B. So, considering the previous definitions, we must calculate

$$\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} R^{\mu\nu}_{\gamma^8\pm} = \sum_{l=0}^{6}\int d\Omega \int \frac{d^3k_2}{(2\pi)^3}\frac{d^3k_1}{(2\pi)^3}\frac{X_l}{(k_1^2+2k_1\cdot p' - c')^8} \ , \tag{4.57}$$

After integrating in the momentum k_1, using the definition of the J_r integrals in Appendix B, the contribution of X_0 is

$$\begin{aligned} X_0' = \int \frac{d^3k_1}{(2\pi)^3}\frac{X_0}{(k_1^2+2k_1\cdot p'-c')^8} = \Big[-J_6^{\theta\omega} g_3^{\chi} p^\xi - J_5^{\theta} g_4^{\xi\chi} p^\omega + J_5^{\theta} g_3^{\chi} p^\omega p^\xi + 2J_6^{\theta\omega} g_3^{\chi\xi\delta} p_\delta \\ -J_6^{\theta\omega} g_2^{\chi\xi} p^2 + 2J_5^{\theta\omega\delta}\left(g_4^{\chi\xi} - 2g_3^{\chi\xi\delta'}p'_\delta + g_2^{\chi\xi}p^2\right)p_\delta - J_4^{\theta\omega}\left(g_4^{\chi\xi} - 2g_3^{\chi\xi\delta}p_\delta + g_2^{\chi\xi}p^2\right)p^2\Big]\eta^{\alpha\beta} \ , \end{aligned} \tag{4.58}$$

where all the J_r are functions of $\alpha = 8$, p' and c'. Since the integral in k_1 have been performed, it remains only the internal momentum k_2, which can be relabeled as k. Furthermore, it is possible to write X_0' as

$$X_0' = \sum_{l'=0}^{10} X_{0l'}' \ . \tag{4.59}$$

Each $X_{0l'}'$ corresponds to the l'-th term in the sum of (4.58). The first term, using (B.7), is

$$\begin{aligned} X_{00}' &= -J_6^{\theta\omega} g_3^{\chi} p^\xi = -\eta_{\phi\phi'}\eta_{\psi\psi'} J_6^{\theta\omega\phi\phi'\psi\psi'}(8,p',c')k^\chi k^2 p^\xi \eta^{\alpha\beta} \\ &= -\eta_{\phi\phi'}\eta_{\psi\psi'} k^\chi k^2 p^\xi C(8)\sum_{j=0}^{3}\left(-\frac{1}{2}\right)^j \Gamma(13/2-j)(c'+p'^2)^{-13/2+j}\times \\ &\quad \times\left(\Omega(p')^{\mu_1\ldots\mu_{6-2j}}_{6-2j}\zeta_{2j}^{\mu_{6-2j+1}\ldots\mu_6} + \text{dist. perm.}\right)\eta^{\alpha\beta} \quad . \end{aligned} \tag{4.60}$$

where $C(8) = i\dfrac{\pi^{3/2}}{\Gamma(8)(2\pi)^3}$ and $(\mu_1,\ldots,\mu_6) = (\theta,\omega,\phi,\phi',\psi,\psi')$ and, as in Appendix A,

$$\begin{aligned} \Omega(p)_n^{\mu_1\ldots\mu_n} &\equiv \begin{cases} p^{\mu_1}\ldots p^{\mu_n} \ , \text{if } n \geq 0 \\ 0, \text{if } n < 0 \ \ ; \end{cases} \\ \zeta_n^{\mu_1\ldots\mu_n} &\equiv \begin{cases} \eta^{\mu_1\mu_2}\ldots\eta^{\mu_{n-1}\mu_n} + \text{dist. perm., if } n \text{ is even,} \\ 0, \ \text{if } n \text{ is odd or } n < 0 \ . \end{cases} \end{aligned} \tag{4.61}$$

Also, using (4.47), we can rewrite p' as

$$p'^{\mu} = Ap^{\mu} + Bk^{\mu} \tag{4.62}$$

where

$$A = -\frac{x_5}{1-x_2} \quad , \qquad B = \frac{x_1+x_2+x_5+x_6-1}{1-x_2} \quad . \tag{4.63}$$

So, we have

$$\Omega(p')^{\mu_1\ldots\mu_{6-2j}}_{6-2j} = p'^{\mu_1}\cdots p'^{\mu_{6-2j}} = \sum_{i=0}^{6-2j} A^i B^{6-2j-i}\{p_i k_{6-2j-i}\}^{\mu_1\ldots\mu_{6-2j}} \quad . \tag{4.64}$$

The previous result can be used in (4.60), and it reads

$$\begin{aligned} X'_{00} =& -\eta_{\phi\phi'}\eta_{\psi\psi'}C(8)\sum_{j=0}^{3}\sum_{i=0}^{6-2j}\left(-\frac{1}{2}\right)^j\Gamma(13/2-j)(c'+p'^2)^{-13/2+j}\times \\ &\times\left(A^iB^{6-2j-i}\{p_ik_{6-2j-i}\}^{\mu_1\ldots\mu_{6-2j}}\zeta_{2j}^{\mu_{6-2j+1}\ldots\mu_6}+\text{dist. perm.}\right)k^{\chi}k^2p^{\xi}\eta^{\alpha\beta} \quad . \\ =& -\eta_{\phi\phi'}\eta_{\psi\psi'}p^{\xi}C(8)\sum_{j=0}^{3}\sum_{i=0}^{6-2j}\left(-\frac{1}{2}\right)^j\Gamma(13/2-j)(c'+p'^2)^{-13/2+j}\times \\ &\times\left(A^iB^{6-2j-i}\{p_ik_{9-2j-i}\}^{\mu_1\ldots\mu_{6-2j}\bar{\chi}_k}\zeta_{2j}^{\mu_{6-2j+1}\ldots\mu_6}+\text{dist. perm.}\right)\eta^{\alpha\beta} \quad . \end{aligned} \tag{4.65}$$

It is also necessary to integrate X'_{00} with respect to the momentum k_2, which has been relabeled as k. For this purpose, the previous result can be put into the form of a J_r integral. The term $(c'+p'^2)^{-13/2+j}$ appears in the denominator, giving $\alpha = -13/2+j$, whereas k_{9-2j-i} appears in the numerator, and the power of k is given by $9-2j-i$. In fact, rewriting the term $(c'+p'^2)$ in the form $(k^2+2k\cdot p''-c'')$, the integration with respect to the momentum k becomes straightforward and consists of replacing k_{9-2j-i} by $J_{9-2j-i}((-13/2+j), p'', c'')$. Thus, $c'+p'^2$ can be rewritten as

$$c'+p'^2 = \frac{a}{(1-x_2)^2}(k^2+2k\cdot p''-c'') \ , \tag{4.66}$$

where

$$\begin{aligned} a &= (x_1+x_2+x_5+x_6-1)^2+(x_1+x_5-1)(1-x_2) \ , \\ p'' &= \frac{x_5(1-x_1-x_2-x_5-x_6)+x_6(1-x_2)}{a}p \ , \\ c'' &= -\frac{[x_5^2-(1-x_2)(x_5+x_6)]p^2+(1-x_1-x_2-x_4-x_5-x_6)(1-x_2)\mu^2}{a}. \end{aligned} \tag{4.67}$$

So, defining

$$M(j) = \left(-\frac{1}{2}\right)^j C(8)\left(\frac{a}{(1-x_2)^2}\right)^{-13/2+j}\Gamma(13/2-j) \ , \tag{4.68}$$

the expression for X'_{00} reads

$$X'_{00} = -\eta_{\phi\phi'}\eta_{\psi\psi'}p^{\xi}\sum_{j=0}^{3}\sum_{i=0}^{6-2j}\frac{\left[M(j)A^iB^{6-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\psi\psi'\bar{\chi}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}\ . \tag{4.69}$$

where ζ, the product of η's, has been absorbed into the expansion operator. As can be noted, the number of indices is less than the sum of the subscripts in the expansion operator and it is due to the presence of a factor k^2. It should be stressed that after integrating X'_{00} with respect to k, no divergence will be encountered. Indeed, J_{9-2j-i} contains $\Gamma(\alpha'-d/2-j')$, where $\alpha'=13/2-j$ and j' ranges from 0 to $\lfloor 9/2-j-i/2\rfloor$. One may wonder whether the argument of the Γ function could be a negative integer or not. However, it can be checked that $\Gamma(\alpha'-d/2-j')=\Gamma(5-j-j')$, and for $j=0$, the maximum value of j' is 4, and as we increase j by one unit, the maximum value of j' decreases by one unit. Therefore, the smallest argument of the Γ function is 1 and there are no divergences.

It remains to determine the other $X'_{0l'}$ terms. Observing the expression for X'_0 in (4.58), it is possible to write the expressions for X'_{01}, ..., $X'_{0,10}$ from X'_{00}. For instance, $X'_{01}=-J_5^{\theta}g_4^{\xi\chi}p^{\omega}$ is

$$X'_{01} = -\,\eta_{\phi\phi'}\eta_{\psi\psi'}p^{\omega}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\psi\psi'\bar{\chi}_k\bar{\xi}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}} \tag{4.70}$$

It is worth noting how the sum over j changed. This was due to X'_{01} having a J_5 instead of a J_6. Consequently, it also affected the sum over i. Moreover, the indices with an upper bar changed due to the modifications in the indices of g, which is $g_4^{\xi\chi}=k^2k^{\xi}k^{\chi}$ in this case. Using a similar procedure, the remaining $X'_{0l'}$ are given by:

$$X'_{02} =\eta_{\phi\phi'}\eta_{\psi\psi'}p^{\omega}p^{\xi}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\psi\psi'\bar{\chi}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{4.71}$$

$$X'_{03} =2\eta_{\phi\phi'}\eta_{\psi\psi'}p_{\delta}\sum_{j=0}^{3}\sum_{i=0}^{6-2j}M(j)\frac{\left[A^iB^{6-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\psi\psi'\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{4.72}$$

$$X'_{04} =-\,\eta_{\phi\phi'}\eta_{\psi\psi'}p^2\sum_{j=0}^{3}\sum_{i=0}^{6-2j}M(j)\frac{\left[A^iB^{6-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\psi\psi'\bar{\xi}_k\bar{\chi}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{4.73}$$

$$X'_{05} =2\eta_{\phi\phi'}p_{\delta}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\omega\delta\bar{\chi}_k\bar{\xi}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{4.74}$$

$$X'_{06} =-\,4\eta_{\phi\phi'}p_{\delta}p_{\delta'}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\delta\bar{\chi}_k\bar{\xi}_k\bar{\delta}'_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{4.75}$$

$$X'_{07} =2\eta_{\phi\phi'}p^2p_{\delta}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{7-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\omega\delta\bar{\chi}_k\bar{\xi}_k}\right]\eta^{\alpha\beta}}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{4.76}$$

$$X'_{08} = -\,\eta_{\phi\phi'} p^2 \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\bar{\chi}_k \bar{\xi}_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k\cdot p'' - c'')^{13/2-j}}; \tag{4.77}$$

$$X'_{09} = 2\eta_{\phi\phi'} p^2 p_\delta \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\bar{\chi}_k \bar{\xi}_k} \bar{\delta}_k \right] \eta^{\alpha\beta}}{(k^2 + 2k\cdot p'' - c'')^{13/2-j}}; \tag{4.78}$$

$$X'_{0,10} = -\,\eta_{\phi\phi'} p^2 p^2 \sum_{j=0}^{1} \sum_{i=0}^{2-2j} M(j) \frac{\left[A^i B^{2-2j-i} \{p_i k_{6-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\bar{\chi}_k \bar{\xi}_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k\cdot p'' - c'')^{13/2-j}}; \tag{4.79}$$

All of the $X'_{0l'}$ will lead to non-divergent and dimensionless terms when integrated with respect to k.

