ÉMERSON DA SILVA MIRANDA

# ON THE QUANTUM PROPERTIES OF GRAPHENE-LIKE QUANTUM ELECTRODYNAMICS

Thesis submitted to the Physics Graduate Program of the Universidade Federal de Viçosa in partial fulfillment of the requirements for the degree of *Doctor Scientiae.*

Adviser: Oswaldo Monteiro Del Cima

Co-adviser: Daniel Heber Theodoro Franco

**VIÇOSA - MINAS GERAIS**
**2021**

**Ficha catalográfica elaborada pela Biblioteca Central da Universidade Federal de Viçosa - Campus Viçosa**

T

Miranda, Émerson da Silva, 1989-

M672o On the quantum properties of graphene-like quantum
2021 electrodynamics / Émerson da Silva Miranda. – Viçosa, MG, 2021.

1 tese eletrônica (124 f.): il. (algumas color.).

Texto em inglês .

Inclui apêndices.

Orientador: Oswaldo Monteiro Del Cima.

Tese (doutorado) - Universidade Federal de Viçosa, Departamento de Física, 2021.

Referências bibliográficas: f.113-124.

DOI: https://doi.org/10.47328/ufvbbt.2022.008

Modo de acesso: World Wide Web.

1. Hall, Efeito quântico de. 2. Grafeno. 3. Espaço e tempo. 4. Eletrodinâmica quântica. I. Cima, Oswaldo Monteiro Del, 1965-. II. Universidade Federal de Viçosa. Departamento de Física. Programa de Pós-Graduação em Física. III. Título.

CDD 22. ed. 530.14

Bibliotecário(a) responsável: Alice Regina Pinto CRB6 2523

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APPROVED: August 31, 2021

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# ACKNOWLEDGEMENTS

My entire gratitude for my Mother always inspired me to do whatever I decided to do giving me the necessary support to face the problems with courage and gratitude.

To my entire family, my brothers: Ronaldo, Reinaldo, Shirley, Luciano, Eliane, and Elimar. To my father Antônio.

To my girlfriend Thais for his pacience and companionship. Your presence was vital to this work.

Thanks to my professor, advisor and I believe can call him in this way, my friend Oswaldo Monteiro Del Cima (Wado). Routinely helping me and the other students, never saying what we do not can do, but motivating every student to do the best in their own time. His capability to aggregate the individual qualities of the students for collective works, which he sometimes develops for personal friendships, is one of his greatest qualities. Of course, there is no need to mention here his professional qualities, everyone who knows him does not doubt them.

Thanks to professor Daniel Franco. Sometimes giving us ear tugs and at the same time, always insisting that we must do our best.

Thanks to the mates Lázaro, Paulo Henrique, Wellison, Matheus, Marlon, Sylvestre, Amilton, Milton, Ricardo, Oscar, Thiago Moura and others whose names I forgot, thank you all. Thanks to the entire Physics Department of the Federal University of Viçosa (Universidade Federal de Viçosa) in special to the professors Winder and Márcio.

To the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), to granting the scholarship.

Thanks CAPES.

"*A miséria das classes baixas é sempre maior que o espírito de fraternidade das classes altas*"

(Victor Hugo)

# ABSTRACT

MIRANDA, Émerson da Silva, D.Sc., Universidade Federal de Viçosa, August, 2021. **On the quantum properties of graphene-like quantum electrodynamics**. Advisor: Oswaldo Monteiro Del Cima. Co-Advisor: Daniel Heber Theodoro Franco

First, in Part I we study a Lorentz invariant version of a mass-gap graphene-like planar quantum electrodynamics, the parity-preserving $U(1) \times U(1)$ massive $\text{QED}_3$. In Chapter 3, was showed it exhibits attractive interaction in low-energy electron-polaron–electron-polaron $s$-wave scattering, favoring quasiparticles bound states, the $s$-wave bipolarons. Still in Part I in Chapter 4, was verified that the model is ultraviolet finiteness – exhibits vanishing $\beta$-functions, associated to the gauge coupling constants (electric and pseudochiral charges) and the Chern-Simons mass parameter, and all the anomalous dimensions of the fields – as well as is parity and gauge anomaly free at all orders in perturbation theory. It was done adopting the Becchi-Rouet-Stora (BRS) algebraic renormalization method in the framework of Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) subtraction scheme. In the sequence, in Part II of the thesis, we deal with the parity-preserving $U(1) \times U(1)$ hybrid $\text{QED}_3$, proposed as a pristine graphene-like planar quantum electrodynamics model. In Chapter 5 was determined the spectrum as well as the four-fold broken degeneracy of the Landau levels, similar as the one experimentally observed in pristine graphene submitted to high applied external magnetic fields, besides that, was verified it exhibits zero-energy Landau level indicating a kind of anomalous quantum Hall effect. Furthermore, the electron-polaron–electron-polaron scattering potentials in $s$- and $p$-wave states mediated by photon and Néel quasiparticles might exhibit attractive ($s$-wave state) or repulsive ($p$-wave state) interactions. Already in Chapter 6 was analyzed the quantum parity conservation at all orders in perturbation theory for the hybrid model. It was proved, by using the Lowenstein-Zimmermann (LZ) subtraction scheme, that the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization method, preserves parity for the model. To finalize the Part II, in Chapter 7, was showed the vanishing of the $\beta$-functions associated to the coupling constants ($\beta_e = 0$ and $\beta_g = 0$), the mass parameter ($\beta_\mu$) and the anomalous dimensions of all fields ($\gamma_{\phi_i} = 0$), as well as the absence of parity and gauge anomalies at all orders of perturbation theory. Closing the analysis of the $U_A(1) \times U_a(1)$ hybrid $\text{QED}_3$ it is ultraviolet and infrared finite at all orders in perturbation theory.

Keywords: Quantum Hall effect. Graphene. Three space-time dimensions. Algebraic renormalization. Quantum Electrodynamics.

# RESUMO

MIRANDA, Émerson da Silva, D.Sc., Universidade Federal de Viçosa, agosto de 2021. **Propriedades quânticas da eletrodinâmica tipo grafeno**. Orientador: Oswaldo Monteiro Del Cima. Coorientador: Daniel Heber Theodoro Franco

Na parte I, estudou-se uma versão de eletrodinâmica quântica planar invariante de Lorentz tipo grafeno com *gap* de massa, uma $QED_3$ massiva Lorentz e paridade invariante com simetria $U(1) \times U(1)$. No Capítulo 3 demonstrou-se que o modelo massivo exibe um potencial de espalhamento (elétron-pólaron–elétron-pólaron) em baixas energias no estado de onda-$s$ atrativo propiciando estados ligados de quasiparículas, os bipolarons no estado de onda-$s$. Ainda na parte I, no Capítulo 4 verificou-se que o modelo massivo é finito no ultravioleta – exibe funções-$\beta$ nulas, associadas as constantes de acoplamento ( cargas elétrica e pseudo quiral) e o parâmetro de massa de Chern-Simons, assim como todas as dimensões anômalas dos campos – como também viu-se ser livre de anomalias de gauge e paridade em todas as ordens da teoria de perturbação onde usou-se o método de renormalização algébrica de BRS (Becchi-Rouet-Stora) tendo o BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann ) como esquema de subtração. Na sequência, parte II do trabalho, trabalhamos com a $QED_3$ híbrida Lorentz e paridade invariante com simetria $U(1) \times U(1)$, uma eletrodinâmica quântica planar proposta tipo grafeno puro (sem *gap de massa*). No Capítulo 5 determinou-se o espectro como também a degenerescência quártica dos níveis de Landau, similar ao que é observado em grafeno puro submetido a intensos campos magnéticos, além disso, verificou-se que esse exibe nível de Landau de nível zero indicando um possível efeito Hall quântico anômalo. Além do mais, calculou-se os potenciais de espalhamento elétron-polaron–elétron-polaron nos estados de onda-$s$ e onda-$p$ mediados pelos campos das quasipartículas de fóton e de Néel podendo exibir comportamento atrativo (estado de onda-$s$) ou repulsivo (estado de onda-$p$). Já no Capítulo 6 analisou-se a conservação de paridade quântica em todas às ordens em teoria de perturbação para o modelo híbrido utilizando-se do método de renormalização de BPHZL (Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein) tendo o esquema de subtração de Lowenstein-Zimmermann (LZ). Finalizando a parte II, no capítulo 7, mostrou-se a nulidade das funçẽos-$\beta$ associadas as constantes de acoplamentos ($\beta_e = 0$ e $\beta_g = 0$), do parâmetro de massa ($\beta_\mu$) e das dimensões anômalas de todos os campos ($\gamma_{\phi_i} = 0$), assim como também a ausência das anomalias de "*gauge*" e paridade em todas as ordens em teoria de perturbação. E por último, mostrou-se que o modelo híbrido é finito nos limites do ultravioleta e infravermelho em todas as ordens da teoria de perturbação.

Palavras-chave: Efeito Hall Quântico. Grafeno. Três dimensões espaço-temporais. Renormalização algébrica. Eletrodinâmica quântica.

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# Chapter 1

# Introduction

The particularities and singularities of $QED_3$ when compared to $QED_4$ show us that the results of a two-dimensional system are unique and that the non-existence of the third spatial component, nor its nullity, brings interesting results that can help in the study of emerging phenomena in physics. In this sense, $QED_3$ is explored as an appropriate theoretical tool to discuss fundamental issues of Fields and Particles Theory confined to the two-dimensional space world, such as quantization, spin and their interactions [1, 2].

Notwithstanding, the rising of new interpretations with possible correlations to physical known systems, in particular to those of Condensed Matter [3] as to quantum Hall effect [4], $SAT_c$ (Superconductivity at high critical temperature) and more recently graphene reinforces the need for a new perspective to planar theories.

Quoting the $SAT_c$ phenomenon discovered by A. Bednorz and K.A. Müller in 1986:' 'Possible High-$T_c$ Superconductivity in the Ba-La-Cu-O system " [5], after confirmation of these results, there was a considerable increase in research on the structure of copper oxide ceramics, revealing, among others things, a planar structure: these oxides are constituted by successive layers of copper-oxygen planes (Cu-O planes), separated from each other by plans of other oxides and this stratification is a reason for the application of the Field Theory, more specifically the $QED_3$ in order to describe some aspects of the " High-$T_c$ Superconductivity.f", as it implies a planning of some fundamental physical quantities of the superconducting state, such as the order and distance of the penetration parameter between Cu-O planes.

Models in $QED_3$ are justified not only by Superconductivity in high-$T_c$ which in itself is a sufficient reason for the studies of planar models, there is also the quantum Hall effect that is shown macroscopically as an effect resulting from planar microscopic chemical interactions. The quantum Hall effect (EHQ) is characterized by the quantization of the Hall conductivity ($\sigma = ne^2/h$, where $n$ ís an integer ) and the almost nullification of the longitudinal conductivity ($\sigma_{xx} \to 0$) of a two-dimensional electron electron subjected to an intense magnetic field ($B > 10T$) orthogonal to the plane, at very low temperatures ($T < 4K$). Hall conductivity is universal and independent of particularities (impurities, structures, etc ...) depending only on fundamental constants (electromagnetic coupling and Planck constant). This universality is a consequence of the quantization of the Landau levels and the fact that the conductivity (in $D = 1 + 2$) does not depend on the spatial extension samples.

In this sense, the study of quantum field theory in special the QED$_3$ is a way of improving the understanding of the purely theoretical, and when is the case, to correlate it with physical phenomena. But before we continue, we just clarify that a study doesn't mean to building models which describe specific phenomenon, like the electromagnetism for electromagnetic phenomena, we just are making correlations, and maybe in some future, the models can be more adjustable for a specific phenomenon.

In this way can be done the extension of Electrodynamics beyond to the traditional $D = 1+3$ dimensions starting from the Lagrangian formulation to the dynamics Maxwell's four-dimensional Lagrangian equations and keeping its shape invariant now at $D = 1 + 2$. Therefore, in this two-dimensional Maxwellian environment, we represent the action by:

$$S_{MAX} = \int d^3x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) . \tag{1.1}$$

where the metric $\eta_{\mu\nu} = (1, -1, -1)$ is used, being $\mu, \nu = 0, 1, 2$ and the gauge vector field $A_\mu$ with the usual field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, but now, with two spatial and one temporal dimensions. The action (1.1) is studied in details in [6] where the author shows the peculiar aspects of electric and magnetic fields, the propagation of these signals (The problem of the associated Green functions) and the typical logarithmic potential of these interactions.

Similar to what is made in $D = 1+3$, the construction of the quantum electrodynamics, a possible natural extension to a lower dimension, in this case ($D = 1 + 2$), is the coupling to Maxwell's terms (1.1) the Dirac's action term representing the fermions dynamics. Beyond that, the parity invariance in ($D = 1 + 2$) requires now two sets of fermions, which we use sub-indices ($+$ and $-$, also known as "flavours") in the wave functions ($\psi_\pm$) to distinguish them. We will see that when the fermions are massive the sub-indices make possible to distinguish the charges associated to the space-time symmetry, that is, the polarization of the spins associated to each fermionic function[1]. In this way, we start from a planar massive parity invariant quantum electrodynamics.

$$S_{qed} = \int d^3x \bigg\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}_+ \partial\!\!\!/ \psi_+ + i\overline{\psi}_- \partial\!\!\!/ \psi_- - m\overline{\psi}_+\psi_+ + m\overline{\psi}_-\psi_- \bigg\} . \tag{1.2}$$

where we use the common definition $C\!\!\!/ \equiv \gamma^\mu C_\mu$ and the Dirac adjoint $\overline{\psi}_\pm = \psi^\dagger_\pm \gamma^0$.

Within the proposal to contextualize the models presented in this work we can talk briefly about the condensed matter phenomena and how they contribute to the development

[1]This will become clearer in the development of Chapters 3 and 5.

of the quantum field theory in $D = 1 + 2$ and vice-versa. Focusing on superconductivity, in special the anionic superconductivity, after proved it to be at finite temperature impossible, new approaches to SAT$_c$ had to be made. The Chern-Simons (CS) formulation emerged as an appropriate tool for the replacement of anionic models in the planar quantum electrodynamics, like as, there was the ZHK model ( Zhang-Hansson-Kivelson) for the Fractional Hall Effect [7] the CS approach would come to superconductivity, since the statistical field ( Nèel field) ) $a_\mu$ representing a non-local interaction, perfectly reproduces the average field approximation of anionic superconductors. Formally represented by the Chern-Simons action.

$$S_{CS} = \int \left( \frac{\mu}{2} \epsilon^{\mu\rho\nu} a_\mu \partial_\rho a_\nu \right) d^3x \tag{1.3}$$

A peculiarity of the Chern-Simons electrodynamics is the generation of a massive photon of a non-local nature. This is due to the fact that the CS term has a topological character, as it does not depend on the metric and does not contribute to the energy-moment tensor of the system [8]. This action also presents gauge invariance, but it does not preserve parity and time-reversal symmetries. In addition, the CS field has no dynamics, always requiring the addition of a term for the propagation of the photon.

However, the parity break is a striking feature in CS theories and, in anionic superconductors for example, there is no definitive experimental evidence to prove the break in the parity symmetry of the superconducting state, we want to discuss a model in which this does not occur. Thus, models without parity breaking constitute an attempt to understand planar superconductivity and can be used in future studies that help to understand SAT$_c$. We therefore use a mixed CS term, coupling the vectorial gauge fields $A_\mu$ and $a_\mu$ and a term that is dynamic for field $a_\mu$ which in this case is Maxwell's term.

$$S_{CSM} = \int \left( \frac{\mu}{2} \epsilon^{\mu\rho\nu} A_\mu \partial_\rho a_\nu - \frac{1}{4} f^{\mu\nu} f_{\mu\nu} \, . \right) d^3x \tag{1.4}$$

In this way, a Maxwell-Chern-Simons (MCS) action is proposed, $S_{MCS} = S_{CSM} + S_{MAX}$, coupled to a system of fermions with "flavours" $(+, -)$ minimally coupled. This model is a way of write a massive electrodynamics parity invariant [9].

$$S = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \mu \epsilon^{\mu\rho\nu} A_\mu \partial_\rho a_\nu + \overline{\psi}_+ \not{D} \psi_+ + \overline{\psi}_- \not{D} \psi_- - m(\overline{\psi}_+ \psi_+ + \overline{\psi}_- \psi_-) \right\} . \tag{1.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the Néel field strength . The covariant derivative is defined by $\not{D} \psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a}) \psi_\pm$ and the gamma matrices $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$ constructed in terms of the Pauli matrices $\{\sigma_i\}$ which have the explicit form given by:

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad \gamma_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} , \quad \gamma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \tag{1.6}$$

In the development of this work, the graphene shows up like an important framework in the quantum electrodynamics as will be seen in the Chapters 3 and 5. For now, we know that the graphene monolayer [10] is a two-dimensional system with no mass-gap in which it behaves like a semi-metallic semi-filled with load carriers (quasiparticles ) being necessary for Dirac fermions loaded without mass. However, for practical applications, such as transistors, a mass gap graphene is more appropriate, and such a mass gap effect is observed in pure graphene monolayer on substrates [11]. Interactions electron-electron (pairing of electrons) [12] includes spreading processes of electron-polarons (electron - phonons) [13, 14], where these quasiparticles, the polarons, which are formed by linked states of electrons (or holes) and phonons, were first introduced by Landau in 1933 [15].

Although, in the context of condensed matter, especially when it comes to graphene [16], we know that we can have collective excitements that have a massless electrical charge, despite they are not elementary particles, they have a quantum number which is similar to spin . Spin, which already has its particularities in tree space-time dimensions [8, 17], does not have such an evident interpretation for massless particles. In four space-time dimensions, this problem is overcome by the magnitude called helicity, which is the projection of the spin in the direction of movement, but there is no two-dimensional analogous for helicity. Still with respect to the spin, the mass signal is responsible for fixing the spin signal in three dimensions, without the mass term, in principle, there would be no distinction between the $\psi_+$ and $\psi_-$ fermions. In fact, the distinction is only made, in the model, at the level of interaction through coupling with the chiral field $a_\mu$. Finally, the non-massive nature of fermions generates serious physical divergences in the model, such as infrared divergences, associated with low-energy fermions (long wavelengths) that must be analyzed.

To improve our comprehension of massless models, it is also proposed an action with fermions without mass, which in principle can be seeing like a subtle modification, it brings a series of notorious physical modifications. Although a charged (electrically) massless particle have not yet been observed in nature, there is still no explanation for it, therefore studies of models of this type can somehow improve the understanding of why a contest of a kind to that configuration. In addition, as massless particles propagate at the speed of light, what is impossible for massive particles, and therefore, they do not present a resting reference frame. Thus, in a similar way, we also write a Maxwell-Chern-Simons invariant by parity with massless fermions,

$$S_{m=0} = \int d^3x \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu + \overline{\psi}_+ \not{D}\psi_+ + \overline{\psi}_- \not{D}\psi_- \right\} . \quad (1.7)$$

The actions (1.5) and (1.7) share some common symmetries . The parity, already mentioned, is given by:

$$
\begin{aligned}
A_\mu &\xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ , \\
\partial_\mu &\xrightarrow{P} \partial_\mu^P = (\partial_0, -\partial_1, \partial_2) \ , \\
a_\mu &\xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ , \\
\psi_\pm &\xrightarrow{P} \psi_\pm^P = -i\gamma^1\psi_\mp \\
\overline{\psi}_\pm &\xrightarrow{P} \overline{\psi}_\pm^P = i\overline{\psi}_\mp\gamma^1 \ , \text{ onde } \overline{\psi} = \psi^\dagger\gamma^0.
\end{aligned} \tag{1.8}
$$

The previous actions are also invariant by local gauge transformations $U_A(1) \times U_a(1)$ which explicitly forms are:

$$
\begin{cases}
\psi_\pm \longrightarrow \psi'_\pm = e^{i(\rho_{(x)} \pm \lambda_{(x)})}\psi_\pm. \\
\overline{\psi}_\pm \longrightarrow \overline{\psi}'_\pm = e^{-i(\rho_{(x)} \pm \lambda_{(x)})}\overline{\psi}_\pm. \\
\not{D}\psi_\pm \longrightarrow (\not{D}\psi_\pm)' = e^{i(\rho_{(x)} \pm \lambda_{(x)})}\not{D}\psi_\pm,
\end{cases} \tag{1.9}
$$

and through the condition $(\not{D}\psi_\pm)' = \not{D}'\psi'_\pm$ we easily obtain the transformations imposed on the gauge fields:

$$
\begin{cases}
A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\rho_{(x)} \ , \\
a'_\mu = a_\mu - \frac{1}{g}\partial_\mu\lambda_{(x)} \ .
\end{cases} \tag{1.10}
$$

In the forthcoming chapters, a detailed discussion is made being organized essentially in two parts: Part I and Part II. However, before this division, we write the Chapter 2 where we discuss some important affirmations used in the process of algebraic renormalization, being the main purpose of this chapter the strengthen the coming chapters and serve as a basis for better comprehension of the text in a general form.

In the Part I we deal with the massive model having the action (1.5) as the starting point. The Chapter 3 have a complete study on the physical spectrum of the Maxwell-Chern-Simons quantum electrodynamics parity invariant model, the calculation of the scattering potential as well as the formation of bound-states of electron-polaron – electron-polaron [16] with the necessary conditions to it happens. To finalize the first part we have in the Chapter 4 the algebraic renormalization of the massive model proving at all orders that the model is renormalizable and it has not any kind of anomaly.

In Part II the parity invariant massless Maxwell-Chern-Simons quantum electrodynamics model (1.7) is studied. In the Chapter 5 we study the physical consistency of the massless model analysing the causality and unitarity at tree level as well as the explicit form of the wave function, together with the form of the scattering potential with the analysis of the possibility of formation of bound states in a possible application in pristine graphene. In the Chapter 6 we realize the renormalization of the massless model using the

Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization scheme finding explicitly values at one-loop of the Feynman diagrams and the expressions to two-loop of the Feynman graphs. In it, we prove the impossibility of the appearance of parity odd counterterm closing the false affirmation that the method breaks parity independently of the model. Still in part II, in the Chapter 7 the algebraic renormalization of the massless model is realized and we prove that there is not renormalization to the fields, masses and coupling constant of the model and in same way the absence of gauge and parity anomalies. Lastly we present the appendix.

# Chapter 2

# A brief talk about algebraic renormalization

In this chapter, we make a brief introduction about some results that will be used in forthcoming chapters. Some results are found in books that deal with quantization using the mechanism of path integrals [18] and also with the method of algebraic renormalization [19], as well as there are results present in course notes [20] . These results will serve as a starting point to the perturbation theory. It is worth mentioning that this chapter is a very brief introduction and does not intend to be an complete algebraic renormalization course, however, I believe it is valid to leave it as a part of the text since it brings an important contribution to the understanding of the coming chapters.

We start by considering classic models composed by a set of $\phi_i(x)$ fields in $D$ dimensions as follows:

$$S[\phi] = \int dx \; \mathcal{L}(x) = S_0[\phi] + S_{int}[\phi] \; , \tag{2.1}$$

where the Lagrangian density has the form,

$$\mathcal{L}(x) = \frac{1}{2}\phi_i(x)K^{ij}(\partial)\phi_j(x) + \mathcal{L}_{int}[\phi] = \mathcal{L}_0[\phi] + \mathcal{L}_{int}[\phi] \tag{2.2}$$

in general $K^{ij}(\partial)$ is a second-order differential operator (higher-order operators almost always lead to nonrenormalisability) and it must be invertible. The term $\mathcal{L}_{int}[\phi]$ corresponds to the Lagrangian interaction terms, that is, they are a polynomial field equal or higher than three.

## 2.1 The generating functional of Green's functions

The expected values of the time-ordered product of field operators in the vacuum, that is, the Green's functions, will be the main objects of study in this section. Being the basis for the study of renormalizations and practically all the relevant physical quantities, namely, the elements of the scatter matrix operator.

We can introduce the Feynman path integral,

$$\int \mathcal{D}\phi \; e^{\frac{i}{\hbar}S[\phi]} \; , \tag{2.3}$$

The mechanisms of Feynman path integral is one of the basis of quantum field theory. In (2.3) the notation $\mathcal{D}\phi$ is representing the product by:

$$\mathcal{D}\phi(x) = \prod_i d\phi(x_i, t_i) \; , \tag{2.4}$$

All Green's functions can be generated by the generating functional $Z[J]$ formally given by Feynman's path integral,

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}(S[\phi] + \int dx J^i(x)\phi_i(x))} \; , \tag{2.5}$$

where $\mathcal{N}$ is a numerical factor. An interesting way to see how the generating functional $Z[J]$ contributes is expanding functionally it in the following way:

$$\begin{aligned} Z[J] = Z[J=0] + \frac{1}{\hbar}\int dx \frac{\delta Z[J]}{\delta J(x)}\bigg|_{J=0} J(x) + \frac{1}{2\hbar^2}\int dx_1 dx_2 \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} J(x_1)J(x_2) + \cdots + \\ + \frac{(-1/\hbar)^n}{n!}\int dx_1 \cdots dx_n \frac{\delta^n Z[J]}{\delta J(x_1)\cdots \delta J(x_n)}\bigg|_{J=0} J(x_1)\cdots J(x_n) + \cdots \end{aligned} \tag{2.6}$$

If we known the expansion coefficients of $Z[J]$ we can completely determine our functional,

$$\frac{\delta^n Z[J]}{\delta J(x_1)\cdots \delta J(x_n)}\bigg|_{J=0} \equiv G_{i_1\cdots i_n}(x_1, \cdots, x_n) = \langle 0|T\phi_{i_1}(x_1)\cdots \phi_{i_n}(x_n)|0\rangle \; . \tag{2.7}$$

We identify (2.7) as $n$-points Green's functions. It should be noted that Green's functions are tempered distributions and the fonts $J(x)$ belong to the Schwarts fast decrease functions of fast decrease $C^\infty$, that is, they are test functions that guarantee convergence of the functional. And we also write $Z[J]$:

$$Z[J] = \sum_{N=0}^{\infty} \frac{(-1/\hbar)^N}{N!}\int dx_1 \cdots dx_N J^{i_1}(x_1)\cdots J^{i_1}(x_N) G_{i_1\cdots i_N}(x_1, \cdots, x_N) \; , \tag{2.8}$$

The case of the free action $S_0[\phi]$, that is, the action without the presence of the interaction term $S_{int}[\phi]$, can be analyzed immediately and it is useful as a base for the general case where the interaction term is present.