The previous calculations account only for the contributions of X'_0. It is still necessary to determine the contributions for the remaining $X'_l = \int \frac{d^3k}{(2\pi)^3} \frac{X_l}{(k_1^2 + 2k_1\cdot p' - c')^8}$. This can be accomplished by using equation (4.51), and we have detailed this procedure in Appendix C. Thus, after integrating each X'_l with respect to k, we obtain

$$X''_l = \int \frac{d^3k}{(2\pi)^3} X'_l = \sum_{l'=0}^{10} \int \frac{d^3k}{(2\pi)^3} X'_{ll'} = \sum_{l'=0}^{10} X''_{ll'}, \tag{4.80}$$

where $X''_{ll'}$ is obtained from $X'_{ll'}$ just by eliminating the denominator and replacing k_r by $J_r(13/2 - j, p'', c'')$, because

$$\int \frac{d^3k}{(2\pi)^3} \frac{\{p_{n_1} k_{n_2} \zeta_{n_3}\}^{\mu_1 \dots}}{(k^2 + 2k\cdot p'' - c'')^{13/2-j}} = \left\{ p_{n_1} J_{n_2} \left(13/2 - j, p'', c''\right) \zeta_{n_3} \right\}^{\mu_1 \dots}. \tag{4.81}$$

Therefore, the renormalized diagram $\gamma_{8\pm}$ yields

$$\int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} R^{\mu\nu}_{\gamma_{8\pm}} = \sum_{l=0}^{6} \sum_{l'=0}^{10} \int d\Omega X''_{ll'}\ . \tag{4.82}$$

The integrals in terms of $d\Omega$, which arise from the Feynman parametrizations, yield just a numerical factor. Furthermore, all of them produce non-local terms, since all $X''_{ll'}$ contain external momenta in the denominators. To prove our last statement, it is necessary to note that all the terms of each $X''_{ll'}$ contain a $J_{n-2j-i}(13/2, p'', c'')$, which can be expressed as [2]

$$J_{n-2j-i}(13/2, p'', c'') = \sum_{j'=0}^{\left\lfloor \frac{n-2j-i}{2} \right\rfloor} f(j')(c'' + p''^2)^{3/2-13/2+j+j'}\ , \tag{4.83}$$

where $f(j')$ includes all the terms that are not important in our analysis here and n varies from 6 to 9, as can be seen in the expressions of $X''_{ll'}$. The power of $(c'' + p''^2)$ is $3/2 - 13/2 + j + j' = -5 + j + j'$ and we can prove that it is always negative. The highest value of j' is attained when $i = 0$ and $n = 9$. In this scenario, $j' = \left\lfloor \frac{9-2j}{2} \right\rfloor = \left\lfloor \frac{8-2j}{2} \right\rfloor = 4 - j$. Consequently, the lowest power of $(c'' + p''^2)$ is

[2] i and j in J_{n-2j-i} are the sum indices that appeared in the expressions of the terms $X'_{ll'}$.

$-5 + j + (4 - j) = -1$, regardless any value of j and we can conclude that all the $X''_{ll'}$ yield only non-local terms.

Therefore, the 2-loops vacuum polarizations do not contribute to the renormalization of any parameter of the theory. Additionally, since the model of the parity-preserving QED_3 is super-renormalizable and has divergent diagrams only up to order of 2-loops, then we completely renormalized the model using the BPHZL procedure. As could be seen explicitly, BPHZL does not break parity in this model and this symmetry is maintained at the quantum level.

Although we have completely renormalized the model by BPHZL, it still remains to verify if the model has an anomaly and how is the structure of the invariant and non-invariant counterterms. This will be performed in the following chapter, in the context of algebraic renormalization.

Chapter 5

Algebraic Renormalization of the Parity-Preserving QED$_3$ Model[1]

Throughout this chapter, we will discuss the algebraic renormalization of the parity-preserving QED$_3$, examining its renormalizability, stability, and verifying the presence or absence of anomalies. As will be shown, the model exhibits quantum scale invariance, mimicking the scale invariance in graphene, which has been suggested experimentally in [38], based on data regarding the strong nonlinear optical response in this material.

In Chapter 3, in equation (3.3), we indicated the transformations of the fields under the BRS symmetry, which will play a fundamental role in the algebraic renormalization procedure in the present chapter. Before applying this method to the model, it is worth providing a brief introduction to this powerful technique. For further details, we refer to [16].

5.1 A Brief Introduction to Algebraic Renormalization

Let us suppose a classical action $\Sigma(\lambda, \phi, \rho)$ obeying the relation

$$\mathcal{S}(\Sigma) = 0 \ , \tag{5.1}$$

where $\mathcal{S}$ is an operator that represents the symmetries in a functional form, λ represents the parameters and coupling constants, ϕ represents the fields and ρ represents the currents. A natural question that arises is whether the previous relation holds at the quantum level, when the operator $\mathcal{S}$ is applied to the quantum action Γ rather than the classical one Σ.

The action Γ can be expressed as power series in terms of of $\hbar$ as follows:

$$\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)} \ . \tag{5.2}$$

Here, $\Gamma^{(0)} \equiv \Sigma$, while $\Gamma^{(i)},\ i > 0$, is the correction given by the i-th loop order. When the operator $\mathcal{S}$ is applied to the quantum action, Eq. (5.1) does not hold in general; instead,

[1]This chapter is based on the letter published in Physics Letters B, Volume 835, 2022, with the reference number 137544, titled 'The Quantum Scale Invariance in Graphene-like Quantum Electrodynamics,' authored by O.M. Del Cima, D.H.T. Franco, L.S. Lima, and E.S. Miranda.

according to the quantum action principle [16, 27], it is replaced by

$$\mathcal{S}(\Gamma) = \hbar^n \Delta + \mathcal{O}(\hbar^{n+1}) \ , \tag{5.3}$$

In the previous equation, Δ represents all the possible symmetry-breaking terms at the n-th order, and the symmetry is supposed to hold up to order $n-1$.

Some comments regarding the possible symmetry-breaking terms must be made. These terms are local functional in the fields and, depending on the form of $\mathcal{S}$, they can be integrated terms or not. Additionally, their UV and IR dimensions, as well as their quantum numbers (such as ghost number), are determined by $\mathcal{S}(\Gamma)$. Furthermore, the possible symmetry-breaking terms can be divided into two categories in general. One category is related to the non-invariant counterterms, which can be reabsorbed into the quantum action, whereas the other is related to the possible anomalies, which express the impossibility of extending a classical symmetry to the quantum action. These two categories of symmetry-breaking terms can be obtained after solving a cohomology problem. The non-invariant counterterms are related to the trivial solutions, whereas the anomalies are related to the non-trivial ones.

After the analysis of non-invariant counterterms to verify the presence or absence of anomalies in the model, the next step is to analyze the invariant counterterms. These counterterms are invariant under symmetry transformations and are included in the counterterms action, denoted as Σ^c. Since they obey the symmetries, it follows that

$$\mathcal{S}(\Sigma^c) = 0 \ . \tag{5.4}$$

We say that the action is stable if, when writing the most general counterterms action, the following relation holds at the n-th order:

$$\Sigma(\lambda, \phi, \rho) + \Sigma^c(\lambda, \phi, \rho) = \Sigma^0(\lambda_0, \phi_0, \rho_0) + \mathcal{O}(\hbar^{n+1}) \ . \tag{5.5}$$

Here, Σ^0 is the bare action, whereas λ_0, ϕ_0 and ρ_0 are given by

$$\lambda_0 = Z_\lambda \lambda \ , \qquad \phi_0 = Z_\phi^{\frac{1}{2}} \phi \ , \qquad \rho_0 = Z_\rho \rho \ , \tag{5.6}$$

and Z_j are given by power series in $\hbar$, and they have to be fixed by proper renormalization conditions. In this sense, the stability condition (5.5) means that the starting action Σ is stable if it has the same form of the most general counterterms action Σ^c. If the stability condition holds, we say that the model is multiplicatively renormalizable.

The previous steps briefly presented in this section constitute the approach that will be followed to investigate the renormalizability of the parity-preserving QED$_3$. Although we have already renormalized the model by BPHZL, it is opportune to analyze whether anomalies exist and to examine the structure of the invariant counterterms.

5.2 Symmetries and Slavnov-Taylor Identity

Let us turn our attention to the parity-preserving QED$_3$ again, with action given by:

$$\begin{aligned}
\Sigma^{(s-1)} = \int d^3x \Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\alpha\nu}A_\mu\partial_\alpha a_\nu + i\overline{\psi}_+ \not{D}\psi_+ \\
& + i\overline{\psi}_- \not{D}\psi_- + \underbrace{- m(s-1)\overline{\psi}_+\psi_+ + m(s-1)\overline{\psi}_-\psi_-}_{\text{Lowenstein-Zimmermann mass term}} \\
& + b\partial^\mu A_\mu + \frac{\alpha}{2}b^2 + \overline{c}\Box c + \pi\partial^\mu a_\mu + \frac{\beta}{2}\pi^2 + \overline{\xi}\Box\xi \Big\} \ ,
\end{aligned} \tag{5.7}$$

where, as previously stated, $D\!\!\!\!/\,\psi_\pm \equiv (\partial\!\!\!/ + ieA\!\!\!/ \pm iga\!\!\!/)\psi_\pm$. The transformations of the fields under the BRS symmetries are

$$
\begin{aligned}
s\psi_+ &= i(c+\xi)\psi_+ \ , \quad s\overline{\psi}_+ = -i(c+\xi)\overline{\psi}_+ \ ; \\
s\psi_- &= i(c-\xi)\psi_- \ , \quad s\overline{\psi}_- = -i(c-\xi)\overline{\psi}_- \ ; \\
sA_\mu = -\frac{1}{e}\partial_\mu c \ , \quad sc &= 0 \ ; \quad sa_\mu = -\frac{1}{g}\partial_\mu \xi \ , \quad s\xi = 0 \ ; \\
s\bar{c} = \frac{b}{e} \ , \quad sb &= 0 \ ; \quad s\bar{\xi} = \frac{\pi}{g} \ , \quad s\pi = 0 \ ;
\end{aligned}
\tag{5.8}
$$

In order to control the renormalization of the BRS transformations, which are non-linear, it is necessary to add an action of external currents

$$
\Sigma_{\rm ext} = \int d^3x \Big\{ \overline{\Omega}_+ s\psi_+ - \overline{\Omega}_- s\psi_- - s\overline{\psi}_+\Omega_+ + s\overline{\psi}_-\Omega_- \Big\} \ . \tag{5.9}
$$

Thus, the complete classical action becomes:

$$
\Gamma_0 \equiv \Gamma_0^{(s-1)} = \Sigma^{(s-1)} + \Sigma_{\rm ext} \ . \tag{5.10}
$$

The BRS transformations given in (5.8) encompass the gauge transformations, but the parameters c and ξ are ghost fields and obey fermionic statistics, and they are governed by a Grassmann Algebra. Beyond that, the BRS transformations are non-linear, as can be seen in the transformation of ψ_+, which involves a product of two fields. It might seem that the BRS symmetry introduces unnecessary additional complications to the model. However, it allows for simplification in the analysis, as it enables the representation of the two gauge symmetries of this model using only one operator, known as the Slavnov-Taylor operator. Without the use of BRS symmetry, two identities known as Ward Identities would be necessary to express the gauge symmetries. Besides, as can be checked in (5.8), the operator s is nilpotent, such that $s(s\phi) = 0$ for ϕ being an arbitrary field of the model.

It is worth noting that the addition of $\Sigma_{\rm ext}$ does not break the BRS symmetry, as $s(s\psi_\pm) = s(s\bar{\psi}_\pm) = 0$. Furthermore, the addition of ghost fields to this model does not introduce any problems since they are free fields and do not participate in any Feynman diagrams. This means that they decouple from the model and should be seen solely as auxiliary fields.

It is also necessary to consider the parity transformations, given by:

$$
\begin{aligned}
&\psi_+ \xrightarrow{P} \psi_+^P = -i\gamma^1\psi_- \ , \quad \psi_- \xrightarrow{P} \psi_-^P = -i\gamma^1\psi_+ \ , \\
&\overline{\psi}_+ \xrightarrow{P} \overline{\psi}_+^P = i\overline{\psi}_-\gamma^1 \ , \quad \overline{\psi}_- \xrightarrow{P} \overline{\psi}_-^P = i\overline{\psi}_+\gamma^1 \ ; \\
&A_\mu \xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ ; \quad \phi \xrightarrow{P} \phi^P = \phi \ , \quad \phi = \{b, c, \bar{c}\} \ ; \\
&a_\mu \xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ ; \quad \chi \xrightarrow{P} \chi^P = -\chi \ , \quad \chi = \{\pi, \xi, \bar{\xi}\} \\
&\Omega_\pm \xrightarrow{P} \Omega_\pm^P = -i\gamma^1\psi_\mp \ , \quad \overline{\Omega}_\pm \xrightarrow{P} \overline{\Omega}_\pm^P = i\overline{\Omega}_\mp\gamma^1 \ ,
\end{aligned}
\tag{5.11}
$$

In the following table, the UV and IR dimensions of the fields and currents of the model are showed, as well as the ghost numbers and Grassmann parities.

	A_μ	a_μ	ψ_+	ψ_-	c	$\bar{c}$	b	ξ	$\bar{\xi}$	π	s	$s-1$	Ω_+	Ω_-
d	1/2	1/2	1	1	0	1	$\frac{3}{2}$	0	1	$\frac{3}{2}$	1	1	2	2
r	1	1	1	1	0	1	1	0	1	1	0	1	2	2
$\Phi\Pi$	0	0	0	0	1	-1	0	1	-1	0	0	0	-1	-1
GP	0	0	1	1	1	1	0	1	1	0	0	0	0	0

Table 5.1: UV dimension (d), IR dimension (r), ghost number ($\Phi\Pi$) and Grassmann parity (GP).

Our aim now is to express the transformations (5.8) in a functional way, by using the Slavnov-Taylor operator. If we consider, for example, the transformation of A_μ by BRS, it can be expressed by the action of the operator $\displaystyle\int d^3x \frac{-1}{e}\partial_\nu c \frac{\delta}{\delta A_\nu}$, because

$$sA_\mu = -\frac{1}{e}\partial_\mu c = -\int d^3x \frac{1}{e}\partial_\nu c \frac{\delta}{\delta A_\nu} A_\mu \tag{5.12}$$

Another example is the transformation of ψ_+. Taking into account $\Sigma_{\rm ext}$ within Γ_0, it follows that

$$\frac{\delta\Gamma_0}{\delta\overline{\Omega}_+} = s\psi_+ \ . \tag{5.13}$$

The action of s in ψ_+ can be expressed functionally as

$$s\psi_+ = \int d^3x \frac{\delta\Gamma_0}{\delta\overline{\Omega}_+}\frac{\delta}{\delta\psi_+}\psi_+ \ . \tag{5.14}$$

If this procedure is repeated to all terms, and taking into account that Γ_0 is BRS invariant, then

$$\begin{aligned}\int d^3x \Big\{ & -\frac{1}{e}\partial^\mu c \frac{\delta\Gamma_0}{\delta A^\mu} + \frac{b}{e}\frac{\delta\Gamma_0}{\delta\bar{c}} - \frac{1}{g}\partial^\mu\xi\frac{\delta\Gamma_0}{\delta a^\mu} + \frac{\pi}{g}\frac{\delta\Gamma_0}{\delta\bar{\xi}} + \\ & + \frac{\delta\Gamma_0}{\delta\overline{\Omega}_+}\frac{\delta\Gamma_0}{\delta\psi_+} - \frac{\delta\Gamma_0}{\delta\Omega_+}\frac{\delta\Gamma_0}{\delta\overline{\psi}_+} - \frac{\delta\Gamma_0}{\delta\overline{\Omega}_-}\frac{\delta\Gamma_0}{\delta\psi_-} + \frac{\delta\Gamma_0}{\delta\Omega_-}\frac{\delta\Gamma_0}{\delta\overline{\psi}_-}\Big\} = 0 \ .\end{aligned} \tag{5.15}$$

The previous equation is known as Slavnov-Taylor identity, and can be summarized in an elegant manner as

$$\mathcal{S}(\Gamma_0) = 0 \ , \tag{5.16}$$

where $\mathcal{S}$ is the Slavnov-Taylor operator, given by

$$\begin{aligned}\mathcal{S}(\mathcal{F}) = \int d^3x \Big\{ & -\frac{1}{e}\partial^\mu c \frac{\delta\mathcal{F}}{\delta A^\mu} + \frac{b}{e}\frac{\delta\mathcal{F}}{\delta\bar{c}} - \frac{1}{g}\partial^\mu\xi\frac{\delta\mathcal{F}}{\delta a^\mu} + \frac{\pi}{g}\frac{\delta\mathcal{F}}{\delta\bar{\xi}} + \\ & + \frac{\delta\mathcal{F}}{\delta\overline{\Omega}_+}\frac{\delta\mathcal{F}}{\delta\psi_+} - \frac{\delta\mathcal{F}}{\delta\Omega_+}\frac{\delta\mathcal{F}}{\delta\overline{\psi}_+} - \frac{\delta\mathcal{F}}{\delta\overline{\Omega}_-}\frac{\delta\mathcal{F}}{\delta\psi_-} + \frac{\delta\mathcal{F}}{\delta\Omega_-}\frac{\delta\mathcal{F}}{\delta\overline{\psi}_-}\Big\} \ .\end{aligned} \tag{5.17}$$

It is still possible to consider the linearized Slavnov-Taylor operator $\mathcal{S}_\mathcal{F}$ related to $\mathcal{S}(\mathcal{F})$, given by

$$\mathcal{S}_{\mathcal{F}} = \int d^3x \Big\{ -\frac{1}{e}\partial^\mu c \frac{\delta}{\delta A^\mu} + \frac{b}{e}\frac{\delta}{\delta \overline{c}} - \frac{1}{g}\partial^\mu \xi \frac{\delta}{\delta a^\mu} + \frac{\pi}{g}\frac{\delta}{\delta \overline{\xi}} +$$
$$+ \frac{\delta \mathcal{F}}{\delta \overline{\Omega}_+}\frac{\delta}{\delta \psi_+} + \frac{\delta \mathcal{F}}{\delta \psi_+}\frac{\delta}{\delta \overline{\Omega}_+} - \frac{\delta \mathcal{F}}{\delta \Omega_+}\frac{\delta}{\delta \overline{\psi}_+} - \frac{\delta \mathcal{F}}{\delta \overline{\psi}_+}\frac{\delta}{\delta \Omega_+} +$$
$$- \frac{\delta \mathcal{F}}{\delta \overline{\Omega}_-}\frac{\delta}{\delta \psi_-} - \frac{\delta \mathcal{F}}{\delta \psi_-}\frac{\delta}{\delta \overline{\Omega}_-} + \frac{\delta \mathcal{F}}{\delta \Omega_-}\frac{\delta}{\delta \overline{\psi}_-} + \frac{\delta \mathcal{F}}{\delta \overline{\psi}_-}\frac{\delta}{\delta \Omega_-} \Big\} . \tag{5.18}$$

The important nilpotency identities hold:

$$\mathcal{S}_{\mathcal{F}}\mathcal{S}(\mathcal{F}) = 0 , \quad \forall \mathcal{F} , \tag{5.19}$$
$$\mathcal{S}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}} = 0 \ \text{ if } \ \mathcal{S}(\mathcal{F}) = 0 . \tag{5.20}$$

In particular, thanks to the Slavnov-Taylor identity (5.16), it follows that $(\mathcal{S}_{\Gamma_0})^2 = 0$.

The following identities also hold:

$$\mathcal{S}_{\Gamma_0}\phi = s\phi , \quad \phi = \{\psi_\pm, \overline{\psi}_\pm, A_\mu, a_\mu, c, \overline{c}, b, \pi, \overline{\xi}, \xi\} ,$$
$$\mathcal{S}_{\Gamma_0}\Omega_+ = -\frac{\delta\Gamma_0}{\delta\overline{\psi}_+} , \quad \mathcal{S}_{\Gamma_0}\overline{\Omega}_+ = \frac{\delta\Gamma_0}{\delta\psi_+} , \quad \mathcal{S}_{\Gamma_0}\Omega_- = \frac{\delta\Gamma_0}{\delta\overline{\psi}_-} , \quad \mathcal{S}_{\Gamma_0}\overline{\Omega}_- = -\frac{\delta\Gamma_0}{\delta\psi_-} . \tag{5.21}$$

Moreover, Γ_0 satisfies the following equations

$$\begin{aligned}
-i\frac{\delta\Gamma_0}{\delta c} &= i\Box\overline{c} + \overline{\Omega}_+\psi_+ - \overline{\Omega}_-\psi_- + \overline{\psi}_+\Omega_+ - \overline{\psi}_-\Omega_- , \\
-i\frac{\delta\Gamma_0}{\delta \xi} &= i\Box\overline{\xi} + \overline{\Omega}_+\psi_+ + \overline{\Omega}_-\psi_- + \overline{\psi}_+\Omega_+ + \overline{\psi}_-\Omega_- , \\
\frac{\delta\Gamma_0}{\delta \overline{c}} &= \Box c , \quad \frac{\delta\Gamma_0}{\delta\overline{\xi}} = \Box\xi , \\
\frac{\delta\Gamma_0}{\delta b} &= \partial^\mu A_\mu + \alpha b , \quad \frac{\delta\Gamma_0}{\delta\pi} = \partial^\mu a_\mu + \beta\pi .
\end{aligned} \tag{5.22}$$

Lastly, two additional rigid symmetries can be considered. They are related to the set of global transformations:

$$\psi_\pm \to e^{i\theta_1}\psi_\pm , \qquad \Omega_\pm \to e^{-i\theta_1}\Omega_\pm ; \tag{5.23}$$

$$\psi_\pm \to e^{\pm i\theta_2}\psi_\pm , \qquad \Omega_\pm \to e^{\mp i\theta_2}\Omega_\pm . \tag{5.24}$$

Here, θ_1 and θ_2 represent parameters that are independent of the spacetime. The rigid symmetries are associated to two Ward rigid operators, in such a way that:

$$W^{(e)}_{\text{rigid}}\Gamma_0 = 0 \ \text{ and } \ W^{(g)}_{\text{rigid}}\Gamma_0 = 0 , \tag{5.25}$$

where the Ward operators, factorizing the parameters of the transformations, are given as follows:

$$W^{(e)}_{\rm rigid} = \int d^3x \Big\{ \psi_+ \frac{\delta}{\delta\psi_+} - \overline{\psi}_+ \frac{\delta}{\delta\overline{\psi}_+} + \Omega_+ \frac{\delta}{\delta\Omega_+} - \overline{\Omega}_+ \frac{\delta}{\delta\overline{\Omega}_+} + \psi_- \frac{\delta}{\delta\psi_-} - \overline{\psi}_- \frac{\delta}{\delta\overline{\psi}_-} + \Omega_- \frac{\delta}{\delta\Omega_-} - \overline{\Omega}_- \frac{\delta}{\delta\overline{\Omega}_-} \Big\} , \quad (5.26)$$

$$W^{(g)}_{\rm rigid} = \int d^3x \Big\{ \psi_+ \frac{\delta}{\delta\psi_+} - \overline{\psi}_+ \frac{\delta}{\delta\overline{\psi}_+} + \Omega_+ \frac{\delta}{\delta\Omega_+} - \overline{\Omega}_+ \frac{\delta}{\delta\overline{\Omega}_+} - \psi_- \frac{\delta}{\delta\psi_-} + \overline{\psi}_- \frac{\delta}{\delta\overline{\psi}_-} - \Omega_- \frac{\delta}{\delta\Omega_-} + \overline{\Omega}_- \frac{\delta}{\delta\overline{\Omega}_-} \Big\} . \quad (5.27)$$

5.3 Searching for Anomalies in the Model

The fact that a model is power-counting renormalizable, or multiplicatively renormalizable does not exclude the possibility of anomalies. There are no guarantees that the Slavnov-Taylor identity will hold at the quantum level, enabling the appearance of a gauge-anomalies, i.e., the electromagnetic and pseudochiral anomalies. These anomalies indicate the impossibility of extending the BRS symmetry at the quantum level, which jeopardizes the unitarity of the model, since BRS symmetry is fundamental ingredient to prove the unitarity of the S-matrix [39]. The parity anomaly is disregarded, as we have proven in the previous chapters that this symmetry holds at the quantum level. Therefore, our attention must to be turned to the Slavnov-Taylor identity and the possible quantum breakings that can arise.

As mentioned earlier, a breaking of a classical symmetry can occur at n-th order in the quantum action. If Γ and $\mathcal{S}$ are the quantum action and the Slavnov-Taylor operator of the parity preserving QED$_3$, respectively, then according to the quantum action principle, if the symmetry holds up to order $n-1$, then a breaking at order n can occur:

$$\mathcal{S}(\Gamma)|_{s=1} = \hbar^n \Delta_n + \mathcal{O}(\hbar^{n+1}) , \quad (5.28)$$

where Δ represents the possible breakings, and in this case, it is given by a integrated local polynomial in the fields, possessing the same quantum numbers of $\mathcal{S}(\Gamma)$. It is straightforward to check that the integrated polynomials of the breakings have ghost number 1, by the structure of $\mathcal{S}$, which has all the terms containing or a c, or a ξ in the numerator, or a derivative with respect to $\bar{c}$, $\bar{\xi}$, Ω_+ or Ω_-, whereas Γ has ghost number 0. In addition, since UV and IR dimensions of Γ are 3, it can be checked that the highest UV dimension [2] in $\mathcal{S}(\Gamma)$ is 7/2, whereas the lowest IR dimension is 3. Then, the most general symmetry breaking terms have to be integrated polynomials in the fields, with ghost number 1 and UV dimension $d \leq 7/2$ and IR dimension $r \geq 3$.