So, making an useful changing of variables:

$$
\begin{aligned}
\phi(x) &\longrightarrow \phi(x) - \int dy K^{-1}(\partial)(x,y)J(y) \ , \\
\mathcal{D}\phi(x) &\longrightarrow \mathcal{D}\phi(x) \ ,
\end{aligned} \tag{2.9}
$$

and keeping in mind the fact that $\int dz K^{ij}(\partial)(x,z)K^{-1}_{jl}(\partial)(z,y) = \delta^i_l \delta^3(x-y)$, we find:

$$
\begin{aligned}
Z_0[J] &= \mathcal{N}_0\bigg(\int \mathcal{D}\phi e^{S_0[\phi]}\bigg) \exp\left[\frac{1}{2\hbar^2}\int dxdy J(x)K^{-1}(\partial)(x,y)J(y)\right] \ , \\
Z_0[J] &= Z[J=0]\exp\left[\frac{1}{2\hbar^2}\int dxdy J(x)K^{-1}(\partial)(x,y)J(y)\right]
\end{aligned} \tag{2.10}
$$

using the normalization factor:

$$
\langle 0|0\rangle = 1 \Longrightarrow Z[J=0] = 1 \ , \tag{2.11}
$$

and therefore,

$$
Z_0[J] = \exp\left[\frac{1}{2\hbar^2}\int dxdy J(x)K^{-1}(\partial)(x,y)J(y)\right] \ , \tag{2.12}
$$

So, by the expression (2.7) we find the Green's functions to the free functional $Z_0[J]$:

$$
\begin{aligned}
\langle 0|T\phi_{i_1}(x_1)|0\rangle_0 &= 0 \ , \\
\langle 0|T\phi_{i_1}(x_1)\phi_{i_2}(x_2)|0\rangle_0 &= -iK^{-1}(\partial)(x_1-x_2) \ , \\
\langle 0|T\phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)|0\rangle_0 &= 0 \ , \\
\langle 0|T\phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)\phi_{i_4}(x_4)|0\rangle_0 &= iK^{-1}(\partial)(x_1-x_2)iK^{-1}(\partial)(x_3-x_4) + \\
&\quad + iK^{-1}(\partial)(x_1-x_3)iK^{-1}(\partial)(x_2-x_4) + \\
&\quad + iK^{-1}(\partial)(x_2-x_3)iK^{-1}(\partial)(x_1-x_4) \ .
\end{aligned}
$$

where $K^{-1}(\partial)(x,y)$ is identified as the fundamental free Stueckelberg-Feynman causal propagator in the study of the physical consistency of the models. We have observed here a pattern known as Wick's theorem which is summarized as follows:

$$
\left.\frac{\delta^n Z_0[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = \langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle_0 = \left\{\begin{array}{cc} 0 & , \ n = \text{odd} \\ iK^{-1}_{1,2}(\partial)\cdots iK^{-1}_{n-1,n}(\partial) & , \ n = \text{even} \end{array}\right\} \tag{2.13}
$$

It is observed that for any $n$ there are $(n-1)!!$ [1] possible numbers of connected pair of points. However, even though it is the case of the interesting free functional, we still need to find the Green's functions in the case where there is interaction $S_{int}[\phi]$ which is of physical interest. For such purpose let's separate the action as in equation (2.1) in the

[1] $(n-1)!! = (n-1)(n-1-2)(n-1-4)\cdots\ 1!$

functional $Z[J]$:

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S_{int}[\phi]} e^{\frac{i}{\hbar}(S_0[\phi] + \int dx J^i(x)\phi_i(x))} \tag{2.14}$$

Before we will present some important results,

$$\begin{aligned} \frac{\delta}{\delta J^{i_1}(y)} \left( e^{\frac{i}{\hbar}\int dx J^i(x)\phi_i(x)} \right) &= \frac{i}{\hbar}\phi_{i_1}(x) e^{\frac{i}{\hbar}\int dx J^i(x)\phi_i(x)} \\ \frac{\delta^2}{\delta J^{i_2}(z)\delta J^{i_1}(y)} \left( \exp \frac{i}{\hbar}\int dx J^i(x)\phi_i(x) \right) &= \left(\frac{i}{\hbar}\right)^2 \phi_{i_2}(x)\phi_{i_1}(x) e^{\frac{i}{\hbar}\int dx J^i(x)\phi_i(x)} \end{aligned} \tag{2.15}$$

what we do in the expression (2.15) is to associate to the fields a respective functional derivative calculated within the generator of Green's functions as follows:

$$\phi_i(x) \longrightarrow \frac{\hbar}{i} \frac{\delta}{\delta J^i(x)} \; , \tag{2.16}$$

Expanding the exponential of the interaction term in a power serie:

$$e^{\frac{i}{\hbar} S_{int}[\phi]} = 1 + \frac{i}{h} S_{int}[\phi] + \frac{i^2}{2!\hbar^2}(S_{int}[\phi])^2 + \frac{i^3}{3!\hbar^3}(S_{int}[\phi])^3 + \cdots \tag{2.17}$$

In the functional $Z[J]$ (2.14):

$$\begin{aligned} Z[J] &= \mathcal{N} \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}(S_0[\phi] + \int dx J^i(x)\phi_i(x))} + \mathcal{N} \int \mathcal{D}\phi \frac{i}{h} S_{int}[\phi] \; e^{\frac{i}{\hbar}(S_0[\phi] + \int dx J^i(x)\phi_i(x))} + \\ &\quad + \mathcal{N} \int \mathcal{D}\phi \; \frac{i^3}{3!\hbar^3}(S_{int}[\phi])^3 \, e^{\frac{i}{\hbar}(S_0[\phi] + \int dx J^i(x)\phi_i(x))} + \cdots \; , \end{aligned} \tag{2.18}$$

this shows us an important result that is the expansion of the functional with interaction in terms of the free functional, that is:

$$Z[J] = \mathcal{N} e^{\frac{i}{\hbar} S_{int}(\frac{\hbar}{i}\frac{\delta}{\delta J})} \, Z_0[J] \; , \tag{2.19}$$

The $Z[J]$ functional generates all Feynman diagrams (graphic representations of Feynman integrals) and it includes connected, disconnected and bubble graphs. However, we are looking for the contributions of the connected graphs that is given by the generating functional of the connected Greens functions $Z^c[J]$.

## 2.2 The connected and vertice functional

The generating functional of the connected Green functions is given by:

$$Z^c[J] = \sum_{N=1}^{\infty} \frac{(-1/\hbar)^N}{N!} \int dx_1 \cdots dx_N J^{i_1}(x_1) \cdots J^{i_N}(x_N) \langle 0 | T \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_N) | 0 \rangle_{CON} \; , \tag{2.20}$$

and it is related to the generating functional of Green functions by the expression

$$Z[J] = e^{-\frac{i}{\hbar}Z^c[J]} \Longrightarrow Z^c[J] = i\hbar \ln Z[J] \ , \tag{2.21}$$

Although we do not go into details about the graphics (Feynman diagrams), it is noticeable when calculating the graphics of the functional $Z^c[J]$ just survive the connected graphics, eliminating disconnected graphics and vacuum bubbles (graphics with loops in self-interaction without external legs).

Beyond the connected functional $Z^c[J]$ we also write the one-particle irreducible graph generator $\Gamma[\phi]$, commonly abbreviated by 1-PI ( 1- Particle irreducible). The contribution to the connected Green functions is that it has the same form but their mathematical and graphic representations are made amputating the external legs, as can be seen, the free propagators to the respective connected Green function.

$$\Gamma[\phi^{clas}] = \sum_{N=2}^{\infty} \frac{1}{N!} \int dx_1 \cdots dx_N \, \phi_{i_1}^{clas}(x_1) \cdots \phi_{i_N}^{clas}(x_N) \Gamma^{i_1 \cdots i_N}(x_1, \cdots, x_N) \tag{2.22}$$

here $\Gamma^{i_1 \cdots i_N}(x_1, \cdots, x_N) = \langle 0|T\phi_{i_1}(x_1) \cdots \phi_{i_n}(x_N)|0\rangle_{1-PI}$ and $\phi_{i_N}^{class}(x)$ are the classical fields, which we usually omit the label $\phi^{clas}(x)$, just writing $\phi(x)$.

The vertex functional ($\Gamma[\phi]$) is related to the connected functional ($Z^c[\phi]$) by a Legendre transformation,

$$\Gamma[\phi] = Z^c[J] - \int dx J^i(x)\phi_i(x) \Big|_{\phi_i(x) = \frac{\delta Z^c}{\delta J^i(x)}} \ , \tag{2.23}$$

and the inverse of the Legendre transformation is:

$$Z^c[J] = \Gamma[\phi] + \int dx J^i(x)\phi_i(x) \Big|_{J^i(x) = -\frac{\delta \Gamma}{\delta \phi_i(x)}} \ . \tag{2.24}$$

Now we will find an important functional relation that can be obtained from (2.21) and which will be important late. That important relation is useful to find out the propagators and the fixation of renormalization constant for example. Making:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x) \Longrightarrow \frac{\delta^2 \Gamma[\phi]}{\delta \phi(z) \delta \phi(x)} = -\frac{\delta J(x)}{\delta \phi(z)} \ , \tag{2.25}$$

Applying on both sides $\frac{\delta \phi(z)}{\delta J(y)}$ of the equation (2.25) and by integrating in the $z$ variable we obtain:

$$\begin{aligned} \int dz\left[\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(x)}\times\frac{\delta\phi(z)}{\delta J(y)}\right] &= -\int dz\left[\frac{\delta J(x)}{\delta\phi(z)}\times\frac{\delta\phi(z)}{\delta J(y)}\right] , \\ \int dz\left[\underbrace{\frac{\hbar}{i}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(x)}}_{\Gamma^{(x,z)}}\underbrace{\frac{-i}{\hbar}\frac{\delta^2 Z^c[J]}{\delta J(y)\delta J(z)}}_{\langle 0|T\phi(y)\phi(z)|0\rangle_{CON}}\right|^{\phi=0}_{J=0} &= \delta(x-y) , \end{aligned} \tag{2.26}$$

and therefore,

$$\langle 0|T\phi(y)\phi(z)|0\rangle_{CON} = \frac{h}{i}\left(\left.\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(x)}\right|_{\phi=0}\right)^{-1} . \tag{2.27}$$

That is, the vertex function of second order is the inverse of the connected Green function. Keeping this in mind, we define:

$$\Gamma^{ik}(x,y) = \frac{\hbar}{i}\left.\frac{\delta^2\Gamma[\phi]}{\delta\phi_i(x)\delta\phi_k(y)}\right|_{\phi=0} , \tag{2.28}$$

Which motivates us to define a more generalized form of the 1-PI $n$ points Green functions,

$$\Gamma^{i_1,\cdots,i_n}(x_1,\cdots,x_n) = \frac{\hbar}{i}\left.\frac{\delta^n\Gamma[\phi]}{\delta\phi_{i_1}(x_1)\cdots\delta\phi_{i_n}(x_m)}\right|_{\phi=0} \tag{2.29}$$

The vertex functional can also be expanded in a Taylor series, as we did for the other functionals having its expansion given by:

$$\Gamma[\phi] = \sum_{n=2}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_n\Gamma^{i_1,\cdots,i_n}(x_1,\cdots,x_n)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n) . \tag{2.30}$$

The $\Gamma^{i_1,\cdots,i_n}(x_1,\cdots,x_n)$ are called proper vertices.

An important point to stress here is the form of the vertex functional expansion which can be given in terms of $\hbar$,

$$\Gamma[\phi] = \sum_{n=0}^{\infty}\hbar^n\Gamma^{(n)}[\phi] , \tag{2.31}$$

where $\Gamma^{(n)}[\phi]$ are the 1-PI Green functions.

We consider the generating functional (2.21) in the classical approximation,

$$Z[J] = \exp\left[\frac{i}{\hbar}\underbrace{(S[\phi]+\int dx J^i(x)\phi_i(x))}_{Z^c[J]}\right] , \tag{2.32}$$

thus implying that $\Gamma^0 = S[\phi]$ is the classical action. "This is obvious since the only 1PI zero-loop graphs – the 1PI tree graphs – are the trivial ones, i.e, those containing a single vertex, and this vertex corresponds to a term of the interaction Lagrangian " [19]. Let $N(\hbar)$ be a function that represents the number of factors of $\hbar$. Consider a vertex diagram consisting of $I$ inner lines, $V$ vertices and $L$ loops. We have that each propagator carries a factor $\hbar$, while the vertices carrier a factor $\hbar^{-1}$ and in addition a global factor $\hbar$ comes from $\Gamma^{(n)}$ . So,

$$N(\hbar) = 1 + I - V \tag{2.33}$$

besides that, from the topological identity $L = I - V + 1 \Longleftrightarrow I - V = L - 1$ we will obtain, by replacing in $N(\hbar)$, the relation:

$$N(\hbar) = L \tag{2.34}$$

The result is of paramount importance, it shows that in each order in the loop, and therefore the order in perturbative expansion, we have a correction in $\hbar$.

## 2.3 Composite fields and external sources

We are interested to know how invariant models under nonlinear field transformations affect Green's functions and how important they are in maintaining symmetries in the renormalization process. For this, we insert in the classical action next to the interaction term the term $S_{\text{ext}}[\rho, \phi]$, that is,

$$S_{\text{int}}[\rho, \phi] = S_{\text{int}}[\phi] + S_{\text{ext}}[\rho, \phi] \ , \tag{2.35}$$

We write $S_{\text{ext}}$ in terms of a field operator $Q^p$, corresponding to a local polynomial in the classical action and also the external sources $\rho_p$ coupled as follows:

$$S_{\text{ext}}[\rho, \phi] = \int dx \rho_p(x) Q^p[\phi(x)] \ , \tag{2.36}$$

When we add such a term within the generating functional of the Green functions, we get a new functional [21] $Z[\rho, J]$,

$$Z[\rho, J] = \int \mathcal{D}\Phi \, \mathrm{e}^{\frac{i}{\hbar}\{S[\phi] + \int dx \ (\rho_p . Q^p + J^i \phi_i)\}} \tag{2.37}$$

The expansion allows us to write:

$$\begin{aligned} Z[\rho, J] \quad = \quad & \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{(i/\hbar)^N}{N!} \frac{(i/\hbar)^M}{M!} \int dx_1 \cdots dx_N \int dy_1 \cdots dy_M \times \\ & \times J^{i_1}(x_1) \cdots J^{i_N}(x_N) \rho_{p_1}(y_1) \cdots \rho_{p_M}(y_M) \times \\ & \langle 0 | T \phi_{i_1}(x_1) \cdots \phi_{i_N}(x_N) Q^{p_1}(y_1) \cdots Q^{p_M}(y_M) | 0 \rangle \ , \end{aligned} \tag{2.38}$$

Calculating the functional derivatives with respect to $\rho^p$ and $J$, we obtain

$$\left.\frac{\delta\delta^n Z[\rho, J]}{\delta\rho^p(x)\delta J_{i_1}(x_1)\cdots\delta J_{i_n}}\right|_{\rho=J=0} = \langle 0|TQ^p(y)\phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)|0\rangle \ , \tag{2.39}$$

As we did previously, we can generalize to the connected generating functional $Z^c[\rho, J]$ and the vertex functional (functions 1-PI) $\Gamma[\rho, J]$ through:

$$Z[\rho, J] = e^{\frac{i}{\hbar}Z^c[\rho,J]} \ , \tag{2.40}$$

in the same way,

$$\Gamma[\rho, \phi] = Z^c[\rho, J] - \int dx \ J^i(x)\phi_i(x)\bigg|_{\phi_i=\frac{\delta Z^c}{\delta J^i(x)}} \ , \tag{2.41}$$

making it for the inverse too,

$$Z^c[\rho, J] = \Gamma[\rho, \phi] + \int dx \ J^i(x)\phi_i(x)\bigg|_{J^i=-\frac{\delta\Gamma}{\delta\phi_i(x)}} \tag{2.42}$$

As a particular case, we write:

$$\left.\frac{\hbar}{i}\frac{\delta Z[\rho, J]}{\delta\rho(y)}\right|_{J=\rho=0} = Q^p(y)Z[J] \ , \tag{2.43}$$

that is, it generates the Green function (2.39), Feynman graphs that contain a new vertex corresponding to the insertion of the $Q^p$ field polynomial that can generate quantum corrections. And in the same way:

$$\left.\frac{\delta Z^c[\rho, J]}{\delta\rho(y)}\right|_{J=\rho=0} = Q^p(y)Z^c[J] \quad \text{and} \quad \left.\frac{\delta\Gamma[\rho, \phi]}{\delta\rho(y)}\right|_{J=\rho=0} = Q^p(y)\Gamma[\phi] \ , \tag{2.44}$$

Recalling the expansion (2.31) we saw that ($\Gamma^{(0)}[\phi] = S[\phi]$) implying that,

$$Q^p(y)\Gamma^{(0)}[\phi] = Q^p_{\text{clas}} \tag{2.45}$$

And the generalization for this expression is given by:

$$Q^p(y)\Gamma[\phi] = Q^p_{\text{clas}} + \mathcal{O}(\hbar) \tag{2.46}$$

This is a crucial point for the quantum insertion in the perturbative construction.

## 2.4 Symmetries and the quantum action principle (QAP)

Symmetries play a fundamental role in understanding nature, in particular, in the study of a more fundamental physics such as field theory. In classical physics we know that, by Noether's theorem, continuous symmetries (local or global) imply conserved currents ( see chapters (3) and (5)) and at the quantum level there is a similar one that join transformations in the fields and leads to the relationship between Green's functions of the theory.

Let's assume that the classical action has $N$ fields $\Phi_i$ and is invariant under infinitesimal transformations,

$$\Phi'_i(x) \;=\; \Phi_i(x) + \varepsilon^a (R_a)_{ij} \Phi_j(x) \Longrightarrow \delta\Phi_i(x) = \varepsilon^a (R_a)_{ij} \Phi_j(x) \ , \tag{2.47}$$

with $a = 1, 2, 3, \cdots$ where $(R_a)_{ij}$ are local functional of the fields and of their derivatives. For our purpose, all continuous symmetries appearing shall belong to a representation of a Lie group $G$. The generators $X_a$ of the Lie Algebra $g$ they satisfy the commutation algebra relation,

$$[X_a, X_b] = i f_{ab}{}^{c} X_c \ , \tag{2.48}$$

being $f_{abc}$ the group structure constants which obey the Jacobi identity,

$$\sum_{cycl.\ perm.\ of\ abc} f_{ab}^{\ d} f_{dc}^{\ e} = 0 \ , \tag{2.49}$$

demanding that the $R_{ai}(x)$ obey the functional relations,

$$\int dy \left( R_{bj}(y) \frac{\delta Rai(x)}{\delta\phi_j(y)} - R_{aj}(y) \frac{\delta Rbi(x)}{\delta\phi_j(y)} \right) = i f_{ab}{}^{c} R_{ci}(x) \ , \tag{2.50}$$

We will see later the motivation of our demanding, for now, to help clarify the relation we can apply it to the case of linear homogeneous transformations,

$$R_{ai}(x) = T_{ai}{}^{j} \phi_j(x) \ , \tag{2.51}$$

where the matrices $T_a$ obey the algebra commutation relation (2.48) and they do not have any dependency of the fields $\phi_i$. Let's see:

$$
\begin{aligned}
&\int dy\Big\{(T_b)_j^k\phi_k(y)(T_a)_i^l\underbrace{\frac{\delta\phi_l(x)}{\delta\phi_j(y)}}_{\delta_l^j\delta(x-y)}-(T_a)_j^k\phi_k(y)(T_b)_i^l\underbrace{\frac{\delta\phi_l(x)}{\delta\phi_j(y)}}_{\delta_l^j\delta(x-y)}\Big\}=\\
&=(T_b)_j^k\phi_k(x)(T_a)_i^j-(T_a)_j^k\phi_k(x)(T_b)_i^j=\\
&=(T_a)_i^j(T_b)_j^k\phi_k(x)-(T_b)_i^j(T_a)_j^k\phi_k(x)=\\
&=[T_a,T_b]_i{}^k\phi_k(x)=if_{ab}^c(T_c)_i^k\phi_k\ .
\end{aligned}
\tag{2.52}
$$

A field functional $F[\phi]$ has the infinitesimal variation of form,

$$
\delta F=i\varepsilon^a\int dxR_{ai}\frac{\delta F}{\delta\phi_i}\ ,
\tag{2.53}
$$

where we define the following operator:

$$
\mathcal{W}_a:=-\int dxR_{ai}\frac{\delta}{\delta\phi_i}\ ,
\tag{2.54}
$$

As we can see by the commutation relation (2.50) the functional operator $\mathcal{W}_a$ was defined with $(-)$ sign so that they obey the same algebra of the generators $X_a$ of the group, that is,

$$
\begin{aligned}
[\mathcal{W}_a,\mathcal{W}_b]F\ =\ &\int dxdy\left\{R_{ai}(x)\frac{\delta}{\delta\phi_i(x)}\left[R_{bj}(y)\frac{\delta F}{\delta\phi_j(y)}\right]-R_{bi}(x)\frac{\delta}{\delta\phi_i(x)}\left[R_{aj}(y)\frac{\delta F}{\delta\phi_j(y)}\right]\right\}=\\
&=\int dxdy\left\{R_{ai}(x)\frac{\delta R_{bj}(y)}{\delta\phi_i(x)}\frac{\delta F}{\delta\phi_j(y)}+R_{ai}(x)R_{bj}(y)\frac{\delta^2F}{\delta\phi_i(x)\delta\phi_j(y)}+\right.\\
&\left.-R_{bi}(x)\frac{\delta R_{aj}(y)}{\delta\phi_i(x)}\frac{\delta F}{\delta\phi_j(y)}-R_{bi}(x)R_{aj}(y)\frac{\delta^2F}{\delta\phi_i(x)\delta\phi_j(y)}\right\}=\\
&=\int dxdy\left\{\left[R_{ai}(x)\frac{\delta R_{bj}(y)}{\delta\phi_i(x)}-R_{bi}(x)\frac{\delta R_{aj}(y)}{\delta\phi_i(x)}\right]\frac{\delta F}{\delta\phi_j(y)}+\right.\\
&\left.+\left[R_{ai}(x)R_{bj}(y)-R_{bi}(x)R_{aj}(y)\right]\frac{\delta^2F}{\delta\phi_i(x)\delta\phi_j(y)}\right\}\ .
\end{aligned}
\tag{2.55}
$$

Working in the expression (2.55) we observe that the second term vanishes and left the first term:

$$
\begin{aligned}
&[\mathcal{W}_a,\mathcal{W}_b]F=\int dxdy\left[R_{ai}(x)\frac{\delta R_{bj}(y)}{\delta\phi_i(x)}-R_{bi}(x)\frac{\delta R_{aj}(y)}{\delta\phi_i(x)}\right]\frac{\delta F}{\delta\phi_j(y)}=\\
&[\mathcal{W}_a,\mathcal{W}_b]F=\int dyif_{ba}{}^cR_{cj}(y)\frac{\delta F}{\delta\phi_j(y)}=-if_{ab}{}^c\int dyR_{cj}(y)\frac{\delta F}{\delta\phi_j(y)}=\\
&[\mathcal{W}_a,\mathcal{W}_b]F=if_{ab}{}^c\,\mathcal{W}_cF
\end{aligned}
\tag{2.56}
$$

(2.57)

And therefore, we can see that the operators, which we usually call Ward operator, has the same algebra of generator of the generators of the group, that is:

$$[\mathcal{W}_a, \mathcal{W}_b] = i f_{ab}{}^{c} \ \ \mathcal{W}_c \ , \tag{2.58}$$

We can classify the symmetries based on the form of the fields transformations saying they are nonlinear on the fields or linearly homogeneous in the fields. The structure of the transformations, and therefore the symmetries, are composed by the parameters that distinguish two different groups, one is called from global transformations where the parameters $\varepsilon^a$ do not have dependency on position $x$ being a constant and a group called local transformations, where the parameters are functions of space-time coordinates $\varepsilon^a = \varepsilon^a(x)$.