From the nilpotency identities and from (5.28), it follows that

$$\mathcal{S}_\Gamma \mathcal{S}(\Gamma) = 0 \ \Rightarrow \mathcal{S}_\Gamma(\hbar^n \Delta + \mathcal{O}(\hbar^{n+1})) = 0 . \quad (5.29)$$

Taking into account (5.18), as $\Gamma = \Gamma_0 + \mathcal{O}(\hbar)$, it is possible to see that

$$\mathcal{S}_\Gamma = \mathcal{S}_{\Gamma_0} + \mathcal{O}(\hbar) . \quad (5.30)$$

[2]Consider, for example, the term of $\mathcal{S}(\Gamma)$ given by $-\frac{1}{e}\int d^3x \partial^\mu c \, \frac{\delta\Gamma}{\delta A^\mu}$. Since the derivative behaves as a momentum, it has UV and IR dimensions equal to 1, whereas c has both equal to zero. Additionally, A^μ has UV dimension 1/2 and IR dimension 1. Therefore, this term has $d = 7/2$ and $r = 3$

Therefore, from the two previous equations, we have

$$\mathcal{S}_{\Gamma_0}\hbar^n\Delta + \mathcal{O}(\hbar^{n+1}) = 0 \ . \tag{5.31}$$

In the previous equation, there is a power series in $\hbar$ which is identically zero. Thus, due to the independence of terms with different powers of $\hbar$, it can be concluded that

$$\mathcal{S}_{\Gamma_0}\Delta = 0 \ . \tag{5.32}$$

The previous equation expresses a condition that the possible breakings must obey, and it is known as the Wess-Zumino consistency condition [16]. It gives rise to a cohomology problem, which will be solved for the parity-preserving QED$_3$ in the following.

When the quantum action is considered in place of Γ_0, the equations (5.22) do not necessarily hold and it must be proven that they can be extended to the quantum-level. Considering the Lagrange multipliers, it is possible to show that

$$\frac{\delta\Gamma}{\delta b} = \partial^\mu A_\mu + \alpha b \ , \qquad \frac{\delta\Gamma}{\delta\pi} = \partial^\mu a_\mu + \beta\pi \ . \tag{5.33}$$

The previous result holds because, if there was a breaking, for example, as

$$\frac{\delta\Gamma}{\delta b} = \partial^\mu A_\mu + \alpha b + \hbar^n\Delta_b + \mathcal{O}(\hbar^{n+1}) \ , \tag{5.34}$$

then it is always possible to write $\Delta_b = \dfrac{\delta\hat{\Delta}_b}{\delta b}$ in such a way that the breaking could be reabsorbed in the quantum action, restoring (5.33). In a similar way, it can be established that

$$\begin{aligned}
\Delta^{(1)}_{\text{clas}} &\equiv -i\frac{\delta\Gamma}{\delta c} = i\Box\bar{c} + \overline{\Omega}_+\psi_+ - \overline{\Omega}_-\psi_- + \overline{\psi}_+\Omega_+ - \overline{\psi}_-\Omega_- \ , \\
\Delta^{(2)}_{\text{clas}} &\equiv -i\frac{\delta\Gamma}{\delta\xi} = i\Box\bar{\xi} + \overline{\Omega}_+\psi_+ + \overline{\Omega}_-\psi_- + \overline{\psi}_+\Omega_+ + \overline{\psi}_-\Omega_- \ , \\
\frac{\delta\Gamma}{\delta\bar{c}} &= \Box c \ , \quad \frac{\delta\Gamma}{\delta\bar{\xi}} = \Box\xi \ .
\end{aligned} \tag{5.35}$$

We will still need of the following results, which are valid for any functional $\mathcal{F}$:

$$\frac{\delta\mathcal{S}(\mathcal{F})}{\delta b} - \mathcal{S}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta b} - \partial^\mu A_\mu - \alpha b\right) = \frac{1}{e}\left(\frac{\delta\mathcal{F}}{\delta\bar{c}} - \Box c\right) \ , \tag{5.36}$$

$$\frac{\delta\mathcal{S}(\mathcal{F})}{\delta\pi} - \mathcal{S}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta\pi} - \partial^\mu a_\mu - \beta\pi\right) = \frac{1}{g}\left(\frac{\delta\mathcal{F}}{\delta\bar{\xi}} - \Box\xi\right) \ , \tag{5.37}$$

The rigid symmetries must also be taken into account. Since their group is non-semisimple, they might be, in principle, anomalous [40, 41]. Nonetheless, the abelian factors are not spontaneously broken, with the electric charge (e) and pseudochiral charge (g) conserved, resulting in [41]:

$$W^{(e)}_{\text{rigid}}\Gamma = W^{(g)}_{\text{rigid}}\Gamma = 0 \ . \tag{5.38}$$

In addition to that, for any functional $\mathcal{F}$, there are the following relations:

$$-i\int d^3x\frac{\delta\mathcal{S}(\mathcal{F})}{\delta c} + \mathcal{S}_{\mathcal{F}}\int d^3x\left(-i\frac{\delta\mathcal{F}}{\delta c} - \Delta^{(1)}_{\text{clas}}\right) = W^{(e)}_{\text{rigid}}\mathcal{F} \ , \tag{5.39}$$

$$-i\int d^3x\frac{\delta\mathcal{S}(\mathcal{F})}{\delta\xi}+\mathcal{S}_{\mathcal{F}}\int d^3x\left(-i\frac{\delta\mathcal{F}}{\delta\xi}-\Delta^{(2)}_{\rm clas}\right)=W^{(g)}_{\rm rigid}\mathcal{F}\ . \tag{5.40}$$

Let us prove (5.39). Taking into account the Grassmann parity of the fields, it can be verified that

$$\frac{\delta\mathcal{S}(\mathcal{F})}{\delta c}+\mathcal{S}_{\mathcal{F}}\frac{\delta\mathcal{F}}{\delta c}=\frac{1}{e}\partial^\mu\frac{\delta\mathcal{F}}{\delta A^\mu}\ . \tag{5.41}$$

As it is a divergence, the previous result when integrated over the whole space is zero. On the other hand, it is possible to see that

$$\int d^3x\mathcal{S}_{\mathcal{F}}\Delta^{(1)}_{\rm clas}=\frac{i}{e}\int d^3x\partial^\mu\partial_\mu b-W^{e}_{\rm rigid}\mathcal{F}\quad . \tag{5.42}$$

The first term in the right hand side of the previous equality is again an integral over a divergence, yielding zero. Thus, the relation (5.39) holds. The poof of (5.40) is analogous.

From (5.36)–(5.40), when considering the functional $\mathcal{F}$ as Γ and Δ, we obtain all the relations that the possible quantum breakings of the Slavnov-Taylor identity must obey:

$$\begin{aligned}
&\frac{\delta\Delta}{\delta b}=\frac{\delta\Delta}{\delta\bar{c}}=0\ ,\ \int d^3x\frac{\delta\Delta}{\delta c}=0\ ,\ \ W^{(e)}_{\rm rigid}\Delta=0\ ,\\
&\frac{\delta\Delta}{\delta\pi}=\frac{\delta\Delta}{\delta\bar{\xi}}=0\ ,\ \int d^3x\frac{\delta\Delta}{\delta\xi}=0\ ,\ \ W^{(g)}_{\rm rigid}\Delta=0\ .
\end{aligned} \tag{5.43}$$

The previous equations must be seen as boundary conditions for the cohomology problem. Furthermore, from the nilpotency identities (5.20), along with the Wess-Zumino consistency condition (5.32), the most general possible breaking is as follows:

$$\Delta=\widehat{\Delta}^{(1)}+\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}\ ,\ . \tag{5.44}$$

Here, $\widehat{\Delta}^{(1)}$ has ghost number 1 and is the cohomology, i.e., the non-trivial solutions, whereas $\widehat{\Delta}^{(0)}$ has ghost number 0 and $\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}$ represents the trivial solutions of the cohomology problem, because $\mathcal{S}_{\Gamma_0}\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}$ is identically zero, as $\mathcal{S}(\Gamma_0)=0$. The possible breaking terms that are given by $\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}$ are referred to as noninvariant counterterms (by the symmetries) and can be reabsorbed in the quantum action. To show the latter, one must consider that

$$\begin{aligned}
\mathcal{S}(\Gamma-\hbar^n\widehat{\Delta}^{(0)})=\mathcal{S}(\Gamma)-\hbar^n\int d^3x\Big\{&-\frac{1}{e}\partial^\mu c\frac{\delta\widehat{\Delta}^{(0)}}{\delta A^\mu}+\frac{b}{e}\frac{\delta\widehat{\Delta}^{(0)}}{\delta\bar{c}}-\frac{1}{g}\partial^\mu\xi\frac{\delta\widehat{\Delta}^{(0)}}{\delta a^\mu}+\\
&+\frac{\pi}{g}\frac{\delta\widehat{\Delta}^{(0)}}{\delta\bar{\xi}}+\frac{\delta\Gamma}{\delta\overline{\Omega}_+}\frac{\delta\widehat{\Delta}^{(0)}}{\delta\psi_+}+\frac{\delta\widehat{\Delta}^{(0)}}{\delta\overline{\Omega}_+}\frac{\delta\Gamma}{\delta\psi_+}-\frac{\delta\Gamma}{\delta\Omega_+}\frac{\delta\widehat{\Delta}^{(0)}}{\delta\overline{\psi}_+}-\frac{\delta\widehat{\Delta}^{(0)}}{\delta\Omega_+}\frac{\delta\Gamma}{\delta\overline{\psi}_+}+\\
&-\frac{\delta\widehat{\Delta}^{(0)}}{\delta\overline{\Omega}_-}\frac{\delta\Gamma}{\delta\psi_-}-\frac{\delta\Gamma}{\delta\overline{\Omega}_-}\frac{\delta\widehat{\Delta}^{(0)}}{\delta\psi_-}+\frac{\delta\Gamma}{\delta\Omega_-}\frac{\delta\widehat{\Delta}^{(0)}}{\delta\overline{\psi}_-}+\frac{\delta\widehat{\Delta}^{(0)}}{\delta\Omega_-}\frac{\delta\Gamma}{\delta\overline{\psi}_-}\Big\}+\mathcal{O}(\hbar^{2n})\\
=\mathcal{S}(\Gamma)-\hbar^n\mathcal{S}_\Gamma\widehat{\Delta}^{(0)}+\mathcal{O}(\hbar^{2n})=\mathcal{S}(\Gamma)-\hbar^n\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}+\mathcal{O}(\hbar^{n+1})\ ,
\end{aligned} \tag{5.45}$$

where we have used the fact that $\mathcal{S}_\Gamma=\mathcal{S}_{\Gamma_0}+\mathcal{O}(\hbar)$. Thus, as long as

$$\mathcal{S}(\Gamma)=\hbar^n\widehat{\Delta}^{(1)}+\hbar^n\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}+\mathcal{O}(\hbar^{n+1})\ , \tag{5.46}$$

it follows from (5.45) that

$$\begin{aligned}
&\mathcal{S}(\Gamma)-\hbar^n\mathcal{S}_{\Gamma_0}\widehat{\Delta}^{(0)}=\hbar^n\widehat{\Delta}^{(1)}+\mathcal{O}(\hbar^{n+1})\Rightarrow\\
&\Rightarrow\mathcal{S}(\Gamma-\hbar^n\widehat{\Delta}^{(0)})=\hbar^n\widehat{\Delta}^{(1)}+\mathcal{O}^{\hbar^{n+1}}\ .
\end{aligned} \tag{5.47}$$

Therefore, it is possible to consider the quantum action not solely as Γ, but as $\Gamma - \hbar^n \widehat{\Delta}^{(0)}$, which proves that the noninvariant counterterms can be reabsorbed into the quantum action. The other terms, however, given by $\widehat{\Delta}^{(1)}$ cannot be reabsorbed and they are genuine anomalies.

If a cohomology problem for a symmetry only admits trivial solutions, then it is possible to ensure that the model is free from anomalies, concerning that symmetry, because it can be restored, order by order, by adding appropriate counterterms to the quantum action. What must be emphasized here is that the procedure is inductive: as long as the breakings at order n can be reabsorbed, the symmetries, which were valid up to order $n-1$, now hold at order n. So, if there is a breaking, it must occur at order $n+1$, and repeating the approach, which will give rise to the same cohomology problem, then it is possible to conclude that the symmetries are not anomalous and they can be restored, order, by order.

As mentioned earlier, the breakings must be integrated local polynomials with UV and IR dimensions bounded by $d \leq 7/2$ and $r \geq 3$, respectively. However, a breaking of the Slavnov-Taylor identities must be, at least, at 1-loop order. According to the power-counting formula (3.10), the 1PI Feynman diagrams have two coupling constants, which changes[3] the UV and IR bounds to $d \leq 5/2$ and $r \geq 4$. From (5.33) and (5.43), it can be conclude that the breaking is independent of the fields b, π, $\bar{c}$ and $\bar{\xi}$. In addition, $\int d^3x \frac{\delta\Delta}{\delta c} = 0$ means that $\dfrac{\delta\Delta}{\delta c}$ is, at most, a divergence of a vector field, with ghost number 0. A similar conclusion can be achieved for $\dfrac{\delta\Delta}{\delta\xi}$. Hence, the most general quantum breaking can be written as

$$\Delta = \int d^3x \, \left(\mathcal{K}_\mu^{(0)} \partial^\mu c + \mathcal{X}_\mu^{(0)} \partial^\mu \xi\right) \,, \tag{5.48}$$

where $\mathcal{K}_\mu^{(0)}$ and $\mathcal{X}_\mu^{(0)}$ are rank-1 tensors, with ghost number 0 and with UV and IR dimensions bounded by $d \leq 3/2$ and $r \geq 3$, respectively, since the derivative ∂^μ has UV and IR dimensions equal to 1. As can be checked, this breaking satisfies all the constraints expressed by (5.33) and (5.43).

We have already proven that parity is not an anomalous symmetry in this model. So, we can consider the breaking even by parity. Due to the transformations of c and ξ under this symmetry, the expressions for $\mathcal{K}_\mu^{(0)}$ and $\mathcal{X}_\mu^{(0)}$ read:

$$\mathcal{K}_\mu^{(0)} = \sum_i v_{k,i} \mathcal{V}_\mu^i \,, \quad \mathcal{X}_\mu^{(0)} = \sum_i v_{x,i} \Pi_\mu^i \,, \tag{5.49}$$

where $\mathcal{V}_\mu^i$ are vectors and Π_μ^i are pseudovectors, while $v_{k,i}$ and $v_{x,i}$ are coefficients to be determined.

To finish the proof that the model is anomaly free, we determine all possible $\mathcal{V}_\mu^i$ and Π_μ^i, respecting the conditions that the breaking must obey, with $d \leq 3/2$ and $r \geq 3$. They read:

$$\begin{aligned} &\mathcal{V}_\mu^1 = A_\mu A^\nu A_\nu \,, \quad \mathcal{V}_\mu^2 = A_\mu a^\nu a_\nu \,, \quad \mathcal{V}_\mu^3 = A_\nu a^\nu a_\mu \,, \\ &\Pi_\mu^1 = a_\mu a^\nu a_\nu \,, \quad \Pi_\mu^2 = a_\mu A^\nu A_\nu \,, \quad \Pi_\mu^3 = a_\nu A^\nu A_\mu . \end{aligned} \tag{5.50}$$

[3] The coupling constants do not have UV ou IR dimension, but they affect the power-counting, as they are related to the number of vertices of a diagram and the model is superrenormalizable.

Thus, the must general breaking is as follows:

$$\Delta = \int d^3x\{(v_{k,1}A_\mu A^\nu A_\nu + v_{k,2}A_\mu a^\nu a_\nu + v_{k,3}A_\nu a^\nu a_\mu)\partial^\mu c + (v_{x,1}a_\mu a^\nu a_\nu + \\ + v_{x,2}a_\mu A^\nu A_\nu + v_{x,3}a_\nu A^\nu A_\mu)\partial^\mu \xi\} \ . \tag{5.51}$$

Due to the Wess-Zumino consistency condition (5.32), we have

$$v_{k,1} = -\frac{4}{e}\lambda_1 \ , \quad v_{k,2} = -\frac{2}{e}\lambda_3 \ , \quad v_{k,3} = -\frac{2}{e}\lambda_4 \ , \\ v_{x,1} = -\frac{4}{g}\lambda_2 \ , \quad v_{x,2} = -\frac{2}{g}\lambda_3 \ , \quad v_{x,3} = -\frac{2}{g}\lambda_4 \ . \tag{5.52}$$

Defining

$$\widehat{\Delta}^{(0)1} = \int d^3x(A_\mu A^\mu)^2 \ , \quad \widehat{\Delta}^{(0)2} = \int d^3x(a_\mu a^\mu)^2 \\ \widehat{\Delta}^{(0)3} = \int d^3x A_\mu A^\mu a_\nu a^\nu, \widehat{\Delta}^{(0)4} = \int d^3x A_\mu A^\nu a_\nu a^\mu \ , \tag{5.53}$$

the most general breaking is given only by noninvariant counterterms as follows:

$$\Delta = \mathcal{S}_{\Gamma_0}(\lambda_1\widehat{\Delta}^{(0)1} + \lambda_2\widehat{\Delta}^{(0)2} + \lambda_3\widehat{\Delta}^{(0)3} + \lambda_4\widehat{\Delta}^{(0)4}) \ . \tag{5.54}$$

Therefore, as the solution of the cohomology problem is trivial, the model is free from anomaly of the BRS symmetry, and the latter can be restored order by order, adding appropriate counterterms to the quantum action. If a BRS breaking occurs, certainly it is originated by a renormalization procedure. A question that might arise is whether the model has infrared anomaly, due to the presence of massless fields in the model. Despite that, none of the terms $\widehat{\Delta}^{(0)i}$, $i = 1, ..., 4$ violates the infrared condition ($r \geq 4$). Therefore, the parity-preserving QED$_3$ is anomaly free and all the symmetries can be extended to the quantum-level.

5.4 Searching for counterterms

At this moment, the stability of the action is analyzed. The stability ensures that terms generated by perturbative quantum corrections are only terms corresponding to the renormalization of the parameters of the model that are already present in the classical action. They can be reabsorbed in the action as invariant counterterms, redefining physical quantities, such as coupling constants, masses and fields. Let us suppose that the action Γ_0 is perturbed by the addition of a counterterms action Σ^c, which is the most general action that fulfills the symmetries of the model. The stability condition is expressed by

$$\Gamma_0[\Phi_i, \rho_i, \lambda_i] + \varepsilon\Sigma^c[\Phi_i, \rho_i, \lambda_i] = \Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0] + \mathcal{O}(\varepsilon^2) \ , \tag{5.55}$$

Here, Σ^0 is the bare action of first order in ε and

$$\Phi_i^0 = \Phi_i(1 + \varepsilon Z_{\Phi_i}) \ , \ \rho_i^0 = \rho_i(1 + \varepsilon Z_{\rho_i}) \text{ e } \lambda_i^0 = \lambda_i(1 + \varepsilon Z_{\lambda_i}) \tag{5.56}$$

The action Σ^0 must obey the same conditions that Γ_0 obeys, given by (5.25), (5.35). This fact induces some constraints to Σ^c. For example,

$$\frac{\delta\Sigma^0}{\delta b} = \frac{\delta\Sigma^{(s-1)}}{\delta b} + \varepsilon\frac{\delta\Sigma^c}{\delta b} = \partial^\mu A_\mu + \alpha b \Rightarrow \frac{\delta\Sigma^c}{\delta b} = 0 \ . \tag{5.57}$$

The remaining constraints are

$$\frac{\delta\Sigma^c}{\delta\bar{c}} = \frac{\delta\Sigma^c}{\delta c} = \frac{\delta\Sigma^c}{\delta\pi} = \frac{\delta\Sigma^c}{\delta\bar{\xi}} = \frac{\delta\Sigma^c}{\delta\xi} = \frac{\delta\Sigma^c}{\delta b} = 0,$$
$$W^{(e)}_{\text{rigid}}\Sigma^c = 0 \ , \quad \text{and} \quad W^{(g)}_{\text{rigid}}\Sigma^c = 0 \tag{5.58}$$

As can be seen, Σ^c is independent of the ghost fields $(c, \bar{c}, \xi, \bar{\xi})$. Furthermore, as Γ_0 and Σ^0 have ghost number 0, then Σ^c also has ghost number 0. Thus, it can be concluded that Σ^c is independent of the external currents $(\Omega_\pm,\ \bar{\Omega}_\pm)$, that is,

$$\frac{\delta\Sigma^c}{\delta\Omega_+} = \frac{\delta\Sigma^c}{\delta\Omega_-} = \frac{\delta\Sigma^c}{\delta\overline{\Omega}_+} = \frac{\delta\Sigma^c}{\delta\overline{\Omega}_-} = 0 \ . \tag{5.59}$$

We can summarize the conditions that Σ^c must obey as follows:

$$\Sigma^c = \Sigma^c[\overline{\psi}_+, \overline{\psi}_-, \psi_+, \psi_-, A_\mu, a_\mu] \ , \tag{5.60}$$
$$W^{(e)}_{\text{rigid}}\Sigma^c = 0 \quad \text{and} \quad W^{(g)}_{\text{rigid}}\ \Sigma^c = 0 \ , \tag{5.61}$$
$$\Sigma^c \xrightarrow{P} \Sigma^c \ , \quad \mathcal{S}_\Sigma\Sigma^c = 0 \ , \quad \Sigma^c|^{d\leq 7/2}_{r\geq 3} \ . \tag{5.62}$$

The most general Σ^c reads

$$\Sigma^c = \int d^3x\{\alpha_1 i\overline{\psi}_+ \not{D}\psi_+ + \alpha_1 i\overline{\psi}_- \not{D}\psi_- + \\ \alpha_2 F^{\mu\nu}F_{\mu\nu} + \alpha_3 f^{\mu\nu}f_{\mu\nu} + \alpha_4\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu\} \ . \tag{5.63}$$

Since Σ^c has contributions of at least 1-loop order, the number of coupling constants for the power-counting has to be taken into account, which changes the UV and IR bounds to $d \leq 5/2$ and $r \geq 4$, as in the previous section. Therefore, none of the terms of (5.63) are allowed, as all of them violate the infrared limit, and we conclude that $\Sigma^c = 0$ and none parameter of the model is renormalized. This fact, along with the absence of anomalies, proves the perturbative finiteness – vanishing all β functions and all anomalous dimensions – of the parity-preserving QED$_3$. This indicates the quantum scale invariance of the model, that is, the parameters do not depend on an energy-scale. The quantum scale invariance, together with the absence of pseudochiral anomaly obtained from this model, seems to agree with experimental data of measures performed in pristine graphene presented in [28, 29, 38].

Final Remarks

As verified, planar quantum electrodynamics models may have wide applications in condensed matter, properly describing certain phenomena, and should not be disregarded based on the argument that the universe is clearly four-dimensional. Especially in condensed matter, the collective behavior may give rise to quasi-particles whose interactions and dynamics are purely planar.

In terms of the fractional quantum Hall effect, in Chapter 2, we analyzed the Kaplan-Sen model [13]. When the action was written in terms of the gauge fields A_μ and Z_μ, it seemed to be non-unitary at the tree-level, as the imaginary parts of current-current transition amplitudes of the mixed propagator were

$$\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{AZ}|_{k^2=\mu_+^2}\}\} = = -C_A C_Z \frac{H(E+G)}{\mu_+^2 - \mu_-^2} \ .$$

for the pole μ_+ and

$$\mathcal{I}m\{\mathcal{R}es\{\mathcal{A}_{AZ}|_{k^2=\mu_-^2}\}\} = C'_A C'_Z \frac{H(E+G)}{\mu_+^2 - \mu_-^2} \ .$$

for the pole μ_-, having the possibility of yielding negative imaginary parts for the current-current transition amplitudes, presenting problems with unitarity, according to the optical theorem. This, along with the fact that the mixed propagator could propagate degrees of freedom, suggested that the boson gauge fields were not fundamental and a deeper investigation was necessary. Regarding causality, there were no problems in the model, since tachyons were not present in its spectrum. We performed a diagonalization procedure, finding the fundamental gauge fields by the transformation

$$\begin{bmatrix} W_+^\mu \\ W_-^\mu \end{bmatrix} = \begin{bmatrix} -\xi & \zeta \\ \zeta & \xi \end{bmatrix} \begin{bmatrix} A^\mu \\ Z^\mu \end{bmatrix} \ .$$

where

$$\xi = \frac{2H}{\sqrt{4H^2 + \left[(E-G) - \sqrt{(E-G)^2 + 4H^2}\right]^2}} \ ,$$

$$\zeta = \frac{(E-G) - \sqrt{(E-G)^2 + 4H^2}}{\sqrt{4H^2 + \left[(E-G) - \sqrt{(E-G)^2 + 4H^2}\right]^2}}$$

The unitarity became explicit when we identified the fundamental boson gauge fields. Furthermore, we verified that no specific conditions regarding fermion flavors and coupling constants were necessary to ensure causality and unitarity. Additionally, we found that the effective action for the fundamental "photon" of the model (W_-), was given by

$$\Sigma_{\rm photon} = \int d^3x \Big\{ -\frac{1}{4} F_-^{\mu\nu} F_{\mu\nu}^- + \nu \frac{e^2}{4\pi} \varepsilon_{\mu\rho\nu} W_-^\mu \partial^\rho W_-^\nu - J_-^\mu W_{-_\mu} \Big\} + \cdots \ ,$$

where $\nu = \left(n_\psi + \frac{n_\chi n_\omega}{n_\chi + n_\omega} \right)$. Furthermore, the fractional Hall conductivity was obtained:

$$\sigma_{xy} = \nu \frac{e^2}{2\pi}, \qquad \text{with } \nu = \left(n_\psi + \frac{\nu_\chi n_\omega}{n_\chi + n_\omega} \right) \ . \tag{5.65}$$

and it was the same as that found by Kaplan and Sen, Moreover, we derived the gauge transformations of the fundamental fields.