### 2.4.1 Linear symmetries

Before proceeding, let's see how the " element" $\mathcal{D}\Phi$ is transformed within the generating functional,

$$\mathcal{D}\Phi(x)' = \left[\det \frac{\delta\Phi_i(x)}{\delta\Phi_j(y)}\right] \mathcal{D}\Phi(x) \ , \tag{2.59}$$

just looking to the determinant,

$$det\left[\det \frac{\delta\Phi_i(x)}{\delta\Phi_j(y)}\right] \ \ = \ \ det[\delta_{ij} + \varepsilon^a tr(R_a)]\mathcal{D}\Phi \ . \tag{2.60}$$

All generators for the Lie groups can be chosen in such a way that $tr \ R_a = 0$. Thus,

$$\mathcal{D}\Phi_i' = \mathcal{D}\Phi_i \implies \delta\mathcal{D}\Phi(x) = 0 \ . \tag{2.61}$$

We want to know how these symmetries of the fields affect the generating functional of Green's functions, and for that, we will calculate

$$\begin{aligned}
\delta Z[J] \ &= \ \mathcal{N}\int \delta\mathcal{D}\Phi \ exp\frac{i}{\hbar}(S[\Phi_i] + J^i\Phi_i) + \mathcal{N}\int \mathcal{D}\Phi \ \delta\{exp\frac{i}{h}(S[\Phi_i] + J^i\Phi_i)\} \ , \\
\delta Z[J] \ &= \ \mathcal{N}\int \mathcal{D}\Phi\left[\frac{i}{\hbar}\int dx \ J^i\varepsilon^a(R_a)_{ij}\Phi_j\right] \ exp\frac{i}{h}(S[\Phi_i] + J^i\Phi_i)
\end{aligned} \tag{2.62}$$

where we use the result (2.61). In addition, as $Z[J]$ is only a function of the sources $J_i(x)$ and using the identification we did in (2.16), we can write:

$$\int dx \left[J^i(x)\varepsilon^a(R_a)_{ij} \ \frac{\delta}{\delta J_j(x)}\right] Z[J] = 0 \ . \tag{2.63}$$

The relation is the source of all Ward identities. Returning to the expression (2.51) with the linearly homogeneous transformations in the fields when the $R_{ai}(x)$ takes the form

$$R_{ai}(x) = T_{ai}{}^{j}\phi_j(x) \ , \tag{2.64}$$

The invariance of the classical action under the $\mathcal{W}_a$ action

$$\mathcal{W}_a S = -\int dx T_{ai}^{\ j}\phi_j \frac{\delta}{\delta\phi_i} S = 0 \ , \tag{2.65}$$

Returning to (2.63) by the linear transformation it is easy to see that

$$\int dx \ \left[ J^i(x)\varepsilon^a (T_a)_{ij} \ \frac{\delta}{\delta J_j(x)} \right] Z[J] = 0 \ . \tag{2.66}$$

From the equations (2.51) and (2.7) we see that the Green functions:

$$\sum_{n=1}^{N} \langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_{n-1}}(x_{n-1}) T_{ai_n}^{\ j_n}\phi_{j_n}(x_n)\phi_{i_{n+1}}(x_{n+1})\cdots\phi_{i_N}(x_N)|0\rangle \tag{2.67}$$

As we can see, we have a sum of linear Green functions, resulting of the linear symmetries. In this way, in the process of renormalization we do not need to regularize the symmetry.

### 2.4.2 Nonlinear symmetries

When we say nonlinear symmetries we are referring to the non-linearity in the fields and assuming a symmetry that is quadratic in the fields like an example we will have,

$$R_{ai}(x) = T_{ai}{}^{jk}\phi_j(x)\phi_k(y) \ , \tag{2.68}$$

Based on the result (2.63), we note that there is a difference between the linear case, because now we do not have a sum of elementary Green functions,

$$\sum_{n=1}^{N} \langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_{n-1}}(x_{n-1}) \underbrace{T_{ai}{}^{jk}\phi_j(x)\phi_k(x)}\,\phi_{i_{n+1}}(x_{n+1})\cdots\phi_{i_N}(x_N)|0\rangle \tag{2.69}$$

One way to restore this is to add to the classical action a term of external source in the following way:

$$S_{ext} = \int dx \rho^i(x)\varepsilon^a R_{ai}(x) \ , \tag{2.70}$$

which results to the total classical action,

$$\Gamma^0[\phi,\rho] = S[\phi] + S_{ext}[\phi,\rho] \tag{2.71}$$

The new form of the action is now not invariant under the transformations (2.65) caused by the external term and a possible solution is:

i) take the infinitesimal parameters of the transformations $\varepsilon^a$ as Grassmann numbers, that is, anti-commuting numbers,

$$\{\varepsilon^a, \varepsilon^b\} = 0 \ , \tag{2.72}$$

ii)the transformations of the parameters takes the form:

$$s\varepsilon^a = -\frac{1}{2} f_{bc}{}^{a} \varepsilon^b \varepsilon^c \ . \tag{2.73}$$

Here again, the $f_{bc}^{\ a}$ are the Lie group structure constants, besides that, the operator $s$ is nilpotent, i.e, $s^2 = 0$.

$$\begin{aligned}
s(s\varepsilon^a) &= \frac{1}{2} f_{bc}{}^{a} s\varepsilon^b \varepsilon^c + \frac{1}{2} f_{bc}{}^{a} \varepsilon^b s\varepsilon^c \ , \\
&= \frac{1}{4} f_{bc}^{a} f_{eg}{}^{b} \varepsilon^e \varepsilon^g \varepsilon^c + \underbrace{\frac{1}{4} f_{bc}{}^{a} f_{eg}{}^{c} \varepsilon^b \varepsilon^e \varepsilon^g}_{b \leftrightarrow c} \ , \\
&= \frac{1}{4} f_{bc}^{a} f_{eg}{}^{b} \varepsilon^e \varepsilon^g \varepsilon^c + \underbrace{\frac{1}{4} f_{cb}{}^{a} f_{eg}^{bc} \varepsilon^c \varepsilon^e \varepsilon^g}_{reordering} \ , \\
s(s\varepsilon^a) &= 0 \ .
\end{aligned} \tag{2.74}$$

Thus, we restore the invariance of the total action (2.71), $s\Gamma^0 = 0$, which we can be written in a functional form as

$$\begin{aligned}
s\Gamma^0[\phi] &= \underbrace{s\Sigma}_{=0} + s\int dx \rho^i(x) s\phi_i(x) \ , \\
&= \underbrace{\int dx s\rho^i(x) s\phi_i(x)}_{=0} + \int dx \rho^i(x) s(s\phi_i(x)) \ , \\
&= i\int dx \rho(x) s(\varepsilon^a R_{ai}(x)) = \frac{i}{2}\int dx \rho(x) f_{bc}{}^{a} \varepsilon^b \varepsilon^c R_{ai}(x) + i\int dx \rho(x) \varepsilon^a s R_{ai}(x) \ , \\
&= \frac{i}{2}\int dx \rho(x) f_{bc}{}^{a} \varepsilon^b \varepsilon^c R_{ai}(x) + i\int dx \rho(x) \underbrace{\varepsilon^a \int dy \underbrace{\delta\phi_j(y)}_{i\varepsilon^b R_{bj}(y)} \frac{\delta R_{ai}(x)}{\delta\phi_j(y)}}_{(*)} \ , \\
&= \frac{i}{2}\int dx \rho(x) f_{bc}{}^{a} \varepsilon^b \varepsilon^c R_{ai}(x) - \int dx \rho(x) \varepsilon^a \varepsilon^b \underbrace{\frac{1}{2}\int dy \left\{ R_{bj}(y) \frac{\delta R_{ai}(x)}{\delta\phi_j(y)} - Raj(y) \frac{\delta R_{bi}}{\delta\phi_j(y)} \right\}}_{\text{The antisymmetric part of } (*) = \frac{i}{2} f_{ab}{}^{c} R_{ci}(x)} \ , \\
s\Gamma^0[\phi] &= \frac{i}{2}\int dx \rho(x) f_{bc}{}^{a} \varepsilon^b \varepsilon^c R_{ai}(x) - \frac{i}{2}\int dx \underbrace{\rho(x) f_{ab}{}^{c} \varepsilon^b \varepsilon^c R_{ci}(x)}_{c \rightarrow a, \ b \longleftrightarrow c} = 0 \ .
\end{aligned} \tag{2.75}$$

The notation used for ($a \rightarrow c$) was the changing of $a$ by $c$ and for ($b \longleftrightarrow c$) the permutation of $b$ and $c$.

We can write the same conclusion above in a functional form in the following way:

$$\mathcal{S}(\Gamma^0) = \int dx \, \underbrace{\frac{\delta\Gamma^0}{\delta\rho^i}}_{\delta\phi_i(x)} \frac{\delta\Gamma^0}{\delta\phi^i} - \underbrace{\frac{1}{2} f_{bc}^{\ a} \varepsilon^b \varepsilon^c}_{s\varepsilon^a} \frac{\partial\Gamma^0}{\partial\varepsilon^a} = 0 \ . \tag{2.76}$$

The operator $\mathcal{S}$ is called of Slavnov-Taylor operator and we shall point out that the variation of the external source is defined as $s\rho(x) = 0$.

### 2.4.3 The QAP

The knowledge of the symmetries in the classical level rises the question: how the symmetries behave at the quantum level? At this point, the quantum action principle (QAP) comes to helps us to answer the question.

At quantum level, the classical Ward identity, by the Quantum Action Principle, establishes its form for the linear symmetries,

$$\int dx T_{\ j}^{ai} \, \phi^j(x) \frac{\delta\Gamma}{\delta\phi_i(x)} = \mathcal{W}^a\Gamma = \Delta^a\Gamma \ , \tag{2.77}$$

and for the nonlinear symmetries,

$$\int dx \frac{\delta\Gamma}{\delta\rho^a(x)} \frac{\delta\Gamma}{\delta\phi_i(x)} = \mathcal{W}^a\Gamma = \Delta^a\Gamma \ , \tag{2.78}$$

where $\Delta$ is an insertion, that is, integrated local polynomial in the fields, sources and their derivatives. The insertion $\Delta = \int dx \Delta(x)$ has the same numbers as $\mathcal{W}$ (global indices, charge conjugation, parity,etc ...) and it is limited by the dimension of space $d$, the mass dimension of the fields $\phi_i$ abbreviated by $d_i$. (**See the chapters 4 and 7**)

### 2.4.4 The Wess-Zumino condition

A symmetry is called anomalous when it is not preserved at the quantum level. The extension of the model by perturbative method and its analysis by the algebraic renormalization consist in part in the analysis of the possible counterterms composing the breaking $\Delta$, searching for a possible anomaly.

Let's supose the Ward identity $\mathcal{W}$ in (2.78) being the nonlinear or the linear Ward operator. We write:

$$\mathcal{W}^a S = \int dx R^a_{ij} \phi^j(x) \frac{\delta S}{\delta\phi_i(x)} = 0 \ , \tag{2.79}$$

In general, we work with power counting renormalizable models, having in mind the power counting theorem [19] like an necessary condition. Assuming it here, we can write,

$$\begin{aligned} \mathcal{W}^a\Gamma &= \Delta^a\Gamma = \mathcal{W}^a\Gamma^0 + h\mathcal{W}^a\Gamma^1 + \mathcal{O}(\hbar^2) \ , \\ \mathcal{W}^a\Gamma &= \hbar\Delta^a + \mathcal{O}(\hbar^2) \ . \end{aligned} \tag{2.80}$$

Refreshing our mind with relation (2.58) and applying it to $\Gamma$, order by order, we find:

$$\begin{aligned} &\mathcal{W}^a\mathcal{W}^b\Gamma - \mathcal{W}^b\mathcal{W}^a\Gamma = f_c^{ab}\mathcal{W}\Gamma \ , \\ &\mathcal{W}^a\Delta^b - \mathcal{W}^b\Delta^b = f_{abc}\Delta^c \end{aligned} \tag{2.81}$$

The equation(2.81) is the Wess-Zumino condition. It resulted from the Ward commutation relation and it states clearly that the breaking $\Delta$ suffer restrictions under the Ward operator, that is, symmetries of the classical action do not allow any counterterm.

## 2.5 The study of the stability

The starting point for the stability analysis is the classical Ward identity [20] associated to the symmetries in question. The classical Ward identity is given by:

$$\mathcal{W} = \int d^Dx \frac{\delta\Sigma}{d\rho^i}\frac{\delta\Sigma}{\delta\Phi_i} \ , \tag{2.82}$$

being $\Sigma$ the classical action that may not be in its complete form and $\rho^i$ the external source. To this purpose, we slightly disturb our action as follows:

$$\Sigma \longrightarrow \widetilde{\Sigma} = \Sigma + \varepsilon\Sigma^c \tag{2.83}$$

applying this to Ward identity,

$$\mathcal{W}(\Sigma + \varepsilon\Sigma^c) = 0 + \mathcal{O}(\varepsilon^2) \tag{2.84}$$

that is,

$$\int d^Dx \frac{\delta(\Sigma + \varepsilon\Sigma^c)}{\delta\rho^i}\frac{\delta(\Sigma + \varepsilon\Sigma^c)}{\delta\Phi_i} = 0 + \mathcal{O}(\varepsilon^2) \tag{2.85}$$

Working the expression above,

$$\int d^Dx \frac{\delta\Sigma}{\delta\rho^i}\frac{\delta\Sigma}{\delta\Phi_i} + \varepsilon \int d^Dx \underbrace{\left(\frac{\delta\Sigma}{d\rho^i}\frac{\delta\Sigma^c}{\delta\Phi_i} + \frac{\delta\Sigma}{d\rho^i}\frac{\delta\Sigma^c}{\delta\Phi_i}\right)}_{B} = 0 + \mathcal{O}(\varepsilon^2) \ . \tag{2.86}$$

from which we obtain:

$$\varepsilon B\Sigma^c = 0 \ . \tag{2.87}$$

The operator $B$ is the $\mathcal{W}$ operator linearized on external $\rho$ sources. The stretched, or more general, form of the action follows an equation with the Ward operator in question.
**Definition**: Given some starting classical action.

$$\Sigma = \Sigma[\phi^i, \rho^i, \lambda^i] \ , \tag{2.88}$$

we say it is renormalizable (multiplicative renormalizable) if the equation's solution,

$$B\Sigma^c = 0 \ , \tag{2.89}$$

can be inserted in the action by a redefinition of the fields, that is,

$$\Sigma + \varepsilon\Sigma^c = \Sigma^0 + \mathcal{O}(\varepsilon^2) \tag{2.90}$$

where $\Sigma^0$ is what we call disturbed classical action obtained by the following transformations:

$$\Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i], \quad \text{com } \Phi_i^0 = \Phi_i(1+\varepsilon Z_{\Phi_i}) \ , \ \rho_i^0 = \rho_i(1+\varepsilon Z_{\rho_i}) \text{ e } \lambda_i^0 = \lambda_i(1+\varepsilon Z_{\lambda_i}) \tag{2.91}$$

The action ($\Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0]$) is the starting action ($\Sigma[\Phi_i, \rho_i, \lambda_i]$) from where we replace all fields $\Phi_i$, external sources $\rho_i$ and the parameters of the model $\lambda_i$ (coupling constants, mass parameters) with their "disturbed" versions ($\Phi_i^0$, $\rho_i^0$, $\lambda_i^0$) and we expand by disregarding the second order terms in $\varepsilon$. Since the classical action is determined by equations of bounds and symmetries, we look for counterterms that also carry such properties.

# Part I

# Massive Case

# Chapter 3

# Electron-polaron–electron-polaron bound states in mass-gap graphene-like planar quantum electrodynamics: $s$-wave bipolarons

## 3.1 Introduction

[1] The seminal works by Deser, Jackiw, Templeton and Schonfeld [2, 23–25] have attracted attention to the quantum electrodynamics in three space-time dimensions ($QED_3$) in view of its potentiality as theoretical foundation for quasi-planar condensed matter phenomena, such as high-$T_c$ superconductors [26–28], quantum Hall effect [29–31], topological insulators [32–34], topological superconductors [35–37] and graphene [16, 38–44]. Since then, the planar quantum electrodynamics has been widely studied in many physical configurations, namely, small (perturbative) and large (non perturbative) gauge transformations, Abelian and non-Abelian gauge groups, fermions families, odd and even under parity, compact space-times, space-times with boundaries, curved space-times, discrete (lattice) space-times, external fields and finite temperatures.

The pure graphene [16, 38–44] monolayer is a gapless (massless gap graphene) bidimensional system which behaves like a half-filling semimetal with its charge carriers (quasiparticles) being described by massless charged Dirac fermions. However, for practical applications like transistors a gap (mass-gap) graphene [45–49] is more appropriate, and such a mass-gap effect is observed in pure monolayer graphene on substrates [11]. Electron-electron interactions (electron pairing) in graphene [12] include electron polarons (electron-phonon) [13] scattering processes [50–52], where this quasiparticle, the polaron, which is formed by a bound state of electron (or hole) and phonon, was first introduced

[1]The content was published in The European Physical Journal B [22].

by Landau [15].

The proposed issue in this chapter about the possibility of $s$-wave bipolarons emerging from the parity-preserving $U(1) \times U(1)$ massive QED$_3$ – a mass-gap graphene-like [45–49] planar quantum electrodynamics – is presented as follows. Initially, the model defined by its discrete and continuous symmetries is introduced and, since the interactions are nonconfining – the vector mesons, the photon and the Néel quasiparticle, are massive – the asymptotic states for the fermions (electron polarons) are established. Hereafter, in the low-energy limit, the $s$- and $p$-wave Møller ($e^-$-polaron–$e^-$-polaron) scattering amplitudes are computed and their respective interaction potentials obtained and analysed. However, from this analysis, it was found conditions on the parameters which, in spite of the $p$-wave scattering potential still remains repulsive, the $s$-wave interaction potential becomes attractive. The latter shall favour $e^-$-polaron–$e^-$-polaron bound states – provided the attractive $s$-wave scattering potential satisfies necessary conditions [53–58] – giving rise to the $s$-wave bipolarons condensates [50–52].

## 3.2 The model

The Lorentz invariant version of mass-gap graphene-like planar quantum electrodynamics, the parity-even $U(1)_A \times U(1)_a$ massive QED$_3$, is defined by the action:

$$\begin{aligned} S = \int d^3x \bigg\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\, \mathrm{e}^{\mu\rho\nu}\, A_\mu \partial_\rho a_\nu + i\overline{\psi}_+ \not{D}\psi_+ + i\overline{\psi}_- \not{D}\psi_- \\ & -m(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-) + \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - \frac{1}{2\beta}(\partial^\mu a_\mu)^2 \bigg\} , \end{aligned} \tag{3.1}$$

where $\not{D}\psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm$, and any object $\not{X} \equiv X^\mu\gamma_\mu$. The coupling constants $e$ and $g$ are dimensionful, with mass dimension $\frac{1}{2}$, and, $m$ and $\mu$ are mass parameters with mass dimension 1. Also, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, are the field strengths associated to the electromagnetic field ($A_\mu$) and the Néel gauge field ($a_\mu$), respectively, $\psi_+$ and $\psi_-$ are two kinds of fermions – each of them describing electron polarons (electron-phonon) and hole polarons (hole-phonon) quasiparticles – where the $\pm$ subscripts are related to their spin sign [17], and the gamma matrices are $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$.

### 3.2.1 The symmetries: parity and $U(1) \times U(1)$

The CPT-even action (3.1) is invariant under:

1. parity symmetry ($P$):

$$
\begin{aligned}
x_\mu &\xrightarrow{P} x_\mu^P = (x_0, -x_1, x_2) \ , \\
\psi_\pm &\xrightarrow{P} \psi_\pm^P = -i\gamma^1\psi_\mp \ , \ \overline{\psi}_\pm \xrightarrow{P} \overline{\psi}_\pm^P = i\overline{\psi}_\mp\gamma^1 \ , \\
A_\mu &\xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ , \\
a_\mu &\xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ .
\end{aligned}
\tag{3.2}
$$

2. gauge $U(1)_A \times U(1)_a$ symmetry ($\delta_{\rm g}$):

$$
\begin{aligned}
\delta_{\rm g}\psi_\pm(x) &= i[\theta(x) \pm \omega(x)]\psi_\pm(x) \ , \\
\delta_{\rm g}\overline{\psi}_\pm(x) &= -i[\theta(x) \pm \omega(x)]\overline{\psi}_\pm(x) \ , \\
\delta_{\rm g}A_\mu(x) &= -\frac{1}{e}\,\partial_\mu\theta(x) \ , \\
\delta_{\rm g}a_\mu(x) &= -\frac{1}{g}\,\partial_\mu\omega(x) \ .
\end{aligned}
\tag{3.3}
$$

### 3.2.2 The spectrum: degrees of freedom, spin, masses and charges

The free Dirac equations associated to $\psi_+$ and $\psi_-$, which stem from the action (3.1), read:

$$
(i\partial\!\!\!/ - m)\psi_+ = 0 \quad \text{and} \quad (i\partial\!\!\!/ + m)\psi_- = 0 \ , \tag{3.4}
$$

So, by expanding the operators $\psi_+$ and $\psi_-$ in terms of the $c$-number plane wave solutions of the Dirac equations, with operator-valued amplitudes, $a_+$, $b_+$, $a_-$ and $b_-$ (annihilation operators), and $a_+^\dagger$, $b_+^\dagger$, $a_-^\dagger$ and $b_-^\dagger$ (creation operators):

$$
\psi_+(x) = \int \frac{d^2\vec{k}}{(2\pi)^2}\frac{m}{k^0}\{a_+(k)u_+(k)e^{-ikx} + b_+^\dagger(k)v_+(k)e^{ikx}\} \ , \tag{3.5}
$$

$$
\psi_-(x) = \int \frac{d^2\vec{k}}{(2\pi)^2}\frac{m}{k^0}\{a_-(k)u_-(k)e^{-ikx} + b_-^\dagger(k)v_-(k)e^{ikx}\} \ , \tag{3.6}
$$

where $\overline{\psi}_\pm = \psi_\pm^\dagger \gamma^0$. Consequently, from (3.4) and (3.5)-(3.6), by assuming $p^\mu = (E, p_x, p_y)$, the wave functions, $u_+$, $v_+$, $u_-$ and $v_-$, are given by:

$$u_+(p) = \frac{(\not{p} + m)}{\sqrt{2m(E+m)}} u_+(m, \vec{0}) \ ,$$
$$v_+(p) = \frac{(-\not{p} + m)}{\sqrt{2m(E+m)}} v_+(m, \vec{0}) \ , \tag{3.7}$$
$$u_-(p) = \frac{(-\not{p} + m)}{\sqrt{2m(E+m)}} u_-(m, \vec{0}) \ ,$$
$$v_-(p) = \frac{(\not{p} + m)}{\sqrt{2m(E+m)}} v_-(m, \vec{0}) \ , \tag{3.8}$$

satisfying the following conditions:

$$\overline{u}_+(p) u_+(p) = 1 \quad \text{and} \quad \overline{v}_+(p) v_+(p) = -1 \ , \tag{3.9}$$
$$\overline{u}_-(p) u_-(p) = -1 \quad \text{and} \quad \overline{v}_-(p) v_-(p) = 1 \ , \tag{3.10}$$

where

$$u_+(m, \vec{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_+(m, \vec{0}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{3.11}$$
$$u_-(m, \vec{0}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_-(m, \vec{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{3.12}$$

are the momenta space solutions of the Dirac equations at the particle rest-frame, $p^\mu = (m, \vec{0})$. The microcausality conditions for $\psi_+$ and $\psi_-$:

$$\left\{ \psi_\pm(x), \psi_\pm^\dagger(y) \right\}_{x^0 = y^0} = \delta^2(\vec{x} - \vec{y}) \ , \tag{3.13}$$

together with the Dirac equations (3.4) and the normalization conditions (5.11)-(5.12), implies that:

$$\left\{ a_\pm(k), a_\pm^\dagger(p) \right\} = (2\pi)^2 \frac{k^0}{m} \delta^2(\vec{k} - \vec{p}) \ , \tag{3.14}$$
$$\left\{ b_\pm(k), b_\pm^\dagger(p) \right\} = (2\pi)^2 \frac{k^0}{m} \delta^2(\vec{k} - \vec{p}) \ , \tag{3.15}$$

where all other anticommutators vanish and, for the vacuum state $|0\rangle$, $a_\pm(k)|0\rangle = b_\pm(k)|0\rangle = 0$.