The subsequent chapters focused on the parity-preserving QED$_3$ [14]. In Chapter 3, the symmetries, UV and IR dimensions of the fields and the power-counting were established. We showed that there were divergent diagrams up to 2-loops order, because the model was super-renormalizable. Additionally, the potentially divergent diagrams at 1 and 2-loops were identified. Motivated by the parity-breaking induced by BPHZL in the Ref. [26], we investigated if BPHZL applied to the parity-preserving QED$_3$ also would induce parity-violating terms. The 1-loop vacuum-polarization divergent diagrams were explicitly renormalized using the BPHZL subtraction scheme, yielding for $\gamma_{1\pm}$

$$\Pi^{(R)\mu\nu}_{\gamma_{1\pm}}(p,1) = -\frac{e^2}{16} \frac{\eta^{\mu\nu} p^2 - p^\mu p^\nu}{\sqrt{p^2}} \mp \frac{e^2 m}{4\pi |m|} \varepsilon^{\mu\alpha\nu} p_\alpha \ .$$

and similar expressions for $\gamma_{2\pm}$ and $\gamma_{3\pm}$, and all of them preserved parity when considering the whole contributions for the vacuum-polarizations. The self-energies also conserved parity at 1-loop. The 2-loop analysis, considering BPHZL, was performed and we conjectured that no parity-violating term could be generated, by a dimensional analysis. Additionally, we proved that the parity is conserved, since the terms with an odd number of γ-matrices should mutually cancel each other. This was different from what was obtained in [26], by applying the BPHZL to the massless QED$_3$. Therefore, the issue of parity, regarding the BPHZL subtraction scheme, is model dependent.

In Chapter 4, aiming to complete the renormalization of the parity-preserving QED$_3$, we applied the BPHZL subtraction scheme to the divergent vacuum-polarization diagrams at 2-loops, which were the remaining divergent diagrams, since the model was super-renormalizable. We determined the divergent subdiagrams and the Zimmermann's Forests. For γ_8 and γ_9 the forests were trivial, since there were no divergent subdiagrams and the BPHZL at 2-loops came down to the same procedure followed for 1-loop diagrams. For γ_{11} and γ_{12}, it was verified the presence of one divergent subdiagram, which yield non-trivial Zimmermann's forests. The renormalized diagrams were calculated explicitly, and it was shown that all the terms resulting from the divergent diagrams at 2-loops were non-local. In addition to that, all the terms generated did not break parity, agreeing with the conclusions obtained in the previous chapter.

In Chapter 5, we established that parity-preserving QED$_3$ was free from anomalies using Algebraic Renormalization. The most general breaking of the Slavnov-Taylor identity that might appear could be reabsorbed, restoring the symmetry, and since the method of the algebraic renormalization is inductive, the symmetry could be restored order, by order, proving the absence of BRS anomaly. We also proved that the model was free from infrared anomalies, as well as free from pseudochiral anomaly. Furthermore, we demonstrated that the most general counterterm action was null, implying the vanishing of all beta functions and anomalous dimensions of the fields. Consequently, we demonstrated one of the most important results of this work: the parity-preserving QED_3 exhibited scale invariance, i.e., the parameters of the theory were independent on the energy scale. This, combined with the absence of any anomalies, such as parity or infrared anomalies, established the perturbative finiteness of the model. The presence of scale invariance, the four-fold degeneracy of the Landau Levels, and the absence of pseudochiral anomaly seem to align with experimental observations in Refs. [28, 29, 38] concerning graphene.

As perspectives, we intend to:

- apply the algebraic renormalization procedure to the Kaplan-Sen model;
- perform an analysis of the unitarity of this model following the procedure of Kugo and Ojima performed in Ref. [39];
- analyze the parity-preserving QED_3 model, but considering massless gauge bosons and massive fermions, verifying possible applications to condensed matter phenomena.

Appendix A

Some notations and useful relations

It is important to establish a notation that facilitates calculations. We define two tensors, Ω and ζ, where the former represents the product of momenta, whereas the latter represents the product of η's:

$$\Omega(p)_n^{\mu_1..\mu_n} \equiv \begin{cases} p^{\mu_1}...p^{\mu_n} \text{ , if } n \geq 0 \\ 0, \text{if } n < 0 \text{ ;} \end{cases}$$
$$\zeta_n^{\mu_1...\mu_n} \equiv \begin{cases} \eta^{\mu_1\mu_2}...\eta^{\mu_{n-1}\mu_n} + \text{dist. perm., if } n \text{ is even,} \\ 0, \text{ if } n \text{ is odd or } n < 0 \text{ .} \end{cases} \tag{A.1}$$

Both tensors are completely symmetric, and by "dist. perm." we mean all distinct permutations. For instance, for the tensor ζ, permutations between indices on the same η are disregarded. In addition, if "dist. perm." is used after a sum of two or more tensors, then permutations between indices of different terms are forbidden. Nevertheless, permutations between indices of a product are allowed. As a final comment about the distinct permutations, if a tensor index has a bar, then it must be fixed and does not participate in the permutations. For example, in the expression $A^{\bar{\rho}\mu\nu\sigma}B^{\omega\delta}$ + dist. perm., the index ρ cannot be permuted, and terms such as $B^{\rho\alpha}$, where α is an arbitrary index, are not allowed.

It can be shown that ζ obeys the following relation:

$$\zeta_{2n+2}^{\mu_1...\mu_{2n+2}} = \zeta_{2n}^{\mu_1...\mu_{2n}}\eta^{\mu_{2n+1}\mu_{2n+2}} + \text{dist. perm.} \tag{A.2}$$

Indeed, suppose $\eta^{\nu_1\nu_2}...\eta^{\nu_{2n+1}\nu_{2n+2}}$ is a term in $\zeta_{2n}^{\mu_1...\mu_{2n+2}}$, where $(\nu_1, ..., \nu_{2n+2})$ is a permutation of $(\mu_1, ..., \mu_{2n+2})$. Since $\eta^{\nu_1\nu_2}...\eta^{\nu_{2n+1}\nu_{2n+2}}$ is also a term of $\zeta_{2n}^{\nu_1...\nu_{2n}}\eta^{\nu_{2n+1}\nu_{2n+2}}$, and the latter is a term of the right-hand side of (A.2), then every term of $\zeta_{2n+2}^{\mu_1...\mu_{2n+2}}$ is a term of ($\zeta_{2n}^{\mu_1...\mu_{2n}}\eta^{\mu_{2n+1}\mu_{2n+2}}$ + dist. perm.). Conversely, by the definition of ζ, every term of ($\zeta_{2n}^{\mu_1...\mu_{2n}}\eta^{\mu_{2n+1}\mu_{2n+2}}$ + dist. perm.) is a term of $\zeta_{2n+2}^{\mu_1...\mu_{2n+2}}$. Therefore, we conclude that (A.2) is true.

Another property of the tensors Ω and ζ is

$$\frac{\partial}{\partial p_\rho}\left(\Omega_n^{\mu_1...\mu_n}\zeta_{2m}^{\mu_{n+1}...\mu_{n+2m}} + \text{dist. perm}\right) = \Omega_{n-1}^{\mu_1...\mu_{n-1}}\zeta_{2m+2}^{\bar{\rho}\mu_n...\mu_{n+2m}} + \text{dist. perm} \tag{A.3}$$

We can prove the previous result by induction. If $m = 0$, then

$$\frac{\partial}{\partial p_\rho}\Omega_n^{\mu_1...\mu_n} = \Omega_{n-1}^{\mu_1...\mu_{n-1}}\zeta_2^{\bar{\rho}\mu_n} + \text{dist. perm.,} \tag{A.4}$$

where the distinct permutations are between the second index of ζ_2 and the indices of Ω_{n-1}. Let us assume that (A.3) is true. Multiplying both sides of (A.3) by $\eta^{\mu_{n+2m+1}\mu_{n+2(m+1)}}$, considering all the distinct permutations between ζ and η, and using (A.2), we have

$$\frac{\partial}{\partial p_\rho}\left(\Omega_n^{\mu_1\ldots\mu_n}\zeta_{2(m+1)}^{\mu_{n+1}\ldots\mu_{n+2(m+1)}}+\text{dist. perm}\right)=\Omega_{n-1}^{\mu_1\ldots\mu_{n-1}}\zeta_{2(m+1)+2}^{\bar{\rho}\mu_n\ldots\mu_{n+2(m+1)}}+\text{dist. perm}\ , \quad (A.5)$$

which proves that (A.3) is valid.

Appendix B

The $\mathbf{J}_r$-integrals

Here are presented the J_r-integrals, frequently encountered in loop calculations. Also, some useful and significant results are shown. These results remain applicable across arbitrary dimensions, whether integers or not. A general J_r-integral expression is

$$J_r^{\mu_1\cdots\mu_r}(\alpha, p, c) = \int \frac{d^D k}{(2\pi)^D} \frac{k^{\mu_1}...k^{\mu_r}}{(k^2 + 2p\cdot k - c)^\alpha} \,. \tag{B.1}$$

The three subsequent results can be found in [42, pp. 83].

$$J_0 \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + 2pk - c)^\alpha} = i(-1)^\alpha \frac{\pi^{d/2}}{(2\pi)^d} (c+p^2)^{d/2-\alpha} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \,. \tag{B.2}$$

$$J_1^\mu \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + 2pk - c)^\alpha} = i(-1)^{\alpha+1} \frac{\pi^{d/2}}{(2\pi)^d} (c+p^2)^{d/2-\alpha}\, p^\mu \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \,. \tag{B.3}$$

$$\begin{aligned} J_2^{\mu\nu} \equiv \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 + 2pk - c)^\alpha} &= i(-1)^\alpha \frac{\pi^{d/2}}{(2\pi)^d} (c+p^2)^{d/2-\alpha} \times \\ &\times \left[\frac{\Gamma(\alpha - d/2) p^\mu p^\nu - \frac{1}{2}\Gamma(\alpha - 1 - d/2)\eta^{\mu\nu}(c+p^2)}{\Gamma(\alpha)} \right] \,. \end{aligned} \tag{B.4}$$

It is important to note that there is a recurrence relation from where $J_{n+1}^{\mu_1\cdots\mu_{n+1}}$ can be obtained, using the derivative of $J_n^{\mu_1\cdots\mu_n}$ in respect to $p^{\mu_{n+1}}$. The recurrence relation is

$$J_{r+1}^{\mu_1\ldots\mu_{r+1}}(\alpha, p, c) = -\frac{1}{2} \frac{1}{\alpha - 1} \frac{\partial J_r^{\mu_1\cdots\mu_r}}{\partial p_{\mu_{r+1}}}(\alpha - 1, p, c) \,. \tag{B.5}$$

Thus, the expression for J_3 is

$$\begin{aligned} J_3^{\mu\nu\beta} \equiv \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\beta}{(k^2 + 2pk - c)^\alpha} &= i(-1)^\alpha \frac{\pi^{d/2}}{(2\pi)^d} (c+p^2)^{d/2-\alpha} \times \\ \times \left[\frac{\Gamma(\alpha - d/2) p^\mu p^\nu p^\beta - \frac{1}{2}\Gamma(\alpha - 1 - d/2)\left(\eta^{\mu\nu}p^\beta + \eta^{\nu\beta}p^\mu + \eta^{\beta\mu}p^\nu\right)(c+p^2)}{\Gamma(\alpha)} \right] \,. \end{aligned} \tag{B.6}$$

There is a general expression[1] for the results of these integrals:

$$
\begin{aligned}
J_r^{\mu_1\cdots\mu_r}(\alpha,p,c) &= i(-1)^{r+\alpha}\frac{\pi^{D/2}}{\Gamma(\alpha)(2\pi)^D}\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\left(-\frac{1}{2}\right)^j\Gamma(\alpha-d/2-j)\times \\
&\times(c+p^2)^{d/2-\alpha+j}\left(\Omega(p)_{r-2j}^{\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right) .
\end{aligned}
\tag{B.7}
$$

where $\lfloor\frac{r}{2}\rfloor$ is the floor function, which returns the closest smaller integer of $\frac{r}{2}$. The previous expression is particularly useful when dealing with J_r-integrals with r bigger than 3. It can be shown the validity of (B.7) by using the inductive method. As can be checked, it is valid for J_0. Let us suppose it is valid for an arbitrary r. Differentiating J_r with respect to $p_{\mu_{r+1}}$ we have

$$
\begin{aligned}
\frac{\partial J_r^{\mu_1\cdots\mu_r}}{\partial p_{\mu_{r+1}}}(\alpha,p,c) &= i(-1)^{r+\alpha}\frac{\pi^{D/2}}{\Gamma(\alpha)(2\pi)^D}\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\left(-\frac{1}{2}\right)^j\Gamma(\alpha-d/2-j)\times \\
&\times\frac{\partial}{\partial p_{\mu_{r+1}}}\left[(c+p^2)^{d/2-\alpha+j}\left(\Omega(p)_{r-2j}^{\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right)\right] .
\end{aligned}
\tag{B.8}
$$

Taking into account (A.3) and

$$
p^{\mu_{r+1}}\left(\Omega(p)_{r-2j}^{\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right) = \Omega(p)_{r-2j+1}^{\bar{\mu}_{r+1}\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.} ,
\tag{B.9}
$$

the derivative in (B.8) can be expanded as

$$
\begin{aligned}
&\frac{\partial}{\partial p_{\mu_{r+1}}}\left[(c+p^2)^{d/2-\alpha+j}\left(\Omega(p)_{r-2j}^{\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right)\right] = \\
&= 2(d/2-\alpha+j)(c+p^2)^{d/2-(\alpha+1)+j}\left(\Omega(p)_{r-2j+1}^{\bar{\mu}_{r+1}\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right)+ \\
&+(c+p^2)^{d/2-\alpha+j}\left(\Omega(p)_{r-2j-1}^{\mu_1\cdots\mu_{r-2j-1}}\zeta_{2(j+1)}^{\bar{\mu}_{r+1}\mu_{r-2j}\cdots\mu_r}+\text{dist. perm.}\right) ,
\end{aligned}
\tag{B.10}
$$

where indices with an upper bar do not permute. Furthermore, since $\Gamma(n+1)=n\Gamma(n)$, then

$$
\begin{aligned}
&\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\left(-\frac{1}{2}\right)^j\Gamma(\alpha-d/2-j)\;\;2(d/2-\alpha+j)(c+p^2)^{d/2-(\alpha+1)+j}\times \\
&\times\left(\Omega(p)_{r-2j+1}^{\bar{\mu}_{r+1}\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right) = \\
&\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\left(-\frac{1}{2}\right)^{j-1}\Gamma(\alpha+1-d/2-j)\;\;(c+p^2)^{d/2-(\alpha+1)+j}\times \\
&\times\left(\Omega(p)_{r-2j+1}^{\bar{\mu}_{r+1}\mu_1\cdots\mu_{r-2j}}\zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r}+\text{dist. perm.}\right) ,
\end{aligned}
\tag{B.11}
$$

[1]To the best of our knowledge, it is not found in the literature.

where $2(d/2-\alpha+j)$ have been written as $-2(\alpha-d/2-j)$ and absorbed in $-(1/2)^j G(\alpha-d/2-j)$. In addition, by performing the change $j \to j-1$, the next summation can be rewritten in the following way:

$$\sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \left(-\frac{1}{2}\right)^j \Gamma(\alpha-d/2-j)(c+p^2)^{d/2-\alpha+j} \left(\Omega(p)_{r-2j-1}^{\mu_1\cdots\mu_{r-2j-1}} \zeta_{2(j+1)}^{\bar{\mu}_{r+1}\mu_{r-2j}\cdots\mu_r} + \text{dist. perm.}\right) =$$

$$= \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor+1} \left(-\frac{1}{2}\right)^{j-1} \Gamma(\alpha+1-d/2-j)(c+p^2)^{d/2-(\alpha+1)+j}$$
$$\left(\Omega(p)_{r-2j+1}^{\mu_1\cdots\mu_{r-2j+1}} \zeta_{2j}^{\bar{\mu}_{r+1}\mu_{r-2j+2}\cdots\mu_r} + \text{dist. perm.}\right) . \tag{B.12}$$

It is worth noting that in the aforementioned expression, if r is even, the term with $j = \lfloor r/2 \rfloor + 1$ vanishes, since it comes from the differentiation, with respect to $p_{\mu_{r+1}}$, of a term that does not depend on p $(\Omega_0\zeta_{2r})$. So, if r is even, the summation must encompass terms up to $\lfloor r/2 \rfloor = \lfloor (r+1)/2 \rfloor$. The last equality is due to r being even. Moreover, if r is odd, then $\lfloor r/2 \rfloor + 1 = \lfloor (r+1)/2 \rfloor$ and in both cases we can change $\sum_{j=1}^{\lfloor \frac{r}{2} \rfloor+1}$ for $\sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor}$. We can also perform, in the summation present in (B.11), the change $\sum_{j=0}^{\lfloor \frac{r}{2} \rfloor}$ for $\sum_{j=0}^{\lfloor \frac{r}{2} \rfloor+1}$ since $\Omega_{-1} = 0$, and then we perform the change $\sum_{j=1}^{\lfloor \frac{r}{2} \rfloor+1}$ for $\sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor}$. Therefore, making the term for $j = 0$ in (B.11) explicit, (B.8) reads

$$\frac{\partial J_r^{\mu_1\cdots\mu_r}}{\partial p_{\mu_{r+1}}}(\alpha,p,c) = i(-1)^{r+\alpha} \frac{\pi^{D/2}}{\Gamma(\alpha)(2\pi)^D} \left[\left(-2\Gamma(\alpha+1-d/2)(c+p^2)^{d/2-(\alpha+1)} \times \right.\right.$$
$$\left. \times\, \Omega(p)_{r+1}^{\mu_1\cdots\mu_{r+1}} \right) + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} \left(-\frac{1}{2}\right)^{j-1} \Gamma(\alpha+1-d/2-j)(c+p^2)^{d/2-(\alpha+1)+j} \times$$
$$\left. \times \left(\Omega(p)_{r-2j+1}^{\bar{\mu}_{r+1}\mu_1\cdots\mu_{r-2j}} \zeta_{2j}^{\mu_{r-2j+1}\cdots\mu_r} + \Omega(p)_{r-2j+1}^{\mu_1\cdots\mu_{r-2j+1}} \zeta_{2j}^{\bar{\mu}_{r+1}\mu_{r-2j+2}\cdots\mu_r} + \text{dist. perm.} \right) \right] . \tag{B.13}$$

In the last term of the previous equation, the missing permutations between the index with an upper bar were obtained, since the referred index appear in both tensors, Ω and ζ. Then, (B.13) reads

$$\frac{\partial J_r^{\mu_1\cdots\mu_r}}{\partial p_{\mu_{r+1}}}(\alpha,p,c) = i(-1)^{r+\alpha} \frac{\pi^{D/2}}{\Gamma(\alpha)(2\pi)^D} \left[\left(-2\Gamma(\alpha+1-d/2)(c+p^2)^{d/2-(\alpha+1)} \times \right.\right.$$
$$\left. \times\, \Omega(p)_{r+1}^{\mu_1\cdots\mu_{r+1}} \right) + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} \left(-\frac{1}{2}\right)^{j-1} \Gamma(\alpha+1-d/2-j)(c+p^2)^{d/2-(\alpha+1)+j} \times \tag{B.14}$$
$$\left. \times \left(\Omega(p)_{r-2j+1}^{\mu_1\cdots\mu_{r-2j+1}} \zeta_{2j}^{\mu_{r+1}\mu_{r-2j+2}\cdots\mu_r} + \text{dist. perm.} \right) \right] .$$

Thus, by the recurrence relation (B.5), in the last expression, multiplying all the terms by $-1/(2(\alpha+1))$ and then performing the change $\alpha \to \alpha - 1$, and due to the fact that $(-1)^{r+a-1} = (-1)^{r+a+1}$, we obtain

$$J_{r+1}^{\mu_1 \cdots \mu_{r+1}}(\alpha, p, c) = i(-1)^{(r+1)+\alpha} \frac{\pi^{D/2}}{\Gamma(\alpha)(2\pi)^D} \sum_{j=0}^{\lfloor \frac{r+1}{2} \rfloor} \left(-\frac{1}{2}\right)^j \Gamma(\alpha - d/2 - j) \times$$
$$\times (c+p^2)^{d/2-\alpha+j} \left(\Omega(p)_{(r+1)-2j}^{\mu_1 \cdots \mu_{(r+1)-2j}} \zeta_{2j}^{\mu_{(r+1)-2j+1} \cdots \mu_{(r+1)}} + \text{dist. perm.} \right) . \tag{B.15}$$

Therefore, it can be concluded that (B.7) is valid.

Another important and useful result that has to be mentioned is regarding the Feynman parametrization. The denominator of a general loop integral is usually not in the form $(k^2 + 2p \cdot k - c)$, but it can be brought into this form by using the Feynman parametrization, a method that consists of introducing some Feynman parameters and substituting the original expression with an integral in these parameters. The simplest case of the Feynman parametrization is the following:

$$\frac{1}{\alpha(\alpha - \beta)} = \int_0^1 d^3x \frac{1}{(\alpha - \beta x)^2} , \tag{B.16}$$

The general case is [42]:

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \frac{\Gamma\left(\sum_i \alpha_i\right)}{\prod_j \Gamma(\alpha_j)} \int_0^1 dx_1 \cdots dx_n \frac{\delta\left(1 - \sum_i x_i\right) \prod_j x_j^{\alpha_j - 1}}{\left(\sum_k A_k x_k\right)^{\sum_l \alpha_l}} . \tag{B.17}$$

Beyond all the results established previously, the following results, found in [43], are also necessary:

$$\int_0^1 dx \, \sqrt{x^2 - x} = i\frac{\pi}{8} , \qquad \int_0^1 dx \, \frac{1}{\sqrt{x^2 - x}} = -i\pi . \qquad \int_0^1 dx \, \frac{x^2}{\sqrt{x^2 - x}} = -i\frac{3\pi}{8} \tag{B.18}$$

Moreover, since $\frac{d}{dx}\sqrt{x^2 - x} = \frac{x}{\sqrt{x^2 - x}} - \frac{1}{2}\frac{1}{\sqrt{x^2 - x}}$, it follows that

$$\int_0^1 dx \frac{x}{\sqrt{x^2 - x}} = -\frac{i\pi}{2} . \tag{B.19}$$

Appendix C

Determination of X'_l

We start from X_0 to determine X_1 and X_2. From (4.51), we have

$$X_1 = -\frac{f_1^{\delta'} g_{1\delta'}}{f_2} X_0 \quad , \qquad X_2 = \frac{g_2}{f_2} X_0 \quad . \tag{C.1}$$

Also, considering the definition

$$X'_0 = \int \frac{d^3k_1}{(2\pi)^3} \frac{X_0}{(k_1^2 + 2k_1 \cdot p' - c')^8} = \sum_{l'=0}^{10} X'_{0l'} \ , \tag{C.2}$$

it was possible to obtain the results for $X'_{0l'}$. In a similar way, we can define

$$X'_l = \int \frac{d^3k_1}{(2\pi)^3} \frac{X_l}{(k_1^2 + 2k_1 \cdot p' - c')^8} = \sum_{l'=0}^{10} X'_{ll'} \ . \tag{C.3}$$

Let us determine X_1. It has one less k_1 and one more k_2 than X_0. When integrated with respect to k_1, each of the terms $X'_{1l'}$ will yield a J_r with a subscript one unit smaller. However, the total power of the internal momenta is not changed. Therefore, to obtain $X'_{1l'}$ from $X'_{0l'}$, some minor changes in the summations and indices are needed. For example, let us check the structure of X'_{10}:

$$X'_{10} = \eta_{\phi\phi'} p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\delta'\bar{\chi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \quad . \tag{C.4}$$