The quantum operators associated to space-time ($SO(1,2)$) symmetry and internal ($U(1)_A \times U(1)_a$) symmetry, spin ($S$), electric charge ($Q_\pm$) and Néel (chiral) charge ($q_\pm$), are

$$S = \frac{1}{2}\sigma_z \ , \tag{3.16}$$
$$Q_\pm = -e\int d^2\vec{x} : \psi^\dagger_\pm(x)\psi_\pm(x) : \tag{3.17}$$
$$Q_\pm = -e\int \frac{d^2\vec{k}}{(2\pi)^2}\frac{m}{k^0}\{a^\dagger_\pm(k)a_\pm(k) - b^\dagger_\pm(k)b_\pm(k)\} \ ,$$
$$q_\pm = \mp g\int d^2\vec{x} : \psi^\dagger_\pm(x)\psi_\pm(x) : \tag{3.18}$$
$$q_\pm = \mp g\int \frac{d^2\vec{k}}{(2\pi)^2}\frac{m}{k^0}\{a^\dagger_\pm(k)a_\pm(k) - b^\dagger_\pm(k)b_\pm(k)\} \ ,$$

respectively, with their action upon the asymptotic fermion (antifermion) states with spin up and spin down, $|f^-_\uparrow\rangle$ ($|f^+_\uparrow\rangle$) and $|f^-_\downarrow\rangle$ ($|f^+_\downarrow\rangle$):

$$S|f^-_\uparrow\rangle = +\tfrac{1}{2}|f^-_\uparrow\rangle \ , \quad S|f^+_\downarrow\rangle = -\tfrac{1}{2}|f^+_\downarrow\rangle \ ,$$
$$S|f^-_\downarrow\rangle = -\tfrac{1}{2}|f^-_\downarrow\rangle \ , \quad S|f^+_\uparrow\rangle = +\tfrac{1}{2}|f^+_\uparrow\rangle \ ; \tag{3.19}$$
$$Q_+|f^-_\uparrow\rangle = -e|f^-_\uparrow\rangle \ , \quad Q_+|f^+_\downarrow\rangle = +e|f^+_\downarrow\rangle \ ,$$
$$Q_-|f^-_\downarrow\rangle = -e|f^-_\downarrow\rangle \ , \quad Q_-|f^+_\uparrow\rangle = +e|f^+_\uparrow\rangle \ ; \tag{3.20}$$
$$q_+|f^-_\uparrow\rangle = -g|f^-_\uparrow\rangle \ , \quad q_+|f^+_\downarrow\rangle = +g|f^+_\downarrow\rangle \ ,$$
$$q_-|f^-_\downarrow\rangle = +g|f^-_\downarrow\rangle \ , \quad q_-|f^+_\uparrow\rangle = -g|f^+_\uparrow\rangle \ ; \tag{3.21}$$

where

$$|f^-_\uparrow\rangle = a^\dagger_+(k)|0\rangle \ , \qquad |f^+_\downarrow\rangle = b^\dagger_+(k)|0\rangle \ , \tag{3.22}$$
$$|f^-_\downarrow\rangle = a^\dagger_-(k)|0\rangle \ , \qquad |f^+_\uparrow\rangle = b^\dagger_-(k)|0\rangle \ , \tag{3.23}$$

which means that, $a^\dagger_+$ ($a^\dagger_-$) creates a spin-up (spin-down) fermion (electron polaron) and $b^\dagger_+$ ($b^\dagger_-$) creates a spin-down (spin-up) antifermion (hole polaron). Moreover, from the results above, for any fermion or antifermion (spin up or down) quantum state $|\psi\rangle$, it is verified that

$$S|\psi\rangle = -\frac{1}{2g}\, q_\pm|\psi\rangle \ , \tag{3.24}$$

which proves the correlation among spin and chiral charge (see TABLE 3.1).

In the low-energy limit (Born approximation), the two-particle scattering potential is given by the Fourier transform of the two-particle $t$-channel scattering amplitude (direct scattering) [59]. However, so as to compute the scattering amplitudes, use has been made of the propagators. Hence, switching off the coupling constants ($e$ and $g$), the tree-level

| state | wave function | electric charge | chiral charge | spin | quasiparticle |
|---|---|---|---|---|---|
| $\vert f_\uparrow^-\rangle$ | $u_+$ | $-e$ | $-g$ | $+\frac{1}{2}$ | electron polaron |
| $\vert f_\downarrow^-\rangle$ | $u_-$ | $-e$ | $+g$ | $-\frac{1}{2}$ | electron polaron |
| $\vert f_\downarrow^+\rangle$ | $v_+$ | $+e$ | $+g$ | $-\frac{1}{2}$ | hole polaron |
| $\vert f_\uparrow^+\rangle$ | $v_-$ | $+e$ | $-g$ | $+\frac{1}{2}$ | hole polaron |

Table 3.1: The quasiparticles, electric charges, chiral charges and spin.

propagators in momenta space, for all the fields, read:

$$\Delta_{++}(k) = i\frac{\not{k}-m}{k^2-m^2} \ , \quad \Delta_{--}(k) = i\frac{\not{k}+m}{k^2-m^2} \ ; \tag{3.25}$$

$$\begin{aligned}
\Delta^{\mu\nu}_{AA}(k) &= -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\alpha}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \ , \\
\Delta^{\mu\nu}_{aa}(k) &= -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\beta}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \ , \\
\Delta^{\mu\nu}_{Aa}(k) &= \Delta^{\mu\nu}_{aA}(k) = \frac{\mu}{k^2(k^2-\mu^2)}\, \mathrm{e}^{\mu\rho\nu}\, k_\rho \ .
\end{aligned} \tag{3.26}$$

From the propagators above, $\Delta_{++}$, $\Delta_{--}$, $\Delta^{\mu\nu}_{AA}$, $\Delta^{\mu\nu}_{aa}$ and $\Delta^{\mu\nu}_{Aa}$, the spectrum and the tree-level unitarity of the model can be be analyzed by coupling them to external currents, $\mathcal{J}_{\Phi_i} = (\mathcal{J}_+, \mathcal{J}_-, \mathcal{J}^\mu_A, \mathcal{J}^\mu_a)$, compatible with the symmetries of the model, where the current-current transition amplitudes in momentum space are written as: $\mathcal{A}_{\Phi_i\Phi_j} = \mathcal{J}^*_{\Phi_i}(k)\langle\Phi_i(k)\Phi_j(k)\rangle\mathcal{J}_{\Phi_j}(k)$. Then, by taking the imaginary part of the residues of the current-current amplitudes, $\mathcal{A}_{\Phi_i\Phi_j}$, at the poles, it can be probed the necessary conditions for unitarity – positive imaginary part of the residues of the transition amplitudes, $\Im\mathrm{Res}\ \mathcal{A}_{\Phi_i\Phi_j} > 0$, as a consequence of the $S$-matrix be unitary – at the tree-level and the counting of the degrees of freedom described by the fields, $\Phi_i = (\psi_+, \psi_-, A_\mu, a_\mu)$. In summary, it has been concluded [60] that the two kind of fermions, $\psi_+$ and $\psi_-$, hold two massive degrees of freedom with mass $m$ – the electron-polaron $|f_\uparrow^-\rangle$ $(u_+)$ and the hole-polaron $|f_\downarrow^+\rangle$ $(v_+)$ associated to the spinor $\psi_+$, and the electron-polaron $|f_\downarrow^-\rangle$ $(u_-)$ and the hole-polaron $|f_\uparrow^+\rangle$ $(v_-)$ associated to the spinor $\psi_-$. Also, the vector fields, the electromagnetic field $(A_\mu)$ and the Néel gauge field $(a_\mu)$, carry each one two massive degrees of freedom with mass $\mu$, moreover, it shall be noticed that the single massless mode in model, displayed in $\Delta^{\mu\nu}_{Aa}$, does not propagate, it decouples. From the results presented above, it can be concluded that the the parity-preserving $U(1)\times U(1)$ massive QED$_3$ is free from tachyons and ghosts at the classical level. Nevertheless, to have full control of the unitarity at tree-level, it is still necessary to study the behaviour of the scattering cross sections in the limit of high center of mass energies, by analyzing the Froissart-Martin bound [53, 61, 62].

## 3.3 The Møller scattering

In order to calculate the scattering amplitudes, it remains the vertex Feynman rules associated to the interaction vertices $-e\overline{\psi}_\pm \not{A}\psi_\pm$ and $\mp g\overline{\psi}_\pm \not{a}\psi_\pm$: $\Upsilon^\mu_{\pm\pm}{=}ie\gamma^\mu$ and $\upsilon^\mu_{\pm\pm}{=}\pm ig\gamma^\mu$, respectively.

Figure 3.1: $e^-$-polaron–$e^-$-polaron (Møller) $t$-channel scattering mediated by electromagnetic ($A_\mu$) and Néel ($a_\mu$) quantum fields.

The $t$-channel $e^-$-polaron–$e^-$-polaron Møller scattering amplitudes mediated by the electromagnetic and the Néel quanta (see FIG. 3.1) are given by:

$$-i\mathcal{M}_{\pm A\mp} = \overline{u}_\pm(p_1')[\Upsilon^\mu_{\pm\pm}]u_\pm(p_1)\Delta^{AA}_{\mu\nu}(k)\overline{u}_\mp(p_2')[\Upsilon^\nu_{\mp\mp}]u_\mp(p_2) \ , \tag{3.27}$$
$$-i\mathcal{M}_{\pm a\mp} = \overline{u}_\pm(p_1')[\upsilon^\mu_{\pm\pm}]u_\pm(p_1)\Delta^{aa}_{\mu\nu}(k)\overline{u}_\mp(p_2')[\upsilon^\nu_{\mp\mp}]u_\mp(p_2) \ , \tag{3.28}$$
$$-i\mathcal{M}_{\pm A\pm} = \overline{u}_\pm(p_1')[\Upsilon^\mu_{\pm\pm}]u_\pm(p_1)\Delta^{AA}_{\mu\nu}(k)\overline{u}_\pm(p_2')[\Upsilon^\nu_{\pm\pm}]u_\pm(p_2) \ , \tag{3.29}$$
$$-i\mathcal{M}_{\pm a\pm} = \overline{u}_\pm(p_1')[\upsilon^\mu_{\pm\pm}]u_\pm(p_1)\Delta^{aa}_{\mu\nu}(k)\overline{u}_\pm(p_2')[\upsilon^\nu_{\pm\pm}]u_\pm(p_2) \ . \tag{3.30}$$

Furthermore, in the center of mass (CM) reference frame, the three-momenta configuration of the two scattered fermions, $p_1$, $p_2$, $p_1'$ and $p_2'$, so as the momentum transfer, $k$, are fixed as

$$p_1 = (E, p, 0) \ , \quad p_1' = (E, p\cos\varphi, p\sin\varphi) \ ; \tag{3.31}$$
$$p_2 = (E, -p, 0) \ , \quad p_2' = (E, -p\cos\varphi, -p\sin\varphi) \ ; \tag{3.32}$$
$$k = p_1 - p_1' = (0, p(1-\cos\varphi), -p\sin\varphi) = (0, \mathbf{k}) \ , \tag{3.33}$$

where $\varphi$ is the CM scattering angle, defined as the angle among the directions in the CM frame of the two incoming (initial state) and outgoing (final state) fermions.

The total $s$- and $p$-wave Møller scattering amplitudes can now be derived from the partial ones (3.27)-(3.30) in the low-energy approximation, $\mathcal{M}_s$ ($|\uparrow\rangle + |\downarrow\rangle \rightarrow |\uparrow\rangle + |\downarrow\rangle$) and $\mathcal{M}_p$ ($|\uparrow\rangle + |\uparrow\rangle \rightarrow |\uparrow\rangle + |\uparrow\rangle$ or $|\downarrow\rangle + |\downarrow\rangle \rightarrow |\downarrow\rangle + |\downarrow\rangle$), where, by assuming the momenta configuration above (3.31)-(3.33), it follows that:

$$\mathcal{M}_s = \frac{1}{\mathbf{k}^2 + \mu^2}\left(e^2 - g^2\right) \ , \tag{3.34}$$
$$\mathcal{M}_p = \frac{1}{\mathbf{k}^2 + \mu^2}\left(e^2 + g^2\right) \ . \tag{3.35}$$

### 3.3.1 Scattering potentials

In the low-energy (nonrelativistic) limit, the two-particle interaction potential, in the Born approximation, is nothing but the two-dimensional Fourier transform of the lowest-order two-particle $\mathcal{M}$ scattering amplitude:

$$\mathcal{V}(r) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, \mathcal{M}\, e^{i\mathbf{k}\cdot\mathbf{r}} \ . \tag{3.36}$$

Accordingly to the Born approximation (3.36), the electron-polaron–electron-polaron $s$- and $p$-wave scattering potentials, mediated by the photon and the Néel quasiparticle, read:

$$\mathcal{V}_s(r) = \frac{1}{2\pi}\left(e^2 - g^2\right) K_0(\mu r) \ , \tag{3.37}$$

$$\mathcal{V}_p(r) = \frac{1}{2\pi}\left(e^2 + g^2\right) K_0(\mu r) \ . \tag{3.38}$$

Thereafter, it can be concluded from (3.38) that, regardless the values of the electromagnetic and the chiral coupling constants – $e$ and $g$, respectively – the $e^-$-polaron–$e^-$-polaron interaction in $p$-wave state ($|\uparrow\rangle$+$|\uparrow\rangle$ or $|\downarrow\rangle$+$|\downarrow\rangle$) is always repulsive. Nevertheless, from (3.37), it shall be stressed about the possibility of attractive $e^-$-polaron–$e^-$-polaron interaction in $s$-wave state ($|\uparrow\rangle$+$|\downarrow\rangle$) provided $g^2 > e^2$. In this case, where $g^2 > e^2$, the $s$-wave interaction potential $\mathcal{V}_s(r)$ is attractive,

$$\mathcal{V}_s(r) = -\frac{1}{2\pi}\left(g^2 - e^2\right) K_0(\mu r) \ , \tag{3.39}$$

however, this is not a sufficient condition for the existence of bound states.

### 3.3.2 Bound states

Beyond the attractive nature, provided that $g^2 > e^2$, of $s$-wave interaction potential (3.39), starting from the "distinguished" extension of the free hamiltonian, and adding to it a potential $V$, does not alter the self-adjointness of the total hamiltonian, provided $V$ is "weak" in the sense of Kato-Redlich [55, 63]. Here, the condition that defines this "weak" class is expressed precisely in the following integrability condition on the potential:

$$\int_0^\infty d\rho\ \rho[1 + |\ln(\rho)|]|\mathcal{V}(\rho)| < \infty \ , \quad \rho = \mu r \ . \tag{3.40}$$

This condition ensure the semi-boundedness of the total hamiltonian,and the finiteness of the number of bound states. As an aside here we give the upper bound of Setô [56] on the number $N$ of bound states for dimension 2, and $l = 0$. This is the 2-dimensional version of the old Bargmann inequality for $d = 3$ [58]. The Setô bound is

$$N^0 < 1 + \frac{1}{2} \frac{\left(\frac{2\mu_\mathrm{r}}{\hbar^2}\right)^2 \int_0^\infty \int_0^\infty \rho\rho' \left|\ln(\frac{\rho}{\rho'})\right| |\mathcal{V}(\rho)||\mathcal{V}(\rho')| d\rho d\rho'}{\mu^2 \left(\frac{2\mu_\mathrm{r}}{\hbar^2}\right) \int_0^\infty \rho|\mathcal{V}(\rho)|d\rho} . \tag{3.41}$$

The fact that there is always a bound state, regardless of how weak an attractive potential $V$ may be, is somehow reflected by the presence of 1 in the right-hand side of Eq. (3.41). Also, an upper bound for their number ($N^l$) for nonvanishing angular momentum ($l \neq 0$):

$$N^l < \frac{1}{2l} \left(\frac{2\mu_\mathrm{r}}{\hbar^2}\right) \frac{1}{\mu^2} \int_0^\infty \rho|\mathcal{V}(\rho)|d\rho , \tag{3.42}$$

respectively, where $\hbar = 1$ and $\mu_\mathrm{r} = \frac{m}{2}$ is the $e^-$-polaron–$e^-$-polaron reduced mass.

It has been proved elsewhere [64] that, whenever an interaction potential of the type $\mathcal{V}(r) = CK_0(\mu r)$ is attractive ($C < 0$), it satisfies the following criteria: the weakness in the sense of Kato (3.40); the Newton-Setô bound (3.41) for $l = 0$; and the Bargmann bound (3.42) for all $l$ such that $l \leq l_\mathrm{m} = \lfloor \frac{\mu_\mathrm{r}|C|}{\hbar^2\mu^2} \rfloor$ (where $\lfloor x \rfloor$ is the floor function of $x$). In the same manner, by means of the effective potential $\mathcal{V}_\mathrm{eff}(r) = \frac{\hbar^2(l^2-\frac{1}{4})}{2\mu_\mathrm{r} r^2} + \mathcal{V}(r)$ with $0 \leq l \leq l_\mathrm{m}$, it can be figured out that bound states arise (see FIG. 3.2). In addition to, it shall be stressed that these fulfilled conditions, (3.40), (3.41) and (3.42), guarantee the existence of bound states for any kind of three-dimensional space-time model which exhibits scattering potential of the type $\mathcal{V}(r) = CK_0(\mu r)$ ($C < 0$).

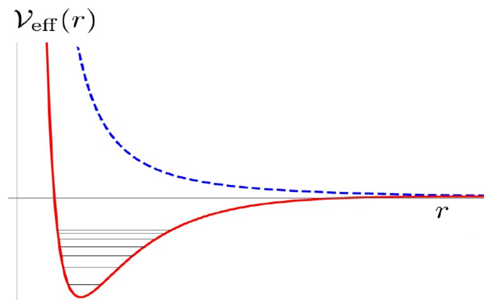


Figure 3.2: The effective $e^-$-polaron–$e^-$-polaron interaction potential, which for the $s$-wave case, $C < 0$, $\mathcal{V}_\mathrm{eff}(r)$ is attrative, if $0 \leq l \leq l_\mathrm{m}$ (solid line), and repulsive, if $l > l_\mathrm{m}$ (dashed line).

## 3.4 Conclusions

The Lorentz invariant parity-preserving $U(1) \times U(1)$ massive QED$_3$, a mass-gap graphene-like planar quantum electrodynamics model, at low-energy limit exhibits electron-polaron–electron-polaron scattering short range non confining potentials, similarly it can be concluded that the same behaviour takes place for hole-polaron–hole-polaron scatterings.

The interactions among electron-polarons and hole-polarons are mediated by two massive vector mesons, the photon (electric charge source) and the Néel quasiparticle (chiral charge source), both stemming from the $U(1)_A \times U(1)_a$ gauge symmetry. It should be noticed that it was disclosed the correlation among the electron-polaron (hole-polaron) spin polarization and correspondent chiral charge. At the tree-level, the absence of tachyons ($k^2 < 0$) and ghosts ($\langle\psi|\psi\rangle < 0$) in the model spectrum guarantees causality and unitarity, respectively, at this level. Notwithstanding, in order to complete the tree-level unitarity analysis, it remains to finish the proof that the scattering cross sections in the limit of high center of mass energies respect the Froissart-Martin bound [53, 61, 62], but since ultraviolet problems are less critical in lower dimensional quantum field models, together with the fact that the four space-time dimensional QED ($QED_4$) [59] satisfies the Froissart-Martin bound, consequently this fulfilment shall be foreseen for parity-even $U(1) \times U(1)$ massive $QED_3$. Also, it shall be pointed out that for condensed matter systems like graphene, the quasiparticles (electron-polaron and hole-polaron) dynamics is in non relativistic regime, so ultraviolet unitarity upper bound violations should not be expected.

Bearing in mind hypothetical applications of the model presented here to graphene, or any other two dimensional system, the orders of magnitude of some theoretical parameters need to be established firstly, namely, a typical mass-gap in graphene is around meV [45–49] whereas the low-energy limit for a condensed matter system is of eV order. In addition to that, the characteristic range of the two interactions, mediated by the both massive photon and the Néel quantum, shall be associated to the pair-coherence length measured in graphene, orders of magnitude in nm [65]. The mass-gap in graphene [45–49], besides of being more realistic, can be either achieved when pure graphene monolayer is settled on substrates [11], increasing its application range and improving device developments.

At the low-energy limit, the non relativistic electron-polaron–electron-polaron (or hole-polaron–hole-polaron) scattering potential, owing to photon and Néel quasiparticle short range exchanges, shows to be always repulsive (3.38) for parallel ($p$-wave) electron-polaron (hole-polaron) spin polarizations ($|\uparrow\rangle + |\uparrow\rangle$ or $|\downarrow\rangle + |\downarrow\rangle$). Nevertheless, for electron-polaron–electron-polaron (or hole-polaron–hole-polaron) scatterings with antiparallel ($s$-wave) spin polarizations ($|\uparrow\rangle + |\downarrow\rangle$), the $s$-wave interaction potential (3.39) might be attractive provided $e^-(e^+)$-polaron–Néel-quasiparticle coupling strength ($|g|$) be stronger than the strength of $e^-(e^+)$-polaron–photon coupling ($|e|$), $g^2 > e^2$. Moreover, the $s$-wave attractive scattering potential (3.39) satisfies the Kato condition [55], the Newton-Setô and the Bargmann upper bounds [56, 58], indicating that $s$-wave bipolarons [50–52] might stem from these electron-polaron–electron-polaron quasiparticles bound states [64]. The possible emergence of such a Cooper-type $e^-$-polaron–$e^-$-polaron condensate (bipolaron) directly calls the issue of superconductivity in graphene [66–68], thus a deep investigation on that deserves special attention.

# Chapter 4

# On the ultraviolet finiteness of parity-preserving $U(1) \times U(1)$ massive QED$_3$

## 4.1 Introduction

[1]The perturbative finiteness in quantum field theory, particularly in Chern-Simons models [2, 24, 25, 70] in three space-time dimensions, has drawn attention since the preliminary results at 1-loop order [2, 71], and afterwards at 2-loops [72]. At all orders in perturbation theory, pure non-Abelian Chern-Simons model in the Landau gauge exhibits ultraviolet finiteness [73–75]. However, even though coupled to bosonic and fermionic matter fields, non-Abelian Chern-Simons model in three-dimensional Riemannian manifolds still manifests at all radiative order vanishing $\beta$-function associated to Chern-Simons coupling constant [76]. The massless $U(1)$ QED$_3$ exhibits ultraviolet and infrared perturbative finiteness, parity and infrared anomaly free at all orders [77, 78]. Moreover, in opposition to some claims in the literature still now defending that parity could spontaneously be broken, even perturbatively, in massless $U(1)$ QED$_3$, known as parity anomaly, has already been discarded by the consistent and correct use of dimension regularization [79, 80], Pauli-Villars regularization [81], algebraic renormalization in the framework of Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) subtraction method [77, 78], and more recently through the Epstein-Glaser method [82]. The exact quantum scale invariance in dimensional reduced to three dimensional space-time massless QED$_4$ models was investigated in [83], and the gauge covariance of the massless fermion propagator was studied in quenched QED$_3$ [84]. The massive $U(1)$ QED$_3$ can be odd (odd fermion families number) or even (even fermion families number) under parity symmetry. The parity-even massive $U(1)$ QED$_3$ is ultraviolet finite, the gauge coupling $\beta$-function

[1]The content was published in Annals of Physics [69]

and the anomalous dimensions of all the fields vanish, furthermore, is infrared and parity anomaly free at all orders [85]. Besides all the latter quantum field theory formal aspects, planar quantum electrodynamics (QED$_3$) has been demonstrated potential applications in condensed matter phenomena and low energy physics, on the other hand in early universe physics and high energy as well.

The main purpose in this chapter is to show the ultraviolet finiteness – vanishing $\beta$-functions of both gauge couplings and all field anomalous dimensions – at all orders in perturbation theory of the parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ [22], and the absence of any kind of anomaly, *e.g.* gauge and parity, as well. The proof is done by using the BRS (Becchi-Rouet-Stora) algebraic renormalization method in the framework of Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) subtraction scheme, which is based on general theorems of perturbative quantum field theory [19, 86–96] thus independent of any regularization scheme. Accordingly, the action of the model and its symmetries, the action for the gauge-fixing and the one which couples antifields to the nonlinear BRS transformations of the fields are established in Section 4.2. The extension of parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ at the classical level to all orders in perturbation theory – its perturbative quantization – is arranged as follows. Prior the stability analysis of the classical action – if the radiative corrections can be reabsorbed by a redefinition of the initial parameters of the model – which is presented in Section 4.4, in Section 4.3 all potential anomalies are identified by means of the analysis of the Wess-Zumino consistency condition, in other words, solving the Slavnov-Taylor cohomology problem in the sector of ghost number one, in addition to, it is checked if the radiatively induced breakings might be fine-tuned by an appropriate choice of local non-invariant counterterms. Final comments and conclusions are left to Section 4.5.

## 4.2 The model and its symmetries

The action for the parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ [22] is defined by:

$$\Sigma_{\rm inv} = \int d^3x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\, \mathrm{e}^{\mu\rho\nu}\, A_\mu \partial_\rho a_\nu + i\overline{\psi}_+ \not{D}\psi_+ + i\overline{\psi}_- \not{D}\psi_- - m(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-) \right\}, \tag{4.1}$$

where $\not{D}\psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm$, $e$ (electric charge) and $g$ (pseudochiral charge) are the coupling constants with mass dimension $\frac{1}{2}$, and, $\mu$ and $m$ are mass parameters with mass dimension 1. The field strengths, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, correspond to the electromagnetic field ($A_\mu$) and the pseudochiral gauge field ($a_\mu$), respectively. $\psi_+$ and $\psi_-$ are two kinds of Dirac spinors where the $\pm$ subscripts are associated to their spin sign [17], and the gamma matrices are $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$.

The action (4.1) was built up assuming invariance under parity, fixed posteriorly, and the gauge $U_A(1) \times U_a(1)$ transformations as follows:

$$\begin{aligned}
&\delta\psi_+(x) = i[\theta_1(x) + \theta_2(x)]\psi_+(x) \ , \quad \delta\psi_-(x) = i[\theta_1(x) - \theta_2(x)]\psi_-(x) \ , \\
&\delta\overline{\psi}_+(x) = -i[\theta_1(x) + \theta_2(x)]\overline{\psi}_+(x) \ , \quad \delta\overline{\psi}_-(x) = -i[\theta_1(x) - \theta_2(x)]\overline{\psi}_-(x) \ , \\
&\delta A_\mu(x) = -\frac{1}{e}\partial_\mu\theta_1(x) \ , \quad \delta a_\mu(x) = -\frac{1}{g}\partial_\mu\theta_2(x) \ ,
\end{aligned} \tag{4.2}$$

from which the BRS field transformations shall be defined. In view of a forthcoming quantization of the model (4.1), a parity-preserving gauge-fixing action is added, beyond that in order to follow the BRS procedure [91–93], two sorts of Lautrup-Nakanishi fields ($b$ and $\pi$) [97–99], ghosts ($c$ and $\xi$) and antighosts ($\overline{c}$ and $\overline{\xi}$), the formers ($b$ and $\pi$) are indeed Lagrange multiplier fields fixing the gauge condition, have to be introduced. Therefore, so as to quantize the model, a gauge-fixing action, belonging to the class of covariant linear gauges [100, 101], is assumed:

$$\Sigma_{\rm gf} = \int d^3x \left\{ b\partial^\mu A_\mu + \frac{\alpha}{2}b^2 + \overline{c}\Box c + \pi\partial^\mu a_\mu + \frac{\beta}{2}\pi^2 + \overline{\xi}\Box\xi \right\} . \tag{4.3}$$

Hereafter, the BRS transformations of the quantum fields are now defined by:

$$\begin{aligned}
s\psi_+ &= i(c+\xi)\psi_+ \ , \quad s\overline{\psi}_+ = -i(c+\xi)\overline{\psi}_+ \ ; \\
s\psi_- &= i(c-\xi)\psi_- \ , \quad s\overline{\psi}_- = -i(c-\xi)\overline{\psi}_- \ ; \\
sA_\mu &= -\frac{1}{e}\partial_\mu c \ , \quad sc = 0 \ ; \quad sa_\mu = -\frac{1}{g}\partial_\mu\xi \ , \quad s\xi = 0 \ ; \\
s\overline{c} &= \frac{b}{e} \ , \quad sb = 0 \ ; \quad s\overline{\xi} = \frac{\pi}{g} \ , \quad s\pi = 0 \ .
\end{aligned} \tag{4.4}$$

Together with the parity-even action term, $\Sigma_{\rm inv} + \Sigma_{\rm gf}$, another parity-even action, $\Sigma_{\rm ext}$, is introduced in order to control at the quantum level the renormalization of the nonlinear BRS transformations by coupling them to the antifields (BRS invariant external fields):

$$\Sigma_{\rm ext} = \int d^3x \left\{ i\overline{\Omega}_+ c_+\psi_+ - i\overline{\Omega}_- c_-\psi_- + ic_+\overline{\psi}_+\Omega_+ - ic_-\overline{\psi}_-\Omega_- \right\} , \tag{4.5}$$

where $c_+ = c + \xi$ and $c_- = c - \xi$. It should be pointed out that in spite of the Faddeev-Popov ghosts be massless, consequently serious infrared divergences could be stemmed from radiative corrections, nevertheless they decouple because of be free fields, there is no need to introduce Lowenstein-Zimmermann mass terms [95, 96] for them. Now, the complete classical action to be perturbatively quantized reads

$$\Gamma^{(0)} = \Sigma_{\rm inv} + \Sigma_{\rm gf} + \Sigma_{\rm ext} \ . \tag{4.6}$$

The propagators are the key ingredient on unitarity and spectral consistency analyses at tree-level of the model[2], even as in the calculation of ultraviolet and infrared dimensions of the fields. The tree-level propagators can be derived for all the quantum fields just by turning off the coupling constants ($e$ and $g$) and picking up the free part of the action $\Sigma_{\rm inv} + \Sigma_{\rm gf}$ ((4.1) and (4.3)). In that case all the propagators in momenta space are given by:

$$\Delta^{++}_{\overline{\psi}\psi}(k) = i\frac{\not{k} - m}{k^2 - m^2} \ , \quad \Delta^{--}_{\overline{\psi}\psi}(k) = i\frac{\not{k} + m}{k^2 - m^2} \ , \tag{4.7}$$

$$\Delta^{\mu\nu}_{AA}(k) = -i\left\{\frac{1}{k^2 - \mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\alpha}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \ , \tag{4.8}$$

$$\Delta^{\mu\nu}_{aa}(k) = -i\left\{\frac{1}{k^2 - \mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\beta}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \ , \tag{4.9}$$

$$\Delta^{\mu\nu}_{Aa}(k) = \frac{\mu}{k^2(k^2 - \mu^2)}\, \mathrm{e}^{\mu\lambda\nu}\, k_\lambda \ , \tag{4.10}$$

$$\Delta^{\mu}_{Ab}(k) = \Delta^{\mu}_{a\pi}(k) = \frac{k^\mu}{k^2} \ , \tag{4.11}$$

$$\Delta_{\overline{c}c}(k) = \Delta_{\overline{\xi}\xi}(k) = -\frac{i}{k^2} \ , \tag{4.12}$$

$$\Delta_{bb}(k) = \Delta_{\pi\pi}(k) = 0 \ . \tag{4.13}$$

It should be emphasized that the non decoupled propagators (4.7)–(4.9) carrying physical degrees of freedom are all massive, so there were no infrared divergences that would arisen during the ultraviolet subtractions in BPHZ method, in other words from (4.11)–(4.13) it can be concluded that all massless degrees of freedom – so potential infrared divergences inducers – do decouple. As a consequence, since no care about possible infrared divergences induced by ultraviolet subtractions is required, the ultraviolet divergences are the only that could spoiled the physical consistency of the model. Accordingly, the ultraviolet (UV) dimension of any fields, $X$ and $Y$, is established through the UV asymptotical behaviour ($d_{XY}$) of their propagator $\Delta_{XY}(k)$, namely $d_{XY} = \overline{\deg}_k \Delta_{XY}(k)$, such that $\overline{\deg}_k$ provides the asymptotic power when $k \to \infty$. In addition to that, in the BPHZ renormalization method sometimes it is opportune to employ more UV subtractions than would indeed be necessary for convergence issue, establishing therefore bounds for UV subtractions of any Feynman graph or subgraph ($\gamma$) independent of either its detailed structure or its order. As a consequence, according to Zimmermann's convergence method [94–96, 102, 103], the UV dimension ($d$) of the fields, $X$ and $Y$, shall fulfill the inequality below:

$$d_X + d_Y \geq d_{XY} + 3 \ , \tag{4.14}$$

in such a manner that, together with the power-counting formula (4.57) presented farther, they set what kind of appropriate UV subtractions have to be performed so as to get a

[2]The issues about unitarity, spectral consistency and two-particle scattering potentials have been discussed in [60].

finite integral associated to a particular graph or subgraph $(\gamma)$ at some perturbative order.

With the aim of determinate the UV dimensions of the vector fields, $A_\mu$ and $a_\mu$, and the spinor fields, $\psi_+$ and $\psi_-$, use has been made of the propagators (4.7)–(4.10) together with the condition (4.14), that results:

$$d_{++} = -1 \;\Rightarrow\; 2d_+ \geq 2 \;\Rightarrow\; d_+ = 1\;;\;\; d_{--} = -1 \;\Rightarrow\; 2d_- \geq 2 \;\Rightarrow\; d_- = 1 \tag{4.15}$$

$$d_{AA} = -2 \;\Rightarrow\; 2d_A \geq 1 \;\Rightarrow\; d_A = \frac{1}{2}\;;\;\; d_{aa} = -2 \;\Rightarrow\; 2d_a \geq 1 \;\Rightarrow\; d_a = \frac{1}{2}\;; \tag{4.16}$$

$$d_{Aa} = -3 \;\Rightarrow\; d_A + d_a \geq 0\;, \tag{4.17}$$

where the latter constraint (4.17)[3] – stemming from the mixed propagator (4.10) and the UV bound (4.14) – is also fulfilled by UV dimensions $d_A$ and $d_a$ obtained in the former condition (4.16). Additionally, it shall be called into question that, despite of the mixed propagator (4.10) carries no on-shell degrees of freedom [22, 60], whenever it enters as an internal (off-shell) line of any 1-particle irreducible Feynman graph or subgraph $(\gamma)$, thus as a virtual quantum, and the graph $(\gamma)$ exhibits non-negative UV degree of divergence $(\delta(\gamma) \geq 0)$, BPHZ ultraviolet subtractions have to be performed. Notice however that, since $d_{Aa} < d_{AA} = d_{aa}$, internal lines containing the mixed propagator (4.10), instead of propagators (4.8) and (4.9), reduce the graph UV degree of divergence, as translated by the power-counting formula (4.57) presented in the following. Furthermore, concerning the fulfillment of the constraint (4.17) by the UV dimensions $d_A$ and $d_a$, stemming from the mixed propagator (4.10), it might imply over-subtractions for some graph or subgraph $(\gamma)$, either related to invariant counterterms or noninvariat counterterms, containing the propagator $\Delta^{\mu\nu}_{Aa}(k)$ as internal lines.

From the propagators (4.11) and the conditions (4.14) and (4.16), the UV dimensions of the Lautrup-Nakanishi fields, $b$ and $\pi$, can be fixed as:

$$d_{Ab} = -1 \;\Rightarrow\; d_A + d_b \geq 2 \;\Rightarrow\; d_b = \frac{3}{2}\;;\;\; d_{a\pi} = -1 \;\Rightarrow\; d_a + d_\pi \geq 2 \;\Rightarrow\; d_\pi = \frac{3}{2}\;. \tag{4.18}$$

By considering the propagators (4.12), the UV dimensions of the Faddeev-Popov ghosts, $c$ and $\xi$, and antighosts, $\bar{c}$ and $\bar{\xi}$, are constrained by:

$$d_{\bar{c}c} = -2 \;\Rightarrow\; d_c + d_{\bar{c}} \geq 1\;;\;\; d_{\bar{\xi}\xi} = -2 \;\Rightarrow\; d_\xi + d_{\bar{\xi}} \geq 1\;. \tag{4.19}$$

Furthermore, by fixing the BRS operator $(s)$ as dimensionless and knowing that the coupling constants $e$ and $g$ have mass dimension $\frac{1}{2}$, from the conditions (4.19), the UV dimensions of the Faddeev-Popov ghosts and antighosts result:

$$d_c = 0 \quad \text{and} \quad d_{\bar{c}} = 1\;;\;\; d_\xi = 0 \quad \text{and} \quad d_{\bar{\xi}} = 1\;. \tag{4.20}$$

[3]The Zimmermann's inequality (4.14) yields likewise the condition (4.17) also satisfied by $d_A$ and $d_a$.

| | $A_\mu$ | $a_\mu$ | $\psi_+$ | $\psi_-$ | $c$ | $\overline{c}$ | $b$ | $\xi$ | $\overline{\xi}$ | $\pi$ | $\Omega_+$ | $\Omega_-$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 1/2 | 1/2 | 1 | 1 | 0 | 1 | $\frac{3}{2}$ | 0 | 1 | $\frac{3}{2}$ | 2 | 2 |
| $\Phi\Pi$ | 0 | 0 | 0 | 0 | 1 | $-1$ | 0 | 1 | $-1$ | 0 | $-1$ | $-1$ |
| $GP$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 |

Table 4.1: The UV dimension ($d$), ghost number ($\Phi\Pi$) and Grassmann parity ($GP$).

After all, from the antifields action ($\Sigma_{\rm ext}$) together with the UV dimensions of all the quantum fields previously computed, it follows that:

$$d_{\Omega_+}=2 \quad \text{and} \quad d_{\Omega_-}=2\ . \tag{4.21}$$

Briefly, the UV dimension ($d$), the ghost number ($\Phi\Pi$) and the Grassmann parity ($GP$) of all fields are displayed in Table 4.1. The statistics is defined in such a way that, the integer spin fields with odd ghost number and the half integer spin fields with even ghost number anticommute among themselves, in any other case the fields commute among themselves.

In a functional way, the Slavnov-Taylor identity expresses the BRS invariance of the action $\Gamma^{(0)}$ (4.6):

$$\mathcal{S}(\Gamma^{(0)})=0\ , \tag{4.22}$$

where, acting on an arbitrary functional $\mathcal{F}$, the Slavnov-Taylor operator $\mathcal{S}$ read

$$\begin{aligned}\mathcal{S}(\mathcal{F})=\int d^3x\bigg\{&-\frac{1}{e}\partial^\mu c\frac{\delta\mathcal{F}}{\delta A^\mu}+\frac{b}{e}\frac{\delta\mathcal{F}}{\delta\overline{c}}-\frac{1}{g}\partial^\mu\xi\frac{\delta\mathcal{F}}{\delta a^\mu}+\frac{\pi}{g}\frac{\delta\mathcal{F}}{\delta\overline{\xi}}+\\ &+\frac{\delta\mathcal{F}}{\delta\overline{\Omega}_+}\frac{\delta\mathcal{F}}{\delta\psi_+}-\frac{\delta\mathcal{F}}{\delta\Omega_+}\frac{\delta\mathcal{F}}{\delta\overline{\psi}_+}-\frac{\delta\mathcal{F}}{\delta\overline{\Omega}_-}\frac{\delta\mathcal{F}}{\delta\psi_-}+\frac{\delta\mathcal{F}}{\delta\Omega_-}\frac{\delta\mathcal{F}}{\delta\overline{\psi}_-}\bigg\}\ .\end{aligned} \tag{4.23}$$

and the linearized Slavnov-Taylor operator $\mathcal{S}_\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{S}_\mathcal{F}\ =\ &\int d^3x\bigg\{-\frac{1}{e}\partial^\mu c\frac{\delta}{\delta A^\mu}+\frac{b}{e}\frac{\delta}{\delta\overline{c}}+-\frac{1}{g}\partial^\mu\xi\frac{\delta}{\delta a^\mu}+\frac{\pi}{g}\frac{\delta}{\delta\overline{\xi}}+\frac{\delta\mathcal{F}}{\delta\overline{\Omega}_+}\frac{\delta}{\delta\psi_+}+\frac{\delta\mathcal{F}}{\delta\psi_+}\frac{\delta}{\delta\overline{\Omega}_+}+\\ &-\frac{\delta\mathcal{F}}{\delta\Omega_+}\frac{\delta}{\delta\overline{\psi}_+}-\frac{\delta\mathcal{F}}{\delta\overline{\psi}_+}\frac{\delta}{\delta\Omega_+}-\frac{\delta\mathcal{F}}{\delta\overline{\Omega}_-}\frac{\delta}{\delta\psi_-}-\frac{\delta\mathcal{F}}{\delta\psi_-}\frac{\delta}{\delta\overline{\Omega}_-}+\frac{\delta\mathcal{F}}{\delta\Omega_-}\frac{\delta}{\delta\overline{\psi}_-}+\frac{\delta\mathcal{F}}{\delta\overline{\psi}_-}\frac{\delta}{\delta\Omega_-}\bigg\}\ .\end{aligned} \tag{4.24}$$

Thenceforward, it follows the nilpotency identities:

$$\mathcal{S}_\mathcal{F}\mathcal{S}(\mathcal{F})=0\ ,\quad \forall\mathcal{F}\ , \tag{4.25}$$

$$\mathcal{S}_\mathcal{F}\mathcal{S}_\mathcal{F}=0\ \text{ if }\ \mathcal{S}(\mathcal{F})=0\ . \tag{4.26}$$

Particularly, the linearized Slavnov-Taylor operator $\mathcal{S}_{\Gamma^{(0)}}$ is nilpotent, namely $\mathcal{S}^2_{\Gamma^{(0)}}=0$, due to the fact that the action $\Gamma^{(0)}$ (4.6) fulfills the Slavnov-Taylor identity (4.22). Moreover, the action of $\mathcal{S}_{\Gamma^{(0)}}$ upon the fields and the antifields (external sources) results

$$\begin{aligned}
&\mathcal{S}_{\Gamma^{(0)}}\phi = s\phi \ , \quad \phi = \{\psi_+, \overline{\psi}_+, \psi_-, \overline{\psi}_-, A_\mu, a_\mu, b, c, \overline{c}, \pi, \xi, \overline{\xi}\} \ , \\
&\mathcal{S}_{\Gamma^{(0)}}\Omega_+ = -\frac{\delta\Gamma^{(0)}}{\delta\overline{\psi}_+} \ , \quad \mathcal{S}_{\Gamma^{(0)}}\overline{\Omega}_+ = \frac{\delta\Gamma^{(0)}}{\delta\psi_+} \ , \\
&\mathcal{S}_{\Gamma^{(0)}}\Omega_- = \frac{\delta\Gamma^{(0)}}{\delta\overline{\psi}_-} \ , \quad \mathcal{S}_{\Gamma^{(0)}}\overline{\Omega}_- = -\frac{\delta\Gamma^{(0)}}{\delta\psi_-} \ .
\end{aligned} \tag{4.27}$$

Besides the Slavnov-Taylor identity (4.22), the classical action $\Gamma^{(0)}$ (4.6) satisfies the gauge conditions, antighost equations and ghost equations as below:

$$\frac{\delta\Gamma^{(0)}}{\delta b} = \partial^\mu A_\mu + \alpha b, \quad -i\frac{\delta\Gamma^{(0)}}{\delta c} = i\Box\overline{c} + \overline{\Omega}_+\psi_+ + \overline{\psi}_+\Omega_+ - \overline{\Omega}_-\psi_- - \overline{\psi}_-\Omega_-, \quad \frac{\delta\Gamma^{(0)}}{\delta\overline{c}} = \Box c \tag{4.28}$$

$$\frac{\delta\Gamma^{(0)}}{\delta\pi} = \partial^\mu a_\mu + \beta\pi, \quad -i\frac{\delta\Gamma^{(0)}}{\delta\xi} = i\Box\overline{\xi} + \overline{\Omega}_+\psi_+ + \overline{\psi}_+\Omega_+ - \overline{\Omega}_-\psi_- - \overline{\psi}_-\Omega_-, \quad \frac{\delta\Gamma^{(0)}}{\delta\overline{\xi}} = \Box\xi \tag{4.29}$$

Furthermore, the action $\Gamma^{(0)}$ (4.6) is invariant under the two rigid symmetries associated to $U_A(1) \times U_a(1)$:

$$W^e_{\text{rig}}\Gamma^{(0)} = 0 \quad \text{and} \quad W^g_{\text{rig}}\Gamma^{(0)} = 0 \ , \tag{4.30}$$

where the Ward operators, $W^e_{\text{rig}}$ and $W^g_{\text{rig}}$, read

$$W^e_{\text{rig}} = \int d^3x \left\{ \psi_+\frac{\delta}{\delta\psi_+} - \overline{\psi}_+\frac{\delta}{\delta\overline{\psi}_+} + \Omega_+\frac{\delta}{\delta\Omega_+} - \overline{\Omega}_+\frac{\delta}{\delta\overline{\Omega}_+} + \psi_-\frac{\delta}{\delta\psi_-} - \overline{\psi}_-\frac{\delta}{\delta\overline{\psi}_-} + \Omega_-\frac{\delta}{\delta\Omega_-} - \overline{\Omega}_-\frac{\delta}{\delta\overline{\Omega}_-} \right\},$$

$$W^g_{\text{rig}} = \int d^3x \left\{ \psi_+\frac{\delta}{\delta\psi_+} - \overline{\psi}_+\frac{\delta}{\delta\overline{\psi}_+} + \Omega_+\frac{\delta}{\delta\Omega_+} - \overline{\Omega}_+\frac{\delta}{\delta\overline{\Omega}_+} - \psi_-\frac{\delta}{\delta\psi_-} + \overline{\psi}_-\frac{\delta}{\delta\overline{\psi}_-} - \Omega_-\frac{\delta}{\delta\Omega_-} + \overline{\Omega}_-\frac{\delta}{\delta\overline{\Omega}_-} \right\}. \tag{4.31}$$

The $U_A(1) \times U_a(1)$ gauge invariant action $\Gamma^{(0)}$ (4.6) being even under the parity transformation ($P$) fixes its operation on the fields and antifields:

$$\begin{aligned}
&\psi_+ \xrightarrow{P} \psi_+^P = -i\gamma^1\psi_- \ , \quad \psi_- \xrightarrow{P} \psi_-^P = -i\gamma^1\psi_+ \ , \\
&\overline{\psi}_+ \xrightarrow{P} \overline{\psi}_+^P = i\overline{\psi}_-\gamma^1 \ , \quad \overline{\psi}_- \xrightarrow{P} \overline{\psi}_-^P = i\overline{\psi}_+\gamma^1 \ ; \\
&\Omega_+ \xrightarrow{P} \Omega_+^P = -i\gamma^1\psi_- \ , \quad \Omega_- \xrightarrow{P} \Omega_-^P = -i\gamma^1\psi_+ \ , \\
&\overline{\Omega}_+ \xrightarrow{P} \overline{\Omega}_+^P = i\overline{\Omega}_-\gamma^1 \ , \quad \overline{\Omega}_- \xrightarrow{P} \overline{\Omega}_-^P = i\overline{\Omega}_+\gamma^1 \ ; \\
&A_\mu \xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ ; \quad \phi \xrightarrow{P} \phi^P = \phi \ , \quad \phi = \{b, c, \overline{c}\} \ ; \\
&a_\mu \xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ ; \quad \chi \xrightarrow{P} \chi^P = -\chi \ , \quad \chi = \{\pi, \xi, \overline{\xi}\} \ .
\end{aligned} \tag{4.32}$$

Next, Section 4.3 is devoted to seek for anomalies, *i.e.* classical symmetries broken at the quantum level. Although the gauge symmetry group $U_A(1) \times U_a(1)$ is Abelian, it is a non-semisimple Lie group, a priori the associated rigid invariance might be broken by radiative corrections, thus anomalous. Therefore, checking potential anomalies is a primary issue in comparison to the stability investigation presented in Section 4.4

## 4.3 The unitarity condition: in search for anomalies

The multiplicative renormalizability, more precisely the stability condition, does not assure the extension of the classical model to quantum level, it still remains to guarantee the non existence of any gauge anomaly, *i.e.* electromagnetic and pseudochiral anomalies, and also the parity anomaly, once the latter is sometimes claimed in the literature as a typical anomaly of three dimensional space-times.

The quantum vertex functional ($\Gamma$) matches the tree-level action ($\Gamma^{(0)}$) at zeroth-order in $\hbar$,

$$\Gamma = \Gamma^{(0)} + \mathcal{O}(\hbar) \ , \tag{4.33}$$

shall fulfill the same conditions (4.28)–(4.30) of the tree-level action.

Actually, as stated by the Quantum Action Principle [86–90], the classical Slavnov-Taylor identity (4.22) acquires a breaking at the quantum level:

$$\mathcal{S}(\Gamma) = \Delta \cdot \Gamma = \Delta + \mathcal{O}(\hbar\Delta) \ , \tag{4.34}$$

where the Slavnov-Taylor breaking $\Delta$ is an integrated local Lorentz invariant functional, with ghost number equal to 1 and UV dimension bounded by $d \leq \frac{7}{2}$.

Taking into consideration the Slavnov-Taylor quantum identity (4.34), the nilpotency identity (4.25) applied to the quantum vertex functional, *i.e.*, $\mathcal{S}_\Gamma \mathcal{S}(\Gamma) = 0$, and the equation $\mathcal{S}_\Gamma = \mathcal{S}_{\Gamma^{(0)}} + \mathcal{O}(\hbar)$ obtained from (4.24) and (4.33), this all together leads to the Wess-Zumino consistency condition for the quantum breaking $\Delta$:

$$\mathcal{S}_{\Gamma^{(0)}} \Delta = 0 \ , \tag{4.35}$$

Moreover, in addition to (4.35), calling into question the Slavnov-Taylor identity (4.22), the gauge, antighost and ghost equations (4.28)–(4.29), so as the rigid conditions (4.30), it is verified that the Slavnov-Taylor quantum breaking ($\Delta$) also satisfies the constraints:

$$\frac{\delta\Delta}{\delta b} = \int d^3x \frac{\delta\Delta}{\delta c} = \frac{\delta\Delta}{\delta\bar{c}} = W^e_{\text{rig}}\Delta = 0 \quad \text{and} \quad \frac{\delta\Delta}{\delta\pi} = \int d^3x \frac{\delta\Delta}{\delta\xi} = \frac{\delta\Delta}{\delta\bar{\xi}} = W^g_{\text{rig}}\Delta = 0 \ . \tag{4.36}$$

At this point it shall be mentioned that, as far as rigid gauge invariance is concerned, since the symmetry group $U_A(1) \times U_a(1)$ is a non-semisimple Lie group, in principle rigid invariance could be broken at the quantum level, in other words, rigid symmetry might be anomalous. Nevertheless, none of both abelian factors are spontaneously broken as well as the conditions displayed in (4.30), $W^e_{\text{rig}}\Gamma^{(0)} = 0$ and $W^g_{\text{rig}}\Gamma^{(0)} = 0$, express indeed the conservation of the electric ($e$) and the pseudochiral ($g$) charges, therefore the conditions exhibited in (4.36), $W^e_{\rm rig}\Delta = 0$ and $W^g_{\rm rig}\Delta = 0$, are valid [103, 104].

Recalling again the Slavnov-Taylor condition (4.52) satisfied by the counterterm (4.56), which is indeed a cohomology problem in the sector of ghost number zero, similarly in the sector of ghost number one, the cohomology problem is represented by Wess-Zumino consistency condition (4.35). As a matter of fact, a general solution to the cohomology problem (4.35) can ever be expressed as a sum of a trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)}$, where $\widehat{\Delta}^{(0)}$ has ghost number zero, and of nontrivial elements possessing ghost number one, $\widehat{\Delta}^{(1)}$, lying in the cohomology of $\mathcal{S}_{\Gamma^{(0)}}$ (4.24):

$$\Delta^{(1)} = \widehat{\Delta}^{(1)} + \mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)} \ , \tag{4.37}$$

reminding that the Slavnov-Taylor quantum breaking $\Delta^{(1)}$ (4.37) has to fulfill the constraints (4.35) and (4.36). It should be highlighted that the trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)}$ can be incorporated order by order into the vertex functional $\Gamma$, namely $\mathcal{S}_{\Gamma^{(0)}}(\Gamma - \widehat{\Delta}^{(0)}) = \widehat{\Delta}^{(1)} + \mathcal{O}(\hbar\Delta^{(1)})$, as a non invariant integrated local counterterm, $-\widehat{\Delta}^{(0)}$ – a vanishing ghost number local field integrated polynomial. The linearized Slavnov-Taylor operator $\mathcal{S}_{\Gamma^{(0)}}$ (4.24) in combination with the Slavnov-Taylor quantum identity (4.34) results that the breaking $\Delta^{(1)}$ exhibits UV dimension bounded by $d \leq \frac{7}{2}$. It shall be stressed that since the insertion $\Delta^{(1)}$ stems from radiative corrections, it possesses a factor, $e^2$, $g^2$ or $eg$, at least, hence its effective UV dimension turns out to be bounded by $d \leq \frac{5}{2}$.