In this example, the contraction between the indices δ' in $f_1^{\delta'} g_{1\delta'}$, after integration with respect to k_1, gives rise to a contravariant index δ', which can be permuted, and a covariant index δ' that is not permuted and is due to a factor $k_{\delta'}$ present in all terms. Moreover, unlike what happened with X'_{00}, the summation over j is up to 2 instead of 3, because it comes from a J_5 and not from a J_6. This is also responsible for the change in the summation over i and the power of B. In a similar way, it is possible to determine the remaining $X'_{1l'}$:

$$X'_{11} = \eta_{\phi\phi'} p^{\omega} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\delta'\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \quad ; \tag{C.5}$$

$$X'_{12} = - \eta_{\phi\phi'} p^{\omega} p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\delta'\bar{\chi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.6}$$

$$X'_{13} = - 2\eta_{\phi\phi'} p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\delta'\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.7}$$

$$X'_{14} = - \eta_{\phi\phi'} p^2 \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\delta'\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.8}$$

$$X'_{15} = - 2p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\delta\delta'\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.9}$$

$$X'_{16} = 4p_{\delta} p_{\delta''} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\delta\delta'\bar{\delta}''_k\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.10}$$

$$X'_{17} = -2p^2 p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\delta'\omega\delta\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.11}$$

$$X'_{18} = p^2 \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\delta'\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.12}$$

$$X'_{19} = - 2p^2 p_{\delta} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\omega\delta'\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.13}$$

$$X'_{1,10} = p^2 p^2 \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{6-2j-i} \zeta_{2j}\}^{\theta\omega\delta'\bar{\chi}_k\bar{\xi}_k}{}_{\bar{\delta}'_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \ ; \tag{C.14}$$

It is possible to repeat the procedure to determine the others $X'_{2l'}$:

$$X'_{20} = -\eta_{\phi\phi'} p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} \frac{\left[M(j) A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\bar{\chi}_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \ . \tag{C.15}$$

$$X'_{21} = - \eta_{\phi\phi'} p^{\omega} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\bar{\chi}_k\bar{\xi}_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.16}$$

$$X'_{22} = \eta_{\phi\phi'} p^{\omega} p^{\xi} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\bar{\chi}_k} \right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.17}$$

$$X'_{23} = 2\eta_{\phi\phi'} p_\delta \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.18}$$

$$X'_{24} = -\eta_{\phi\phi'} p^2 \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\bar{\xi}_k\bar{\chi}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.19}$$

$$X'_{25} = 2p_\delta \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\delta\bar{\chi}_k\bar{\xi}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.20}$$

$$X'_{26} = -4p_\delta p_{\delta'} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\delta\bar{\chi}_k\bar{\xi}_k\bar{\delta'}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.21}$$

$$X'_{27} = 2p^2 p_\delta \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\omega\delta\bar{\chi}_k\bar{\xi}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.22}$$

$$X'_{28} = -p^2 \sum_{j=0}^{1} \sum_{i=0}^{2-2j} M(j) \frac{\left[A^i B^{2-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.23}$$

$$X'_{29} = 2p^2 p_\delta \sum_{j=0}^{1} \sum_{i=0}^{2-2j} M(j) \frac{\left[A^i B^{2-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.24}$$

$$X'_{2,10} = -p^2 p^2 \sum_{j=0}^{1} \sum_{i=0}^{2-2j} M(j) \frac{\left[A^i B^{2-2j-i} \{p_i k_{6-2j-i} \zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k}\right] \eta^{\alpha\beta}}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}; \tag{C.25}$$

The structure of $X'_{3l'}$ is the same as that of $X'_{0l'}$, except for a -1 factor and for the tensor indices α and β, which are in a J_r function instead of a metric tensor. This also occurs with $X'_{4l'}, X'_{5l'}$, and $X'_{1l'}$, as well as with $X'_{6l'}$ and $X'_{2l'}$. Therefore, we are now presenting the remaining $X'_{ll'}$:

$X'_{3l'}$:

$$X'_{30} = \eta_{\phi\phi'} p^\xi \sum_{j=0}^{3} \sum_{i=0}^{6-2j} \frac{\left[M(j) A^i B^{6-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\alpha\beta\bar{\chi}_k}\right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}}\ . \tag{C.26}$$

$$X'_{31} =\eta_{\phi\phi'}p^{\omega}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\alpha\beta\bar{\chi}_k\bar{\xi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}} \tag{C.27}$$

$$X'_{32} = -\eta_{\phi\phi'}p^{\omega}p^{\xi}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\alpha\beta\bar{\chi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.28}$$

$$X'_{33} =-2\eta_{\phi\phi'}p_{\delta}\sum_{j=0}^{3}\sum_{i=0}^{6-2j}M(j)\frac{\left[A^iB^{6-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\alpha\beta\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.29}$$

$$X'_{34} = \eta_{\phi\phi'}p^{2}\sum_{j=0}^{3}\sum_{i=0}^{6-2j}M(j)\frac{\left[A^iB^{6-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\alpha\beta\bar{\xi}_k\bar{\chi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.30}$$

$$X'_{35} =-2p_{\delta}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\alpha\beta\omega\delta\bar{\chi}_k\bar{\xi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.31}$$

$$X'_{36} =4p_{\delta}p_{\delta'}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\alpha\beta\delta\bar{\chi}_k\bar{\xi}_k\bar{\delta}'_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.32}$$

$$X'_{37} =-2p^2p_{\delta}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{7-2j-i}\zeta_{2j}\}^{\theta\alpha\beta\omega\delta\bar{\chi}_k\bar{\xi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.33}$$

$$X'_{38} =p^2\sum_{j=0}^{2}\sum_{i=0}^{4-2j}M(j)\frac{\left[A^iB^{4-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\alpha\beta\bar{\chi}_k\bar{\xi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.34}$$

$$X'_{39} =-2p^2p_{\delta}\sum_{j=0}^{2}\sum_{i=0}^{4-2j}M(j)\frac{\left[A^iB^{4-2j-i}\{p_ik_{7-2j-i}\zeta_{2j}\}^{\theta\omega\alpha\beta\bar{\chi}_k\bar{\xi}_k}\bar{\delta}_k\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.35}$$

$$X'_{3,10} =p^2p^2\sum_{j=0}^{1}\sum_{i=0}^{2-2j}M(j)\frac{\left[A^iB^{2-2j-i}\{p_ik_{6-2j-i}\zeta_{2j}\}^{\theta\omega\alpha\beta\bar{\chi}_k\bar{\xi}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.36}$$

$X'_{4l'}$:

$$X'_{40} =-\eta_{\phi\phi'}p^{\xi}\sum_{j=0}^{2}\sum_{i=0}^{5-2j}M(j)\frac{\left[A^iB^{5-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\omega\phi\phi'\alpha\bar{\chi}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}} \ ; \tag{C.37}$$

$$X'_{41} =-\eta_{\phi\phi'}p^{\omega}\sum_{j=0}^{2}\sum_{i=0}^{4-2j}M(j)\frac{\left[A^iB^{4-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\phi\phi'\alpha\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}} \ ; \tag{C.38}$$

$$X'_{42} = \eta_{\phi\phi'} p^{\omega} p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\alpha\bar{\chi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.39}$$

$$X'_{43} = 2\eta_{\phi\phi'} p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\alpha\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.40}$$

$$X'_{44} = -\eta_{\phi\phi'} p^2 \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\alpha\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.41}$$

$$X'_{45} = 2p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\delta\alpha\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.42}$$

$$X'_{46} = -4p_{\delta} p_{\delta''} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\delta\alpha\bar{\delta}''_k\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.43}$$

$$X'_{47} = 2p^2 p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\alpha\omega\delta\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.44}$$

$$X'_{48} = -p^2 \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\alpha\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.45}$$

$$X'_{49} = 2p^2 p_{\delta} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\omega\alpha\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.46}$$

$$X'_{4,10} = -p^2 p^2 \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{6-2j-i} \zeta_{2j}\}^{\theta\omega\alpha\bar{\chi}_k\bar{\xi}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.47}$$

$X'_{5l'}$:

$$X'_{50} = -\eta_{\phi\phi'} p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\beta\bar{\chi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.48}$$

$$X'_{51} = -\eta_{\phi\phi'} p^{\omega} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\beta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.49}$$

$$X'_{52} = \eta_{\phi\phi'} p^{\omega} p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\phi\phi'\beta\bar{\chi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.50}$$

$$X'_{53} = 2\eta_{\phi\phi'} p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\beta\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.51}$$

$$X'_{54} = -\eta_{\phi\phi'} p^2 \sum_{j=0}^{2} \sum_{i=0}^{5-2j} M(j) \frac{\left[A^i B^{5-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\phi\phi'\beta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.52}$$

$$X'_{55} = 2p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\delta\beta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.53}$$

$$X'_{56} = -4p_{\delta} p_{\delta''} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\delta\beta\bar{\delta}''_k\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.54}$$

$$X'_{57} = 2p^2 p_{\delta} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} M(j) \frac{\left[A^i B^{4-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\beta\omega\delta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.55}$$

$$X'_{58} = -p^2 \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{8-2j-i} \zeta_{2j}\}^{\theta\omega\beta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.56}$$

$$X'_{59} = 2p^2 p_{\delta} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{7-2j-i} \zeta_{2j}\}^{\theta\omega\beta\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.57}$$

$$X'_{5,10} = -p^2 p^2 \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{6-2j-i} \zeta_{2j}\}^{\theta\omega\beta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} \; ; \tag{C.58}$$

$X'_{6l'}$:

$$X'_{60} = p^{\xi} \sum_{j=0}^{2} \sum_{i=0}^{4-2j} \frac{\left[M(j) A^i B^{4-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\alpha}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \; . \tag{C.59}$$

$$X'_{61} = p^{\omega} \sum_{j=0}^{1} \sum_{i=0}^{3-2j} M(j) \frac{\left[A^i B^{3-2j-i} \{p_i k_{9-2j-i} \zeta_{2j}\}^{\theta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k\bar{\beta}_k} \right]}{(k^2 + 2k \cdot p'' - c'')^{13/2-j}} ; \tag{C.60}$$

$$X'_{62} = - p^{\omega}p^{\xi}\sum_{j=0}^{1}\sum_{i=0}^{3-2j} M(j)\frac{\left[A^iB^{3-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\bar{\chi}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.61}$$

$$X'_{63} = - 2p_{\delta}\sum_{j=0}^{2}\sum_{i=0}^{4-2j} M(j)\frac{\left[A^iB^{4-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.62}$$

$$X'_{64} = p^2\sum_{j=0}^{2}\sum_{i=0}^{4-2j} M(j)\frac{\left[A^iB^{4-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\bar{\xi}_k\bar{\chi}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.63}$$

$$X'_{65} = - 2p_{\delta}\sum_{j=0}^{1}\sum_{i=0}^{3-2j} M(j)\frac{\left[A^iB^{3-2j-i}\{p_ik_{9-2j-i}\zeta_{2j}\}^{\theta\omega\delta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.64}$$

$$X'_{66} = 4p_{\delta}p_{\delta'}\sum_{j=0}^{1}\sum_{i=0}^{3-2j} M(j)\frac{\left[A^iB^{3-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\delta\bar{\chi}_k\bar{\xi}_k\bar{\delta'}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.65}$$

$$X'_{67} = - 2p^2p_{\delta}\sum_{j=0}^{1}\sum_{i=0}^{3-2j} M(j)\frac{\left[A^iB^{3-2j-i}\{p_ik_{7-2j-i}\zeta_{2j}\}^{\theta\omega\delta\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.66}$$

$$X'_{68} = p^2\sum_{j=0}^{1}\sum_{i=0}^{2-2j} M(j)\frac{\left[A^iB^{2-2j-i}\{p_ik_{8-2j-i}\zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.67}$$

$$X'_{69} = - 2p^2p_{\delta}\sum_{j=0}^{1}\sum_{i=0}^{2-2j} M(j)\frac{\left[A^iB^{2-2j-i}\{p_ik_{7-2j-i}\zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k\bar{\delta}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}; \tag{C.68}$$

$$X'_{6,10} = p^2p^2\sum_{j=0}^{1}\sum_{i=0}^{2-2j} M(j)\frac{\left[A^iB^{2-2j-i}\{p_ik_{6-2j-i}\zeta_{2j}\}^{\theta\omega\bar{\chi}_k\bar{\xi}_k\bar{\alpha}_k\bar{\beta}_k}\right]}{(k^2+2k\cdot p''-c'')^{13/2-j}}. \tag{C.69}$$

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