Now, it has been verified that, as displayed in (4.36), from the antighost equations (4.28)–(4.29), the quantum breaking $\Delta^{(1)}$ (4.37) fulfill the constraints:

$$\int d^3x \frac{\delta\Delta^{(1)}}{\delta c} = 0 \quad \text{and} \quad \int d^3x \frac{\delta\Delta^{(1)}}{\delta\xi} = 0 \ , \tag{4.38}$$

then it follows that $\Delta^{(1)}$ reads

$$\Delta^{(1)} = \int d^3x \Big\{ \mathcal{K}^{(0)}_\mu \partial^\mu c + \mathcal{X}^{(0)}_\mu \partial^\mu \xi \Big\} \ , \tag{4.39}$$

where $\mathcal{K}^{(0)}_\mu$ and $\mathcal{X}^{(0)}_\mu$ are rank-1 tensors with zero ghost number and UV dimension bounded by $d \leq \frac{3}{2}$. Beyond that, the breaking $\Delta^{(1)}$ may be expressed by two linearly independent terms, where one is even under parity while the other is odd, thus $\mathcal{K}^{(0)}_\mu$ and $\mathcal{X}^{(0)}_\mu$ can be written as

$$\mathcal{K}^{(0)}_\mu = \sum_{i=1} v_{k,i}\mathcal{V}^i_\mu + \sum_{i=1} p_{k,i}\mathcal{P}^i_\mu \quad \text{and} \quad \mathcal{X}^{(0)}_\mu = \sum_{i=1} v_{x,i}\Upsilon^i_\mu + \sum_{i=1} p_{x,i}\Pi^i_\mu \ , \tag{4.40}$$

with $v_{k,i}$, $p_{k,i}$, $v_{x,i}$ and $p_{x,i}$ being fixed coefficients to be further determined. Moreover, $\mathcal{V}^i_\mu$ and $\Upsilon^i_\mu$ are defined as vectors, while $\mathcal{P}^i_\mu$ and $\Pi^i_\mu$ as pseudo-vectors, in such a way that

$\mathcal{V}^i_\mu \partial^\mu c$ and $\Pi^i_\mu \partial^\mu \xi$ are parity-even, whereas $\mathcal{P}^i_\mu \partial^\mu c$ and $\Upsilon^i_\mu \partial^\mu \xi$ are parity-odd, since $\partial^\mu c$ is a vector and $\partial^\mu \xi$ a pseudo-vector. Taking into consideration that $\mathcal{K}^{(0)}_\mu$ and $\mathcal{X}^{(0)}_\mu$ have their UV dimensions given by $d \leq \frac{3}{2}$ and $\Delta^{(1)}$ must fulfill the conditions (4.35) and (4.36), it is verified that even though there are $\mathcal{P}^i_\mu$ and $\Upsilon^i_\mu$ surviving all of these constraints, namely, $\mathcal{P}^1_\mu = \mathrm{e}_{\mu\rho\nu}\, \partial^\rho A^\nu$ and $\Upsilon^1_\mu = \mathrm{e}_{\mu\rho\nu}\, \partial^\rho a^\nu$, their contributions in $\Delta^{(1)}$ (4.39) are all ruled out by partial integration, therefore effectively for the anomaly analysis purposes, $\{\mathcal{P}^i_\mu\} = \emptyset$ and $\{\Upsilon^i_\mu\} = \emptyset$, thereby leaving only the parity invariant part ($\Delta^{(1)}_{\text{even}}$) of the Slavnov-Taylor quantum breaking $\Delta^{(1)}$ (4.39):

$$\Delta^{(1)}_{\text{even}} = \int d^3x \Big\{ \sum_{i=1} v_{k,i}\; \mathcal{V}^i_\mu \partial^\mu c + \sum_{i=1} p_{x,i}\; \Pi^i_\mu \partial^\mu \xi \Big\} \;, \tag{4.41}$$

ending the proof that parity is not broken at the quantum level, there is no parity anomaly at all.

It still remains to verify if the parity-even breaking $\Delta^{(1)}_{\text{even}}$ (4.41) is a genuine gauge anomaly or just a trivial cocycle that could be reabsorbed into the quantum action as noninvariant counterterms. However, it lacks to find the candidates for $\mathcal{V}_\mu$ (vector) and $\Pi_\mu$ (pseudo-vector) with UV dimensions given by $d \leq \frac{3}{2}$ provided that $\Delta^{(1)}_{\text{even}}$ (4.41) satisfies the constraints (4.35) and (4.36). So, they read as follows:

$$\begin{aligned} &\mathcal{V}^1_\mu = A_\mu A^\nu A_\nu \;, \quad \mathcal{V}^2_\mu = A_\mu a^\nu a_\nu \;, \quad \mathcal{V}^3_\mu = A_\nu a^\nu a_\mu \;, \\ &\Pi^1_\mu = a_\mu a^\nu a_\nu \;, \quad \Pi^2_\mu = a_\mu A^\nu A_\nu \;, \quad \Pi^3_\mu = a_\nu A^\nu A_\mu \;, \end{aligned} \tag{4.42}$$

as a consequence the quantum breaking $\Delta^{(1)}_{\text{even}}$ (4.41) becomes expressed by

$$\begin{aligned} \Delta^{(1)}_{\text{even}} = \int d^3x \Big\{ &\; v_{k,1} A_\mu A^\nu A_\nu \partial^\mu c + v_{k,2} A_\mu a^\nu a_\nu \partial^\mu c + v_{k,3} A_\nu a^\nu a_\mu \partial^\mu c + \\ &+ p_{x,1} a_\mu a^\nu a_\nu \partial^\mu \xi + p_{x,2} a_\mu A^\nu A_\nu \partial^\mu \xi + p_{x,3} a_\nu A^\nu A_\mu \partial^\mu \xi \Big\} \;. \end{aligned} \tag{4.43}$$

At this moment, from the Wess-Zumino consistency condition (4.35), whether any gauge anomaly thanks to radiative corrections shows up or not shall be demonstrated by analyzing the breaking $\Delta^{(1)}_{\text{even}}$ (4.43), consequently if it could be written unambiguously as a trivial cocycle $\mathcal{S}_{\Gamma^{(0)}} \widehat{\Delta}^{(0)}$, with $\widehat{\Delta}^{(0)}$ being a zero ghost number integrated local monomials, the noninvariant counterterms, there is no gauge anomaly and the unitarity is guaranteed, otherwise, if there is at least one nontrivial element $\widehat{\Delta}^{(1)}$ pertaining to the cohomology of $\mathcal{S}_{\Gamma^{(0)}}$ (4.24) in the sector of ghost number one, the gauge symmetry is anomalous and unitarity is definitely jeopardized. In order to give the sequel on the issue of gauge anomaly, it can be straightforwardly verified that:

$$\begin{aligned}
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)1} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ A^\mu A_\mu A^\nu A_\nu = -\frac{4}{e}\int d^3x\ A_\mu A^\nu A_\nu \partial^\mu c\ , &(4.44)\\
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)2} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ a^\mu a_\mu a^\nu a_\nu = -\frac{4}{g}\int d^3x\ a_\mu a^\nu a_\nu \partial^\mu \xi\ , &(4.45)\\
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)3} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ A^\mu A_\mu a^\nu a_\nu = \\
&= -\frac{2}{e}\int d^3x\ A_\mu a^\nu a_\nu \partial^\mu c - \frac{2}{g}\int d^3x\ a_\mu A^\nu A_\nu \partial^\mu \xi\ , &(4.46)\\
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)4} &= -\mathcal{S}_{\Gamma^{(0)}}\int d^3x\ A^\mu A_\nu a^\nu a_\mu = \\
&= -\frac{2}{e}\int d^3x\ A_\nu a^\nu a_\mu \partial^\mu c - \frac{2}{g}\int d^3x\ a_\nu A^\nu A_\mu \partial^\mu \xi\ . &(4.47)
\end{aligned}$$

Besides that, by taking into consideration the four trivial cocycles above (4.44)–(4.47), it follows that the quantum breaking (4.41) might be rewritten as:

$$\begin{aligned}
\Delta^{(1)}_{\text{even}} &= \mathcal{S}_{\Gamma^{(0)}}\left\{\lambda_1\widehat{\Delta}^{(0)1} + \lambda_2\widehat{\Delta}^{(0)2} + \lambda_3\widehat{\Delta}^{(0)3} + \lambda_4\widehat{\Delta}^{(0)4}\right\}\\
\Delta^{(1)}_{\text{even}} &= \mathcal{S}_{\Gamma^{(0)}}\left\{\lambda_1\int d^3x\ A^\mu A_\mu A^\nu A_\nu + \lambda_2\int d^3x\ a^\mu a_\mu a^\nu a_\nu + \right.\\
&\qquad \left. + \lambda_3\int d^3x\ A^\mu A_\mu a^\nu a_\nu + \lambda_4\int d^3x\ A^\mu A_\nu a^\nu a_\mu\right\}\\
\Delta^{(1)}_{\text{even}} &= \int d^3x\left\{-\frac{4}{e}\lambda_1 A_\mu A^\nu A_\nu\partial^\mu c - \frac{2}{e}\lambda_3 A_\mu a^\nu a_\nu \partial^\mu c - \frac{2}{e}\lambda_4 A_\nu a^\nu a_\mu\partial^\mu c + \right.\\
&\qquad \left. -\frac{4}{g}\lambda_2 a_\mu a^\nu a_\nu\partial^\mu\xi - \frac{2}{g}\lambda_3 a_\mu A^\nu A_\nu\partial^\mu\xi - \frac{2}{g}\lambda_4 a_\nu A^\nu A_\mu \partial^\mu\xi\right\}\ , \qquad (4.48)
\end{aligned}$$

where it can be checked that

$$\begin{aligned}
v_{k,1} &= -\frac{4}{e}\lambda_1\ , \quad v_{k,2} = -\frac{2}{e}\lambda_3\ , \quad v_{k,3} = -\frac{2}{e}\lambda_4\ ,\\
p_{x,1} &= -\frac{4}{g}\lambda_2\ , \quad p_{x,2} = -\frac{2}{g}\lambda_3\ , \quad p_{x,3} = -\frac{2}{g}\lambda_4\ , \qquad (4.49)
\end{aligned}$$

thus finally demonstrating that the quantum breaking $\Delta^{(1)}_{\text{even}}$ (4.41) is actually a trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)}$:

$$\Delta^{(1)}_{\text{even}} = \mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)} = \mathcal{S}_{\Gamma^{(0)}}\left\{\lambda_1\widehat{\Delta}^{(0)1} + \lambda_2\widehat{\Delta}^{(0)2} + \lambda_3\widehat{\Delta}^{(0)3} + \lambda_4\widehat{\Delta}^{(0)4}\right\}\ . \qquad (4.50)$$

Therefore, as a final result, the integrated local monomials $\widehat{\Delta}^{(0)1}$, $\widehat{\Delta}^{(0)2}$, $\widehat{\Delta}^{(0)3}$ and $\widehat{\Delta}^{(0)4}$ can be absorbed as noninvariant counterterms order by order into the quantum action, *e.g.*, at $n$-order in $\hbar$:

$$\mathcal{S}_{\Gamma^{(0)}}(\Gamma - \hbar^n\widehat{\Delta}^{(0)}) \equiv \mathcal{S}_{\Gamma^{(0)}}\left(\Gamma - \hbar^n\lambda_1\widehat{\Delta}^{(0)1} - \hbar^n\lambda_2\widehat{\Delta}^{(0)2} - \hbar^n\lambda_3\widehat{\Delta}^{(0)3} - \hbar^n\lambda_4\widehat{\Delta}^{(0)4}\right) = 0\hbar^n + \mathcal{O}(\hbar^{n+1})\ , \qquad (4.51)$$

which concludes the proof on the absence of gauge anomaly, meaning that the $U_A(1)\times U_a(1)$ local symmetry is not anomalous at the quantum level.

In summary, it is finally proved that the parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ [22] is free from any anomalies, stemming from either continuous or discrete symmetries, at all orders in perturbation theory. However, in the meantime in order to complete the full quantum level analysis, it still remains to demonstrate the multiplicative renormalizability of the model, which is presented in the following.

## 4.4 The stability condition: in search for counterterms

The stability condition, *i.e.* the multiplicative renormalizability, is ensured if perturbative quantum corrections do not produce local counterterms corresponding to renormalization of parameters which are not already present in the classical theory, therefore thus those radiative corrections can be reabsorbed order by order through redefinitions of the initial physical quantities – fields, coupling constants and masses – of the theory. Consequently, so as to verify if the classical action $\Gamma^{(0)}$ (4.6) is stable under radiative corrections, it is perturbed by an arbitrary counterterm (integrated local functional) $\Sigma^c$, namely $\Gamma^\varepsilon = \Gamma^{(0)} + \varepsilon\Sigma^c$, such that $\varepsilon$ is an infinitesimal parameter and the counterterm action $\Sigma^c$ has the same quantum numbers as the tree-level action $\Gamma^{(0)}$ (4.6). Reminding the results of Section 4.3 about the absence of any sort of anomaly, which means that all the classical symmetries are preserved at the quantum level, it can be straightforwardly concluded that the deformed action $\Gamma^\varepsilon$ satisfies all the conditions fulfilled by the classical action $\Gamma^{(0)}$ (4.6), this leads to the counterterm $\Sigma^c$ be subjected to the following set of constraints:

$$\mathcal{S}_{\Gamma^{(0)}}\Sigma^c = 0 \ ; \tag{4.52}$$

$$W^e_{\rm rig}\Sigma^c = 0 \ , \quad W^g_{\rm rig}\Sigma^c = 0 \ ; \tag{4.53}$$

$$\frac{\delta\Sigma^c}{\delta b} = \frac{\delta\Sigma^c}{\delta c} = \frac{\delta\Sigma^c}{\delta\overline{c}} = 0 \ , \quad \frac{\delta\Sigma^c}{\delta\pi} = \frac{\delta\Sigma^c}{\delta\xi} = \frac{\delta\Sigma^c}{\delta\overline{\xi}} = 0 \ ; \tag{4.54}$$

$$\frac{\delta\Sigma^c}{\delta\Omega_+} = \frac{\delta\Sigma^c}{\delta\overline{\Omega}_+} = 0 \ , \quad \frac{\delta\Sigma^c}{\delta\Omega_-} = \frac{\delta\Sigma^c}{\delta\overline{\Omega}_-} = 0 \ . \tag{4.55}$$

The most general Lorentz invariant and vanishing ghost number field polynomial ($\Sigma^c$) fulfilling the conditions (4.52)–(4.55) with ultraviolet dimension bounded by $d \leq 3$, reads:

$$\Sigma^c = \int d^3x \Big\{ \alpha_1 i\overline{\psi}_+ \not{D}\psi_+ + \alpha_2 i\overline{\psi}_- \not{D}\psi_- + \alpha_3\overline{\psi}_+\psi_+ + \alpha_4\overline{\psi}_-\psi_- + \\ + \alpha_5 F^{\mu\nu}F_{\mu\nu} + \alpha_6 f^{\mu\nu}f_{\mu\nu} + \alpha_7\, \mathrm{e}^{\mu\rho\nu}\, A_\mu\partial_\rho a_\nu \Big\} \ , \tag{4.56}$$

where $\alpha_i$ ($i = 1, \ldots, 7$) are parameters to be fixed later by normalization conditions. Beyond that, there are other constraints due to superrenormalizability and parity invariance. Since all quantum fields are massive no infrared divergences arise from the ultraviolet subtractions in the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization procedure, thereby there is no need to use the Lowenstein-Zimmermann subtraction scheme [95, 96] – which explicitly breaks parity in three space-time dimensions [77, 78, 85] while the infrared divergences are subtracted – in order to subtract those infrared divergences. In this way, as a by-product the BPHZ method is an available parity-preserving subtraction procedure guaranteeing that at each loop order the counterterm ($\Sigma^c$) shall be parity-even. The superrenormalizability of the model is responsible for the following coupling-constant-dependent power-counting formula:

$$\delta(\gamma) = 3 - \sum_{\Phi} d_\Phi N_\Phi - N_{Aa} - \frac{1}{2}N_e - \frac{1}{2}N_g \; . \tag{4.57}$$

is defined for the UV degree of divergence ($\delta(\gamma)$) of a 1-particle irreducible Feynman diagram ($\gamma$), whith $N_\Phi$ being the number of external lines of $\gamma$ associated to the field $\Phi$, $d_\Phi$ the UV dimension of $\Phi$ (see Table 4.1), $N_{Aa}$ the number of internal lines of $\gamma$ associated to the mixed propagator $\Delta_{Aa}$(4.10), and $N_e$ and $N_g$ are the powers of the coupling constants $e$ and $g$, respectively, in the integral representing the Feynman graph $\gamma$. Thanks to fact that counterterms stem from loop corrections, thus they are at least of order two in the coupling constants, namely, $e^2$, $g^2$ or $eg$. Beyond that as previously mentioned, any topologically equivalent graphs with the same type of external legs, among those which have internal lines as the mixed propagator $\Delta_{Aa}$(4.10) instead of the pure ones $\Delta_{AA}$(4.8) or $\Delta_{aa}$(4.9), exhibit smaller UV degree of divergence due to the third term, $-N_{Aa}$, displayed in (4.57).

Accordingly, the effective UV dimension of the counterterm ($\Sigma^c$) is bounded by $d \leq 2$, implying that, $\alpha_1 = \alpha_2 = \alpha_5 = \alpha_6 = 0$. Furthermore, since a parity-even subtraction scheme is available thus the counterterm has to be parity invariant, resulting that $\alpha_3 = -\alpha_4 = \alpha$, and the counterterm expressed by

$$\Sigma^c = \int d^3x \Big\{ \alpha(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-) + \alpha_7 \, \mathrm{e}^{\mu\rho\nu} \, A_\mu \partial_\rho a_\nu \Big\} = z_m m \frac{\partial}{\partial m}\Gamma^{(0)} + z_\mu \mu \frac{\partial}{\partial \mu}\Gamma^{(0)} \; , \tag{4.58}$$

where $z_m = -\frac{\alpha}{m}$ and $z_\mu = \frac{\alpha_7}{\mu}$ are arbitrary parameters to be fixed order by order through the vacuum-polarization tensor and fermion self-energies normalization conditions:

$$\frac{i}{6} \, \mathrm{e}_{\mu\rho\nu} \frac{\partial}{\partial p_\rho} \Gamma^{\mu\nu}_{Aa}(p) \Big|_{p^2=\kappa^2} = \mu \; , \quad \Gamma_{\overline{\psi}_+\psi_+}(\not{p}) \Big|_{\not{p}=+m} = 0 \quad \text{and} \quad \Gamma_{\overline{\psi}_-\psi_-}(\not{p}) \Big|_{\not{p}=-m} = 0 \; , \tag{4.59}$$

with $\kappa$ being an energy scale. The counterterm (4.58) shows that, in principle, only the fermions mass $m$ and the Chern-Simons mass $\mu$ shall acquire radiative corrections, implying that, at all orders in $\hbar$, the $\beta$-functions associated to the gauge coupling constants, $e$ and

$g$, vanish, $\beta_e = 0$ and $\beta_g = 0$, respectively, the anomalous dimensions ($\gamma$) of any field as well.

In time, a subtle property of the Chern-Simons piece of action in $\Gamma^{(0)}$ (4.6):

$$\Sigma_{\rm CS} = \mu \int d^3x \ \mathrm{e}^{\mu\rho\nu} \, A_\mu \partial_\rho a_\nu \ , \tag{4.60}$$

shall be put in evidence. The Chern-Simons action $\Sigma_{\rm CS}$ (4.60) is not BRS local invariant, thus its invariance under BRS transformations is up to a surface term, *i.e.* a total derivative,

$$s\Sigma_{\rm CS} = s\Big\{\mu \int d^3x \ \mathrm{e}^{\mu\rho\nu} \, A_\mu \partial_\rho a_\nu\Big\} = -\frac{\mu}{e} \int d^3x \ \mathrm{e}^{\mu\rho\nu} \, \partial_\mu \left(c \partial_\rho a_\nu\right) \ , \tag{4.61}$$

suggesting a vanishing at the quantum level of the $\beta$-function [76, 105–107] associated to the Chern-Simons mass parameter ($\mu$), $\beta_\mu = 0$. In summary, the counterterm finally reads

$$\Sigma^c = \int d^3x \Big\{\alpha(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-)\Big\} = z_m m \frac{\partial}{\partial m}\Gamma^{(0)} \ , \tag{4.62}$$

yielding that all $\beta$-functions associated to the gauge coupling constants, electric charge ($e$) and pseudochiral charge ($g$), the Chern-Simons mass parameter ($\mu$), and all anomalous dimensions ($\gamma$) of the fields, are vanishing, excepting the $\beta$-function corresponding to the fermions mass parameter ($m$).

In summary, by taking the last result (4.62), the stability condition analysis, together with the former one (4.51), about the Wess-Zumino condition, ends the proof of vanishing $\beta$-functions associated to the gauge coupling constants ($e$ and $g$) and the Chern-Simons mass parameter ($\mu$), and all anomalous dimensions ($\gamma$) of the fields, as well as the absence of parity and gauge anomaly at all orders in perturbation theory. Finally, as a by-product, it is demonstrated the all orders ultraviolet finiteness of the parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ [22].

## 4.5 Conclusion

In conclusion, the parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ [22] is free from any gauge anomaly and parity anomaly at all orders in perturbation theory. Beyond that, it exhibits vanishing $\beta$-functions associated to the gauge coupling constants ($e$ and $g$) and the Chern-Simons mass parameter ($\mu$), and all the anomalous dimensions ($\gamma$) of the fields as well. The proof is independent of particular diagrammatic calculations or regularization schemes, since the BRS (Becchi-Rouet-Stora) algebraic renormalization method together with the BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) subtraction scheme [19, 86–96] is grounded in the general theorems of perturbative quantum field theory. Furthermore, once the quantum perturbative physical consistency of the mass-gap graphene-like planar quantum electrodynamics has been proven from the results demonstrated here together with those presented in [22], it should be newsworthy to deepen its analysis so as to apply in graphene-like electronic systems [108]. As a final comment, the vanishing of all $\beta$-functions associated to the electric charge ($e$), the pseudochiral charge ($g$) and the Chern-Simons mass parameter ($\mu$) – with the exception of that associated to the fermions mass parameter ($m$) – foresees the independence of those system parameters with respect to the temperature, on the other hand for instance the mass-gap in graphene, which can be described by the fermions mass parameter, shall be temperature dependent.

# Part II

# Massless Case

# Chapter 5

# On the electron-polaron–electron-polaron scattering and Landau levels in pristine graphene-like quantum electrodynamics

## 5.1 Introduction

[1]The quantum electrodynamics in three dimensional space-time ($QED_3$) has drawn attention since the groundbreaking works by Schonfeld, Jackiw, Templeton and Deser [2, 24, 25, 70, 110] owing to the viability of taking planar quantum electrodynamics models as theoretical foundation for quasiplanar condensed matter phenomena, such as high-$T_c$ superconductors [26–28], quantum Hall effect [29–31], topological insulators [32–34], topological superconductors [35–37] and graphene [16, 38–44, 111]. Since then, planar quantum electrodynamics models have been investigated in many physical arrangements, namely, small (perturbative) and large (non perturbative) gauge transformations, Abelian and non-Abelian gauge groups, fermions families, odd and even under parity, compact space-times, space-times with boundaries, curved space-times, discrete (lattice) space-times, external fields and finite temperatures.

The pristine graphene, a monolayer of pure graphene [16, 38–44, 111], is a gapless quasibidimensional system behaving like a half-filling semimetal where the quasiparticles charge carriers are described by massless charged Dirac fermions. The electron-electron interactions in graphene [12] include electron-polarons [13] scattering processes [14, 50–52, 112], where the quasiparticle electron-polaron (or hole-polaron) is formed by a bound state

[1]The content was published in The European Physical Journal B [109]

of electron (or hole) and phonon [15].

In this chapter a pristine graphene-like planar quantum electrodynamics model, the parity-preserving $U(1)\times U(1)$ massless QED$_3$, is proposed and introduced as follows. In Section 5.2, the model defined by its discrete and continuous symmetries is presented and its spectrum – degrees of freedom, spin, masses and charges of all the quanta particles content of the model – is discussed. In graphene the interactions among the massless fermion quasiparticles (electron-polaron and hole-polaron) are nonconfining, so the vector meson mediated quasiparticles contained in the model, namely the photon and the Néel quasiparticles, must be massive – massless mediated quanta in three space-time dimensions yield logarithm-type (confining) interaction potentials [113] – consequently the asymptotic states for the massless fermion quasiparticles might be determined. Also, similar effect as the four-fold broken degeneracy of the Landau levels experimentally observed [114–129] in pristine graphene under high applied external magnetic fields is noticed. Next, in Section 5.3, the $s$- and $p$-wave Møller (electron-polaron–electron-polaron) scattering amplitudes are computed and their respective interaction potentials obtained and analyzed. Conclusions and final comments are left to Section 5.4.

## 5.2 The model

The proposed Lorentz invariant model for a pristine graphene-like planar quantum electrodynamics, the parity-even $U(1)_A\times U(1)_a$ massless QED$_3$, is defined by the action:

$$\begin{aligned} S = \int d^3x \Big\{ & i\overline{\psi}_+ \not{D}\psi_+ + i\overline{\psi}_- \not{D}\psi_- - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \\ & +\mu\, \mathrm{e}^{\mu\rho\nu} A_\mu \partial_\rho a_\nu - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - \frac{1}{2\beta}(\partial^\mu a_\mu)^2 \Big\} , \end{aligned} \tag{5.1}$$

where $\not{D}\psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm$, and any object $\not{X} \equiv X^\mu\gamma_\mu$. The coupling constants $e$ (electric charge) and $g$ (chiral charge) carry mass dimension $\frac{1}{2}$, and the mixed Chern-Simons (CS) mass parameter $\mu$ has mass dimension 1. The field strengths, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, are associated to the electromagnetic field ($A_\mu$) and the Néel (pseudo)chiral field ($a_\mu$), respectively. The spinors $\psi_+$ and $\psi_-$ are two kinds of massless fermions, each of them representing electron-polaron (electron-phonon) and hole-polaron (hole-phonon) quasiparticles, where the subscripts + (sublattice **A**) and − (sublattice **B**) are related to their coupling to the Néel gauge field, or alternatively, to the two inequivalent **K** and **K′** points in the Brillouin zone of a monolayer graphene. Also, the gamma matrices are fixed by $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$. It should be pointed out that possible parity-odd local counterterms as radiative corrections to the classical action (5.1) might appear for the vacuum polarization tensor associated to the both gauge fields, $A_\mu$ and $a_\mu$, at 1-loop with linear ultraviolet degree of divergence ($\delta = 1$), and at 2-loops with logarithm ultraviolet degree of divergence ($\delta = 0$). In addition to that, at 1-loop parity-odd local counterterms might also arise for the self-energy related to the fermion fields, $\psi_+$ and $\psi_-$, with logarithm ultraviolet degree of divergence ($\delta = 0$). Nevertheless, by adopting the BPHZL (Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein) subtraction scheme and computing the contribution of those possible parity-violating local counterterms – at 1- and 2-loops to the vacuum polarization tensor for the electromagnetic field ($A_\mu$) and the Néel field ($a_\mu$), as well as, at 1-loop to the self-energy for the sublattice **A** fermion ($\psi_+$) and the sublattice **B** fermion ($\psi_-$) – the final conclusion is that they vanish [130]. Furthermore, if parity symmetry could be broken or not – at the stage of infrared subtractions [77, 78, 85] induced by subtracting ultraviolet divergences during the BPHZL renormalization procedure at 1- and 2-loops orders, where potential ultraviolet divergences in vacuum polarization tensor and fermion self-energy Feynman graphs shall show up – has also been verified. Also, it was demonstrated in [130] that, neither parity-odd CS pure-like terms, $\mathrm{e}^{\mu\rho\nu}\, A_\mu\partial_\rho A_\nu$ or $\mathrm{e}^{\mu\rho\nu}\, a_\mu\partial_\rho a_\nu$, nor parity-even CS mixed-like term, $\mathrm{e}^{\mu\rho\nu}\, A_\mu\partial_\rho a_\nu$, just as neither fermion parity-odd monomials, $\overline{\psi}_+\psi_+$ or $\overline{\psi}_-\psi_-$, nor fermion parity-even binomial, $\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-$, are radiatively generated. Finally, the action (5.1) shows to be stable under quantum perturbation, thus multiplicative renormalizable, however it still lacks to prove its full renormalizability, namely the absence of any kind of anomaly at all orders in perturbation theory, which is now under investigation.

### 5.2.1 The symmetries: charge conjugation, parity, time reversion and $U(1) \times U(1)$

The CPT-even action (5.1) is invariant under the following discrete and continuous symmetries:

1. charge conjugation symmetry ($C$):

$$\begin{aligned}
\psi_\pm &\xrightarrow{C} \psi_\pm^C = -\gamma^2\overline{\psi}_\pm^\top \ , \ \overline{\psi}_\pm \xrightarrow{C} \overline{\psi}_\pm^C = -\psi_\pm^\top\gamma^2 \ , \\
A_\mu &\xrightarrow{C} A_\mu^C = (-A_0, -A_1, -A_2) \ , \\
a_\mu &\xrightarrow{C} a_\mu^C = (-a_0, -a_1, -a_2) \ .
\end{aligned} \tag{5.2}$$

2. parity symmetry ($P$):

$$\begin{aligned}
x_\mu &\xrightarrow{P} x_\mu^P = (x_0, -x_1, x_2) \ , \\
\psi_\pm &\xrightarrow{P} \psi_\pm^P = -i\gamma^1\psi_\mp \ , \ \overline{\psi}_\pm \xrightarrow{P} \overline{\psi}_\pm^P = i\overline{\psi}_\mp\gamma^1 \ , \\
A_\mu &\xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ , \\
a_\mu &\xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ .
\end{aligned} \tag{5.3}$$

3. time reversion symmetry ($T$):

$$
\begin{aligned}
x_\mu &\xrightarrow{T} x_\mu^T = (-x_0, x_1, x_2)\ , \\
\psi_\pm &\xrightarrow{T} \psi_\pm^T = -i\gamma^2\psi_\mp^*\ ,\ \overline{\psi}_\pm \xrightarrow{T} \overline{\psi}_\pm^T = i\overline{\psi}_\mp^*\gamma^2\ , \\
A_\mu &\xrightarrow{T} A_\mu^T = (A_0, -A_1, -A_2)\ , \\
a_\mu &\xrightarrow{T} a_\mu^T = (-a_0, a_1, a_2)\ .
\end{aligned}
\tag{5.4}
$$

4. gauge $U(1)_A \times U(1)_a$ symmetry ($\delta_{\rm g}$):

$$
\begin{aligned}
\delta_{\rm g}\psi_\pm(x) &= i[\theta(x) \pm \omega(x)]\psi_\pm(x)\ , \\
\delta_{\rm g}\overline{\psi}_\pm(x) &= -i[\theta(x) \pm \omega(x)]\overline{\psi}_\pm(x)\ , \\
\delta_{\rm g}A_\mu(x) &= -\frac{1}{e}\,\partial_\mu\theta(x)\ , \\
\delta_{\rm g}a_\mu(x) &= -\frac{1}{g}\,\partial_\mu\omega(x)\ .
\end{aligned}
\tag{5.5}
$$

### 5.2.2 The spectrum: charges, spin, Landau levels, masses and degrees of freedom

The free Dirac equations associated to massless spinors, $\psi_+$ and $\psi_-$, derived from the action (5.1), read:

$$
i\partial\!\!\!/\psi_+ = 0 \quad \text{and} \quad i\partial\!\!\!/\psi_- = 0\ . \tag{5.6}
$$

Thus, expanding the spinor field operators $\psi_+$ and $\psi_-$ in terms of the $c$-number plane wave solutions of the Dirac equations, with operator-valued amplitudes, $a_+$, $b_+$, $a_-$ and $b_-$ (annihilation operators), and $a_+^\dagger$, $b_+^\dagger$, $a_-^\dagger$ and $b_-^\dagger$ (creation operators), it follows that:

$$
\psi_+(x) = \int \frac{d^2\vec{p}}{(2\pi)\sqrt{2E}}\{a_+(p)u_+(p)e^{-ipx} + b_+^\dagger(p)v_+(p)e^{ipx}\}\ ; \tag{5.7}
$$

$$
\psi_-(x) = \int \frac{d^2\vec{p}}{(2\pi)\sqrt{2E}}\{a_-(p)u_-(p)e^{-ipx} + b_-^\dagger(p)v_-(p)e^{ipx}\}\ , \tag{5.8}
$$

where $\overline{\psi}_\pm = \psi_\pm^\dagger\gamma^0$. Consequently, taking into account (5.6) and (5.7)-(5.8), and by adopting $p^\mu = (E, p_x, p_y)$ where $E = \sqrt{p_x^2 + p_y^2}$, since $p^\mu p_\mu = 0$, the wave functions, $u_+$, $v_+$, $u_-$ and $v_-$, are given by:

$$
u_+(p) = \frac{p\!\!\!/}{\sqrt{E}}u'_+\ ,\quad u_-(p) = \frac{p\!\!\!/}{\sqrt{E}}u'_-\ ; \tag{5.9}
$$

$$
v_+(p) = \frac{-p\!\!\!/}{\sqrt{E}}v'_+\ ,\quad v_-(p) = \frac{-p\!\!\!/}{\sqrt{E}}v'_-\ , \tag{5.10}
$$

fulfilling the conditions below:

$$u_+^\dagger(p)u_+(p) = v_+^\dagger(p)v_+(p) = 2E \ ; \tag{5.11}$$
$$u_-^\dagger(p)u_-(p) = v_-^\dagger(p)v_-(p) = 2E \ ; \tag{5.12}$$
$$\overline{u}_\pm(p)u_\pm(p) = \overline{v}_\pm(p)v_\pm(p) = 0 \ , \tag{5.13}$$

where

$$u'_+ = u'_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v'_+ = v'_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{5.14}$$

From the microcausality conditions for the massless fermions $\psi_+$ and $\psi_-$:

$$\left\{\psi_\pm(x), \psi_\pm^\dagger(y)\right\}_{x^0=y^0} = \delta^2(\vec{x}-\vec{y}) \ , \tag{5.15}$$

together with the Dirac equations (5.6) and the normalization conditions (5.11)-(5.12), it stems that:

$$\left\{a_\pm(p), a_\pm^\dagger(k)\right\} = \delta^2(\vec{p}-\vec{k}) \ , \tag{5.16}$$
$$\left\{b_\pm(p), b_\pm^\dagger(k)\right\} = \delta^2(\vec{p}-\vec{k}) \ , \tag{5.17}$$

where all other anticommutators vanish and, for the vacuum state $|0\rangle$, $a_\pm(k)|0\rangle = b_\pm(k)|0\rangle = 0$.

### Charges

The quantum operators associated with the internal $U(1)_A \times U(1)_a$ symmetry, namely electric charge ($Q_\pm$) and Néel (chiral) charge ($q_\pm$) operators, are

$$Q_\pm = -e\int d^2\vec{x} : \psi_\pm^\dagger(x)\psi_\pm(x) : \tag{5.18}$$
$$Q_\pm = -e\int d^2\vec{p}\{a_\pm^\dagger(p)a_\pm(p) - b_\pm^\dagger(p)b_\pm(p)\} \ ,$$
$$q_\pm = \mp g\int d^2\vec{x} : \psi_\pm^\dagger(x)\psi_\pm(x) : \tag{5.19}$$
$$q_\pm = \mp g\int d^2\vec{p}\{a_\pm^\dagger(p)a_\pm(p) - b_\pm^\dagger(p)b_\pm(p)\} \ ,$$

respectively. Therefore, the electric charges and chiral charges of the asymptotic massless fermion (antifermion) states, $|f_\uparrow^-\rangle$ ($|f_\uparrow^+\rangle$) and $|f_\downarrow^-\rangle$ ($|f_\downarrow^+\rangle$) read:

$$Q_+|f_\uparrow^-\rangle = -e|f_\uparrow^-\rangle \ , \quad Q_+|f_\downarrow^+\rangle = +e|f_\downarrow^+\rangle \ ;$$
$$Q_-|f_\downarrow^-\rangle = -e|f_\downarrow^-\rangle \ , \quad Q_-|f_\uparrow^+\rangle = +e|f_\uparrow^+\rangle \ ; \tag{5.20}$$
$$q_+|f_\uparrow^-\rangle = -g|f_\uparrow^-\rangle \ , \quad q_+|f_\downarrow^+\rangle = +g|f_\downarrow^+\rangle \ ;$$
$$q_-|f_\downarrow^-\rangle = +g|f_\downarrow^-\rangle \ , \quad q_-|f_\uparrow^+\rangle = -g|f_\uparrow^+\rangle \ ; \tag{5.21}$$

where

$$|f_\uparrow^-\rangle = a_+^\dagger(k)|0\rangle \ , \qquad |f_\downarrow^+\rangle = b_+^\dagger(k)|0\rangle \ , \tag{5.22}$$
$$|f_\downarrow^-\rangle = a_-^\dagger(k)|0\rangle \ , \qquad |f_\uparrow^+\rangle = b_-^\dagger(k)|0\rangle \ , \tag{5.23}$$

meaning that the creation operators $a_+^\dagger$ and $a_-^\dagger$ create a fermion (electron-polaron) whereas the creation operators $b_+^\dagger$ and $b_-^\dagger$ creates an antifermion (hole-polaron). The electron-polaron and hole-polaron electric and chiral charges are displayed in TAB. 5.1.

### (Pseudo)spin

Whenever dealing with massless particles in three space-time dimensions, since there is no rest frame, spin cannot be straightforwardly defined, nonetheless, spin (in the generalized sense of a quantum number labelling the representation of the little group [17]) is still a fundamental quantum number. In the present case of massless fermions, it is verified that

$$\left[H_\pm^{(0)}, L + \frac{1}{2}\sigma_z\right] = 0 \ , \tag{5.24}$$

where $H_\pm^{(0)} = \vec{\alpha}\cdot\vec{p}$ (with $\vec{\alpha} = \gamma^0\vec{\gamma}$) is the free Hamiltonian operator associated to the massless spinors, $\psi_+$ and $\psi_-$, and $L = xp_y - yp_x$ is the angular momentum operator. Thus, accordingly to (5.24) it shall be concluded that $\frac{1}{2}\sigma_z$ is the (pseudo)spin operator.

### Landau levels

The issue of the (pseudo)spin can also be investigated by computing the quantum Landau levels of the model. Therefore, subjecting this pristine graphene-like system to high external and static magnetic field, $B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$ and $A^\mu = (0, \vec{A})$, somehow inducing within the bulk of the system a static magnetic-chiral field, $b = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}$ and $a^\mu = (0, \vec{a})$, the hamiltonians for the both massless spinors, $\psi_+$ and $\psi_-$, are respectively given by:

$$H_+ = \vec{\alpha} \cdot (\vec{p} - e\vec{A} - g\vec{a}) \; ; \tag{5.25}$$

$$H_- = \vec{\alpha} \cdot (\vec{p} - e\vec{A} + g\vec{a}) \; , \tag{5.26}$$

whose spectrum, the quantum Landau levels, reads as follows:

$$E_{n,+,s} = \pm\sqrt{2(eB + gb)}\sqrt{n + \frac{1}{2} - \underbrace{(\pm\frac{1}{2})}_{s}} \; , \tag{5.27}$$

$$E_{n,-,s} = \pm\sqrt{2(eB - gb)}\sqrt{n + \frac{1}{2} - \underbrace{(\pm\frac{1}{2})}_{s}} \; , \tag{5.28}$$

with $n$ being a non-negative integer number, $n \in \mathbb{N}$ $(n = 0, 1, 2, ...)$, where $E_{n,+,s}$ and $E_{n,-,s}$ are the Landau levels associated to $\psi_+$ (at sublattice **A**) and $\psi_-$ (at sublattice **B**) for electron-polarons (if $+$ sign in $E_{n,+,s}$ and $E_{n,-,s}$) or hole-polarons (if $-$ sign in $E_{n,+,s}$ and $E_{n,-,s}$) and $s = \pm\frac{1}{2}$ are the (pseudo)spin eigenvalues. The obtention of the spectrum for $g = 0$ is well known [16, 38–44, 111] and the calculation for the present case [131][2] is mathematically the same. The conceptual novelty is that the presence of the two types of fermions leads to two different cyclotron frequencies. This can be traced back to the difference in sign of the couplings of the two fermion flavors with the (pseudo)chiral field and possibly sheds light in the current debate [114–129] concerning the role played by spin and valley symmetries when graphene is submitted to a magnetic field. For example, it shall be stressed that the result displayed in (5.27)-(5.28) mimics the four-fold broken degeneracy effect of the Landau levels (see FIG. 5.1) experimentally observed in pristine graphene under high applied magnetic fields [114–129]. Also, it yields from the equations (5.27)-(5.28) that the lowest Landau level $(n = 0)$ appears at $E_{0,+,s} = E_{0,-,s} = 0$ and accommodates electron-polarons or hole-polarons with only one pseudospin eigenvalue, namely $s = +\frac{1}{2}$, signalizing a possible anomalous-type quantum Hall effect. All other levels $n \geq 1$ are occupied by electron-polarons or hole-polarons with both $(s = \pm\frac{1}{2})$ pseudospin eigenvalues. Therefore, this implies that for the lowest Landau level $n = 0$ the degeneracy is half of that for any other $n \geq 1$, likewise, all Landau levels $(n \geq 1)$ have the same degeneracy (a number of electron-polaron or hole-polaron states with a given energy) but the zero-energy $(n = 0)$ Landau level is shared equally by electron-polarons and hole-polarons, *i.e.* depending on the sign of the applied magnetic field there is only sublattice **A** or sublattice **B** states which contribute to the zero-energy (lowest) Landau level.

[2] In [131], the pristine graphene quantum electrodynamics model is introduced adopting the International System of Units, the quantum Landau levels computations are discussed in details and the results compared with the experimental data of [114–129].

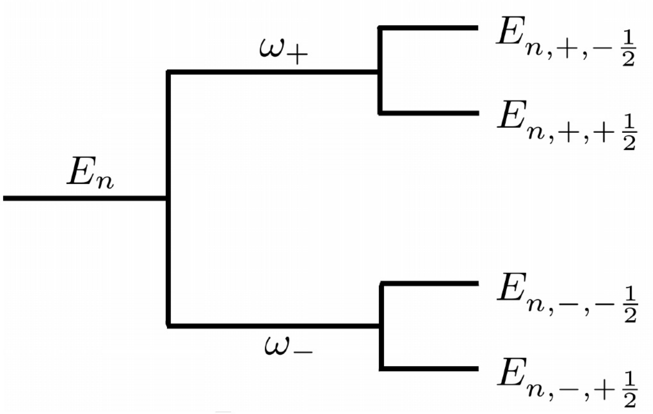


Figure 5.1: Four-fold Landau levels of electron-polarons and hole-polarons at sublattices **A** and **B** with cyclotron frequencies $\omega_+ = \sqrt{2(eB+gb)}$ and $\omega- = \sqrt{2(eB-gb)}$, respectively, provided $n \geq 1$.

An interesting issue comes to light, how the magnetic-chiral bulk-induced quantity $(gb)$ could be measured in terms of physical quantities like external applied magnetic field $(B)$, electron-polaron (hole-polaron) electric charge $(e)$, pseudospin quantum number $(s)$ and Landau levels state energies ($E_{n,+,s}$ and $E_{n,-,s}$). One way would be by measuring the energy gaps of sublattices **A** $(\psi_+)$ and **B** $(\psi_-)$ electron-polarons (or hole-polarons) with pseudospin eigenvalue $s = +\frac{1}{2}$, from the first excited Landau state $(n = 1)$ to zero-energy state $(n = 0)$, $\Delta E_{10,+,+\frac{1}{2}} = E_{1,+,+\frac{1}{2}} - E_{0,+,+\frac{1}{2}}$ and $\Delta E_{10,-,+\frac{1}{2}} = E_{1,-,+\frac{1}{2}} - E_{0,-,+\frac{1}{2}}$, respectively, then the cyclotron frequencies $\omega_+ = \sqrt{2(eB+gb)}$ and $\omega_- = \sqrt{2(eB-gb)}$ can be written as:

$$\omega_+ =\mid \Delta E_{10,+,+\frac{1}{2}} \mid \quad \text{and} \quad \omega_- =\mid \Delta E_{10,-,+\frac{1}{2}} \mid \ , \tag{5.29}$$

and the bulk-induced quantity $gb$ reads:

$$gb = \frac{(\Delta E_{10,+,+\frac{1}{2}})^2 - (\Delta E_{10,-,+\frac{1}{2}})^2}{4} \ . \tag{5.30}$$

In the other way around, the bulk-induced quantity $(gb)$ could be measured, for fixed Landau level quantum number $(n \geq 1)$ and pseudospin eigenvalue $(s)$, by means of the sublattice **A** energy $(E_{n,+,s})$ and the sublattice **B** energy $(E_{n,-,s})$, such that from (5.27)-(5.28), it yields that:

$$gb = \frac{(E_{n,+,s})^2 - (E_{n,-,s})^2}{4(n + \frac{1}{2} - s)} \ . \tag{5.31}$$

### Masses and degrees of freedom

For further computation on the electron-polaron–electron-polaron scattering amplitudes, the tree-level propagators in momenta space for all the fields have to be obtained and this can be achieved by switching off the coupling constants $e$ and $g$ in action (5.1). Thus, it can be verified that:

$$\Delta_{++}(k) = \Delta_{--}(k) = i\frac{\not{k}}{k^2} \; ; \tag{5.32}$$

$$\begin{aligned}
\Delta_{AA}^{\mu\nu}(k) &= -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\alpha}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} , \\
\Delta_{aa}^{\mu\nu}(k) &= -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\beta}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} , \\
\Delta_{Aa}^{\mu\nu}(k) &= \Delta_{aA}^{\mu\nu}(k) = \frac{\mu}{k^2(k^2-\mu^2)}\, \mathrm{e}^{\mu\rho\nu}\, k_\rho \; .
\end{aligned} \tag{5.33}$$

From the propagators, $\Delta_{++}$ and $\Delta_{--}$ (5.32), $\Delta_{AA}^{\mu\nu}$, $\Delta_{aa}^{\mu\nu}$ and $\Delta_{Aa}^{\mu\nu}$ (5.33), the tree-level unitarity of the model, so as its spectrum, shall be verified by coupling them to conserved external currents $\mathcal{J}_{\Phi_i} = (\mathcal{J}_+, \mathcal{J}_-, \mathcal{J}_A^\mu, \mathcal{J}_a^\mu)$ compatible with the symmetries of the model. Thereby, the current-current transition amplitudes in momentum space are written as $\mathcal{A}_{\Phi_i\Phi_j} = \mathcal{J}^*_{\Phi_i}(k)\langle\Phi_i(k)\Phi_j(k)\rangle\mathcal{J}_{\Phi_j}(k)$. Furthermore, picking up the imaginary part of the residues of the current-current amplitudes $(\mathcal{A}_{\Phi_i\Phi_j})$ at the poles, it can be verified the necessary conditions for unitarity of the $S$-matrix at the tree-level – positive imaginary part of the residues of the transition amplitudes, $\Im\mathrm{Res}\ \mathcal{A}_{\Phi_i\Phi_j} > 0$ at all the poles – besides the degrees of freedom counting of all the quantum fields presented in the model, $\Phi_i = (\psi_+, \psi_-, A_\mu, a_\mu)$. Briefly, it has been concluded [132] that the two spinors, $\psi_+$ and $\psi_-$, hold two degrees of freedom – the electron-polaron $|f_\uparrow^-\rangle$ $(u_+)$ and the hole-polaron $|f_\downarrow^+\rangle$ $(v_+)$ associated to the spinor $\psi_+$, and the electron-polaron $|f_\downarrow^-\rangle$ $(u_-)$ and the hole-polaron $|f_\uparrow^+\rangle$ $(v_-)$ associated to the spinor $\psi_-$. Moreover, the gauge fields, the electromagnetic field $(A_\mu)$ and the Néel field $(a_\mu)$, carry each one two massive degrees of freedom with mass $\mu$. Also, the single massless mode in $\Delta_{Aa}^{\mu\nu}$ (5.33) presented in the interaction sector does not propagate, it decouples. Summarizing all results presented above, it is concluded that the parity-even $U(1)_A \times U(1)_a$ massless QED$_3$, a pristine graphene-like planar quantum electrodynamics model, is free from tachyons and ghosts at the classical level. Notwithstanding, to guarantee the unitarity at tree-level, it is still necessary to investigate the behaviour of the scattering cross sections by testing the fulfillment of the Froissart-Martin bound [53, 61, 62] in the ultraviolet regime, as also in the infrared limit due to the presence of massless quantum fermions.

## 5.3 The Møller scattering: electron-polaron–electron-polaron

In order to calculate the scattering amplitudes, so as to obtain the scattering potentials, use has been made of the vertex Feynman rules associated to the interaction vertices

$-e\overline{\psi}_{\pm}\not{A}\psi_{\pm}$ and $\mp g\overline{\psi}_{\pm}\not{a}\psi_{\pm}$: $\Upsilon^{\mu}_{\pm\pm}$=$ie\gamma^{\mu}$ and $\upsilon^{\mu}_{\pm\pm}$=$\pm ig\gamma^{\mu}$, respectively.

Figure 5.2: $e^{-}$-polaron–$e^{-}$-polaron (Møller) $t$-channel scattering mediated by electromagnetic ($A_{\mu}$) and Néel ($a_{\mu}$) quantum fields.

The $t$-channel in $s$- and $p$-wave states of electron-polaron–electron-polaron scattering amplitudes owing to electromagnetic and Néel quanta exchange (see FIG. 5.2) are given by:

$$
\begin{aligned}
-i\mathcal{M}_{\pm A\mp} &= ovu_{\pm}(p_1')[\Upsilon^{\mu}_{\pm\pm}]u_{\pm}(p_1)\Delta^{AA}_{\mu\nu}(k)\overline{u}_{\mp}(p_2')[\Upsilon^{\nu}_{\mp\mp}]u_{\mp}(p_2)\ , && (5.34)\\
-i\mathcal{M}_{\pm a\mp} &= \overline{u}_{\pm}(p_1')[\upsilon^{\mu}_{\pm\pm}]u_{\pm}(p_1)\Delta^{aa}_{\mu\nu}(k)\overline{u}_{\mp}(p_2')[\upsilon^{\nu}_{\mp\mp}]u_{\mp}(p_2)\ , && (5.35)\\
-i\mathcal{M}_{\pm A\pm} &= \overline{u}_{\pm}(p_1')[\Upsilon^{\mu}_{\pm\pm}]u_{\pm}(p_1)\Delta^{AA}_{\mu\nu}(k)\overline{u}_{\pm}(p_2')[\Upsilon^{\nu}_{\pm\pm}]u_{\pm}(p_2)\ , && (5.36)\\
-i\mathcal{M}_{\pm a\pm} &= \overline{u}_{\pm}(p_1')[\upsilon^{\mu}_{\pm\pm}]u_{\pm}(p_1)\Delta^{aa}_{\mu\nu}(k)\overline{u}_{\pm}(p_2')[\upsilon^{\nu}_{\pm\pm}]u_{\pm}(p_2)\ , && (5.37)
\end{aligned}
$$

with $\mathcal{M}_{\pm A\mp}$ and $\mathcal{M}_{\pm a\mp}$ being the scattering amplitudes in $s$-wave state, whereas $\mathcal{M}_{\pm A\pm}$ and $\mathcal{M}_{\pm a\pm}$ being those in $p$-wave state.

The three-momenta configuration of the two scattered massless electron-polarons in the center of momenta (CM) reference frame; the incoming momenta, $p_1$ and $p_2$; the outgoing momenta, $p_1'$ and $p_2'$, as well as the momentum transfer, $k$, are defined below

$$
\begin{aligned}
&p_1 = (E, p, 0)\ ,\quad p_1' = (E, p\cos\varphi, p\sin\varphi)\ ; && (5.38)\\
&p_2 = (E, -p, 0)\ ,\quad p_2' = (E, -p\cos\varphi, -p\sin\varphi)\ ; && (5.39)\\
&k = p_1 - p_1' = (0, p(1-\cos\varphi), -p\sin\varphi) = (0, \vec{k})\ , && (5.40)
\end{aligned}
$$

where $\varphi$ is the CM scattering angle, defined as the angle among the directions in the CM frame of the two incoming (initial state) and outgoing (final state) massless electron-polarons.

Assuming the momenta configuration above (5.38)-(5.40) and taking into consideration

| state | wave function | electric charge | chiral charge | quasiparticle |
|---|---|---|---|---|
| $\lvert f^{-}_{\uparrow}\rangle$ | $u_{+}$ | $-e$ | $-g$ | electron-polaron |
| $\lvert f^{-}_{\downarrow}\rangle$ | $u_{-}$ | $-e$ | $+g$ | electron-polaron |
| $\lvert f^{+}_{\downarrow}\rangle$ | $v_{+}$ | $+e$ | $+g$ | hole-polaron |
| $\lvert f^{+}_{\uparrow}\rangle$ | $v_{-}$ | $+e$ | $-g$ | hole-polaron |

Table 5.1: The electron-polaron and hole-polaron electric and chiral charges.

the conditions (5.9)-(5.14) on the wave functions $u_+$ and $u_-$, the total $s$- and $p$-wave Møller scattering amplitudes can be obtained from the partial ones (5.34)-(5.37), $\mathcal{M}_{\pm\mp}$ ($|\uparrow\rangle+|\downarrow\rangle \rightarrow |\uparrow\rangle+|\downarrow\rangle$) and $\mathcal{M}_{\pm\pm}$ ($|\uparrow\rangle+|\uparrow\rangle \rightarrow |\uparrow\rangle+|\uparrow\rangle$ or $|\downarrow\rangle+|\downarrow\rangle \rightarrow |\downarrow\rangle+|\downarrow\rangle$), then it follows that:

$$\mathcal{M}_{\pm\mp} = -(e^2 - g^2)\left[\left(\frac{s-u}{t-\mu^2}\right) - (u \leftrightarrow t)\right] \; ; \tag{5.41}$$

$$\mathcal{M}_{\pm\pm} = (e^2 + g^2)e^{\pm i\varphi}\left[\left(\frac{s-u}{t-\mu^2}\right) + (u \leftrightarrow t)\right] \; , \tag{5.42}$$

where, $s$, $t$ and $u$ are the Lorentz invariant Mandelstam variables evaluated at the CM frame:

$$\begin{aligned} s &= (p_1 + p_2)^2 = 4E^2 \; , \\ t &= (p_1 - p_1')^2 = -2p^2(1 - \cos\varphi) = -4p^2 \sin^2\left(\frac{\varphi}{2}\right) \; , \\ u &= (p_1 - p_2')^2 = -2p^2(1 + \cos\varphi) = -4p^2 \cos^2\left(\frac{\varphi}{2}\right) \; . \end{aligned}$$

### 5.3.1 The scattering potentials

The two-particle interaction potential for two distinguishable electron-polarons (fermions), 1 and 2, in the tree approximation [133–137] at the CM frame, reads:

$$V(\vec{r}) = \int \frac{d^2\vec{k}}{(2\pi)^2}\, e^{i\vec{k}\cdot\vec{r}} \beta_1\beta_2 F(\vec{k}) \; , \tag{5.43}$$

with the product $\beta_1\beta_2$ being a spinorial factor in the space of the electron-polarons 1 and 2, where $\beta_1 = \gamma_1^0$ and $\beta_2 = \gamma_2^0$. Also, in addition to that, taking into account only the $t$-channel part of the electron-polaron–electron-polaron total scattering amplitude ($\mathcal{M}$) evaluated at the CM frame, it follows that

$$\mathcal{M} = \overline{u}_1(p_1')\overline{u}_2(p_2')F(k)u_1(p_1)u_2(p_2) \; . \tag{5.44}$$

Moreover, in accordance to (5.34)-(5.37), (5.43) and (5.44), the scattering potentials among two electron-polarons, firstly, in $s$-wave state – one situated at **K** point and the other at **K′** point in the Brillouin zone – and secondly, in $p$-wave state – the both located at either **K** point or **K′** point in the Brillouin zone – mediated by photon and Néel quasiparticles, can be respectively written as:

$$V_s(r) = (1 - \vec{\alpha}_1 \cdot \vec{\alpha}_2)\frac{(e^2 - g^2)}{2\pi}K_0(\mu r) \; , \tag{5.45}$$

$$V_p(r) = (1 - \vec{\alpha}_1 \cdot \vec{\alpha}_2)\frac{(e^2 + g^2)}{2\pi}K_0(\mu r) \; , \tag{5.46}$$

where $\vec{\alpha} = \gamma^0\vec{\gamma}$. Thereafter, from (5.46) it is concluded that, regardless the values of the electromagnetic ($e$) and the chiral coupling ($g$) constants, the electron-polaron–electron-polaron interaction in $p$-wave state ($|\mathbf{K}\rangle + |\mathbf{K}\rangle$ or $|\mathbf{K}'\rangle + |\mathbf{K}'\rangle$) is invariably repulsive. Nevertheless, drawing attention to (5.45) it takes in evidence about the possibility of attractive electron-polaron–electron-polaron interaction in $s$-wave state ($|\mathbf{K}\rangle + |\mathbf{K}'\rangle$) provided $g^2 > e^2$. Notwithstanding, in spite of attractive potential be a necessary condition, although not a sufficient one, for the existence of $s$-wave ($\mathbf{K}$-$\mathbf{K}'$) massless bipolarons bound states, it still remains to verify relativistic conditions similar as the ones whose were already verified for the non relativistic massive case [22] – where the non relativistic $s$-wave attractive scattering potential [64] was proved to satisfy the Kato condition [55], the Newton-Setô and the Bargmann upper bounds [56–58, 138] – which, in turn, guarantees that $s$-wave ($\mathbf{K}$-$\mathbf{K}'$) massive bipolarons bound states exist.

It shall be also mentioned that the presence of Breit-Darwin-type term $\vec{\alpha}_1 \cdot \vec{\alpha}_2$ in (5.45) and (5.46), where in the non-zero gap (mass-gap) graphene-type [22] is of order $\frac{v_1 v_2}{c^2}$ (in real graphene units $\frac{v_1 v_2}{v_F^2}$) and can be therefore neglected at low energies (low speeds), here in the pristine (gapless) graphene-type case, wherein the electron-polarons and hole-polarons are massless, cannot. In summary, it has yet to be investigated in relativistic regime if the attractive ($g^2 > e^2$) $s$-wave state potential (5.45) favours massless bipolarons, namely, electron-polaron–electron-polaron or hole-polaron–hole-polaron $s$-wave bound states. Furthermore, in order to avoid the continuum dissolution problem [133–137], the Dirac equation for the wave function ($\Psi$) associated to a two-particle-interaction system, $H_{\mathrm{D}}\Psi = E\Psi$, has to be rewritten as:

$$H_{\mathrm{D}}\Psi = H_{\mathrm{D}1}\Psi + H_{\mathrm{D}2}\Psi + V_{++}\Psi = E\Psi \ , \tag{5.47}$$

where $V_{++} = \Lambda_{++} V \Lambda_{++}$, with $\Lambda_{++} = \Lambda(1)\Lambda(2)$ being the product of Casimir-type positive energy projection operators $\Lambda(i) = \frac{1}{2}\left(\mathbb{I} + \frac{H_{\mathrm{D}i}}{E}\right)$ with $H_{\mathrm{D}i} = \vec{\alpha}_i \cdot \vec{p}_i$ ($i = 1, 2$). Besides that, bearing in mind that at the CM frame $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$, thus

$$\vec{\alpha}_1 \cdot \vec{p}\Psi - \vec{\alpha}_2 \cdot \vec{p}\Psi + \Lambda_{++} V \Lambda_{++}\Psi = E\Psi \ . \tag{5.48}$$

Recently, the question if wheter or not, whenever $g^2 > e^2$, the attractive $s$-wave state potential (5.45) favours $s$-wave masssless bipolarons (two-fermion bound states) has been answeredby the authors of [139], by investigating into details the Dirac equation (5.48). Analyzing the stability of the matter, they have obtained the value of the critical constant of the model. As a by product, they have shown that there is no bound state in the subcritical region.

## 5.4 Conclusions

The model proposed, a gapless pristine graphene-like planar quantum electrodynamics model, the parity-preserving $U(1) \times U(1)$ massless QED$_3$, exhibits two-fermion scattering short range non confining potentials originated by two massive vector-mediated quanta, the photon (electric charge) and the Néel (chiral charge) quasiparticle, both stemming from the gauging of the $U(1) \times U(1)$ global symmetry. At the tree-level, the absence in the spectrum of tachyons ($k^2 < 0$) and ghosts ($\langle\psi|\psi\rangle < 0$) assures respectively, causality and unitarity, at this level. Additionally, the charges of the quasiparticles (electron-polaron, hole-polaron, photon and Néel quasiparticles), their masses, degrees of freedom and (pseudo)spin are determined and discussed. As a by-product, it is obtained the four-fold broken degeneracy of the Landau levels, reminding those experimentally observed in pristine graphene subjected to high external magnetic fields [114–129], moreover, the system presents zero-energy Landau level suggesting a kind of anomalous quantum Hall effect – detailing results and discussions shall appear further [131].

The $p$-wave state fermion–fermion (or antifermion–antifermion) scattering potential shows to be repulsive (5.46) whatever the values of the electric ($e$) and chiral ($g$) charges. Nevertheless, for $s$-wave scattering of fermion–fermion (or antifermion–antifermion), the interaction potential (5.45) might be attractive provided $g^2 > e^2$. In summary, if two electron-polarons (or hole-polarons) lie in the inequivalent $\mathbf{K}$ and $\mathbf{K}'$ points in the Brillouin zone the interaction might be attractive, otherwise the interaction is always repulsive if those two electron-polarons (or hole-polarons) rest both either in $\mathbf{K}$ or in $\mathbf{K}'$ points.

In view of possible applications of this quantum electrodynamics three space-time dimensional model to pristine (gapless) graphene, or any other planar system, the orders of magnitude of some theoretical parameters have to be estimated. The typical energy-scale in graphene – for instance $E = v_F|\vec{p}|$, $C_s = \frac{1}{2\pi}(e^2 - g^2)$ or $C_p = \frac{1}{2\pi}(e^2 + g^2)$ – is around meV [16, 38–44, 111], while the length-scale interaction $\lambdabar = \frac{2\pi\hbar}{\mu c}$ – the reduced Compton wavelength of the quantum-mediated photon and Néel massive quasiparticles – is orders of magnitude in nm [65].

To end this conclusions, it is in progress the proof, analogously to the relativistic massive case [140], if whether the attractive $s$-wave scattering potential can lead to bound states, that is, if the potential (5.45), provided $g^2 > e^2$, could favour $s$-wave massless bipolarons. The possible emergence of such a kind of Cooper-type electron-polaron–electron-polaron (hole-polaron–hole-polaron) condensate draws attention to superconductivity in graphene [66–68].

# Chapter 6

# Quantum parity conservation in planar quantum electrodynamics

## 6.1 Introduction

[1]The quantum electrodynamics in three space-time dimensions ($\mathrm{QED}_3$) [2, 23–25, 110] has been considered as a potential theoretical framework for some condensed matter phenomena, namely high-temperature superconductivity [26–28], quantum Hall effect [29], graphene [16, 22, 38–40, 42, 44, 141, 142], topological insulators [32–34] and topological superconductors [35–37]. Some interesting properties may arise in massless, mixed or massive $\mathrm{QED}_3$, as parity violation, anyons, topological gauge fields, superrenormalizability and the appearance of infrared divergences. The ordinary massless $U(1)$ $\mathrm{QED}_3$ is infrared and ultraviolet perturbatively finite, parity and infrared anomaly free at all orders [78], however at 1-loop parity is explicitly broken in the course of Lowenstein-Zimmermann (LZ) infrared subtractions in the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization scheme [77], signalizing that the 1-loop radiatively induced parity-odd Chern-Simons term to the vacuum-polarization tensor is nothing but a counterterm owing to parity-violating LZ infrared subtractions in the BPHZL program[2]. In the meantime, a fundamental question arises if regardless of model the LZ subtraction scheme necessarily violates parity in three space-time dimensions, more specifically, if whether or not parity is broken at any order throughout the infrared subtraction in the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization procedure. Accordingly, the latter issue is dealt in this work by considering a massless parity-even $U(1) \times U(1)$ Maxwell-Chern-Simons $\mathrm{QED}_3$ model [109], with two massless fermions, $\psi_+$ and $\psi_-$, where the gauge mediating bosons, $A_\mu$ (electromagnetic field) and $a_\mu$ (pseudochiral field), associated to the both $U(1)$ symmetries, are massive through a mixed Chern-Simons term.

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[1]The content was published in International Journal of Theoretical Physics [130]

[2]In perturbation theory, the proof on the absence of a parity anomaly in massless $U(1)$ $\mathrm{QED}_3$ has also been performed by the Epstein-Glaser renormalization method [82].

The proof presented in this chapter is organized as follows. In Section 6.2 the action of the model is introduced and some useful gamma matrices relations are established. Moreover, in Subsections 6.2.1 and 6.2.2, the continuous and discrete classical symmetries, gauge and parity, the propagators and the interactions Feynman rules are presented, the ultraviolet and infrared power countings are fixed for the model, the 1-loop Feynman graphs are identified and the BPHZL subtraction operator defined. In Subsections 6.2.3 and 6.2.4, the 1-loop vacuum-polarization tensor and self-energy graphs are presented, and among those ones, the divergents are renormalized. The 2-loop graphs and their BPHZL analyses are left to Section 6.3.

## 6.2 BPHZL: 1-loop

In this section the Bogoliubov-Parasiuk-Hepp-Zimmermann momentum space subtraction scheme (BPHZ) [94, 143], which does not use any regularization procedure, is applied to 1-loop vacuum-polarization tensor and self-energy divergent graphs. However, due to the presence of massless fermions, $\psi_+$ and $\psi_-$, the momentum subtraction scheme modified by Lowenstein-Zimmermann (BPHZL) [95, 96] has to be adopted in order to deal with the infrared (IR) divergences that shall arise in the process of ultraviolet (UV) subtractions.

The action for the massless parity-even $U(1) \times U(1)$ Maxwell-Chern-Simons QED$_3$ model [109][3], with the parity and gauge invariant Lowenstein-Zimmermann mass term added, is given by:

$$\Sigma^{(s-1)} = \int d^3x \Big\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\alpha\nu}A_\mu\partial_\alpha a_\nu + i\overline{\psi}_+ \not{D}\psi_+ + i\overline{\psi}_- \not{D}\psi_- + \underbrace{-m(s-1)\overline{\psi}_+\psi_+ + m(s-1)\overline{\psi}_-\psi_-}_{\text{Lowenstein-Zimmermann mass term}} + b\partial^\mu A_\mu + \frac{\alpha}{2}b^2 + \overline{c}\Box c + \pi\partial^\mu a_\mu + \frac{\beta}{2}\pi^2 + \overline{\xi}\Box\xi \Big\} \tag{6.1}$$

where $\not{D}\psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm$, $m$ and $\mu$ are mass parameters with mass dimension 1 and the coupling constants $e$ (electric charge) and $g$ (pseudochiral charge) are dimensionful with mass dimension $\frac{1}{2}$. The field strengths, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, are related to the electromagnetic field ($A_\mu$) and the pseudochiral gauge field ($a_\mu$), respectively. The Dirac spinors $\psi_+$ and $\psi_-$ are two kinds of fermions where the $\pm$ subscripts are related to their pseudospin sign [17, 109]. Also, the fields $c$ and $\xi$ are two kind of ghosts[4] and, $\overline{c}$ and $\overline{\xi}$, the two antighosts, whereas $b$ and $\pi$ are the Lautrup-Nakanishi fields [97–99]

[3]A quantum electrodynamics model describing electron-polaron–electron-polaron scattering and four-fold broken degeneracy of the Landau levels in pristine graphene.

[4]It is appropriated to stress that neither the ghosts ($c$ and $\xi$) nor the antighosts ($\overline{c}$ and $\overline{\xi}$) take part of vacuum-polarization tensor, self energy or vertex function Feynman diagrams at any perturbative order, since they are free quantum fields, thus they decouple.

playing the role of Lagrange multiplier fields for the gauge conditions. The adopted gamma matrices are $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$. Finally, the Lowenstein-Zimmermann parameter $s$ lies in the interval $0 \leq s \leq 1$ and has the same status of an additional subtraction variable (as the external momentum) in the BPHZL renormalization scheme, in such a way that the massless model [109] is recovered by taking $s = 1$ at the end of calculations. Furthermore, some conventions and useful relations that shall be used in subsequent calculations follow:

$$\begin{aligned}
&\eta^{\mu\nu} = \mathrm{diag}(+--) \ , \quad \gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbb{I} + i\varepsilon^{\mu\nu\alpha}\gamma_\alpha \ , \\
&\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 2\eta^{\mu\nu} \ , \quad \mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\alpha\} = 2i\varepsilon^{\mu\nu\alpha} \ , \\
&\mathrm{Tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_n}\} = \eta^{\mu_{n-1}\mu_n}\mathrm{Tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_{n-2}}\} + i\varepsilon^{\mu_{n-1}\mu_n\alpha}\mathrm{Tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_{n-2}}\gamma_\alpha\} \ . 
\end{aligned} \tag{6.2}$$

It should be pointed out that the trace (Tr) of product of an even number of gamma matrices does not exhibit the Levi-Civita symbol, on the other hand, the trace of product of an odd number (greater than one) of gamma matrices does.

### 6.2.1 Classical symmetries: BRS and parity

The action $\Sigma^{(s-1)}$ (6.1) is invariant under the Becchi-Rouet-Stora (BRS) transformations [91–93, 144]:

$$\begin{aligned}
s\psi_+ &= i(c+\xi)\psi_+ \ , \quad s\overline{\psi}_+ = -i(c+\xi)\overline{\psi}_+ \ ; \\
s\psi_- &= i(c-\xi)\psi_- \ , \quad s\overline{\psi}_- = -i(c-\xi)\overline{\psi}_- \ ; \\
sA_\mu &= -\frac{1}{e}\partial_\mu c \ , \quad sc = 0 \ ; \quad sa_\mu = -\frac{1}{g}\partial_\mu \xi \ , \quad s\xi = 0 \ ; \\
s\overline{c} &= \frac{b}{e} \ , \quad sb = 0 \ ; \quad s\overline{\xi} = \frac{\pi}{g} \ , \quad s\pi = 0 \ ;
\end{aligned} \tag{6.3}$$

as well as under the parity transformations:

$$\begin{aligned}
&\psi_+ \xrightarrow{P} \psi_+^P = -i\gamma^1\psi_- \ , \quad \psi_- \xrightarrow{P} \psi_-^P = -i\gamma^1\psi_+ \ , \\
&\overline{\psi}_+ \xrightarrow{P} \overline{\psi}_+^P = i\overline{\psi}_-\gamma^1 \ , \quad \overline{\psi}_- \xrightarrow{P} \overline{\psi}_-^P = i\overline{\psi}_+\gamma^1 \ ; \\
&A_\mu \xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ ; \quad \phi \xrightarrow{P} \phi^P = \phi \ , \quad \phi = \{b, c, \overline{c}\} \ ; \\
&a_\mu \xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ ; \quad \chi \xrightarrow{P} \chi^P = -\chi \ , \quad \chi = \{\pi, \xi, \overline{\xi}\} \ .
\end{aligned} \tag{6.4}$$

The tree-level propagators are obtained by taking the free part of the action $\Sigma^{(s-1)}$ (6.1), *i.e.*, by switching off the coupling constants $e$ and $g$, thence the propagators in momenta space read:

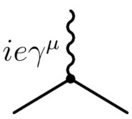
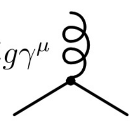
$$
\begin{aligned}
\Delta^{\mu\nu}_{AA}(k) &= -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+\frac{\alpha}{k^2}\frac{k^\mu k^\nu}{k^2}\right\}\,, \\
\Delta^{\mu\nu}_{aa}(k) &= -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+\frac{\beta}{k^2}\frac{k^\mu k^\nu}{k^2}\right\}\,, \\
\Delta_{++}(k) &= i\frac{\not{k}-m(s-1)}{k^2-m^2(s-1)^2}\,,\ \ \Delta_{--}(k) = i\frac{\not{k}+m(s-1)}{k^2-m^2(s-1)^2}\,, \\
\Delta^{\mu\nu}_{Aa}(k) &= \frac{\mu}{k^2(k^2-\mu^2)}\,\mathrm{e}^{\mu\alpha\nu}\,k_\alpha\,,\ \ \Delta^{\mu}_{Ab}(k)=\Delta^{\mu}_{a\pi}(k)=\frac{k^\mu}{k^2}\,, \\
\Delta_{bb}(k) &= \Delta_{\pi\pi}(k)=0\,,\ \ \Delta_{\overline{c}c}(k)=\Delta_{\overline{\xi}\xi}(k)=-\frac{i}{k^2}\,,
\end{aligned}
\tag{6.5}
$$

Notice that from this point forward, all 1- and 2-loops Feynman graphs calculations will be performed in the Landau gauge, $\alpha=\beta=0$.

The graphical conventions for the propagators are assumed as below:

$\Delta^{\mu\nu}_{AA} \equiv$  , $\Delta^{\mu\nu}_{aa} \equiv$  , $\Delta^{\mu\nu}_{Aa} \equiv$  , $\Delta_{\pm\pm} \equiv$ , (6.6)

and the Feynman rules for the interaction vertices are given by:

$V_{\pm A^\mu \pm} \equiv$ $ie\gamma^\mu$  , $V_{\pm a^\mu \pm} \equiv$ $\pm ig\gamma^\mu$  . (6.7)

### 6.2.2 The BPHZL scheme: power counting, subtraction operator, vacuum-polarization and self-energy

For the purpose of renormalizing the ultraviolet (UV) and infrared (IR) divergences of all divergent graphs, UV and IR subtraction degrees have to be fixed, to do so the UV and IR dimensions of all the fields shall be determined firstly. For any propagator $\Delta_{XY}(k,s)$, the UV ($d$) and IR ($r$) dimensions of the fields, $X$ and $Y$, are defined by means of the asymptotical UV and IR behaviour of the propagator, $d_{XY}$ (for $k,s\to\infty$) and $r_{XY}$ (for $k,(s-1)\to 0$), respectively, furthermore the following inequalities hold [94, 143]:

$$
d_X + d_Y \geq 3 + d_{XY} \quad \text{and} \quad r_X + r_Y \leq 3 + r_{XY}\,, \tag{6.8}
$$

where, in the Landau gauge, $\alpha=\beta=0$, the UV ($d$) and IR ($r$) dimensions of all the fields are summarized in the Table 6.1. Thus, by taking into account all previous results, the UV ($d(\gamma)$) and IR ($r(\gamma)$) superficial degrees of divergence of a 1-particle irreducible Feynman

| | $\psi_+$ | $\psi_-$ | $A_\mu$ | $a_\mu$ | $b$ | $\pi$ | $c$ | $\bar{c}$ | $\xi$ | $\bar{\xi}$ | $s$ | $s-1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | 0 | 1 | 0 | 1 | 1 | 1 |
| $r$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

Table 6.1: UV ($d$) and IR ($r$) dimensions of the fields.

diagram $\gamma$ stems:

$$\begin{pmatrix} d(\gamma) \\ r(\gamma) \end{pmatrix} = 3 - \sum_f \begin{pmatrix} d_f \\ r_f \end{pmatrix} N_f - \sum_b \begin{pmatrix} d_b \\ \frac{3}{2} r_b \end{pmatrix} N_b + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_e + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_g - N_{Aa} \ , \quad (6.9)$$

where $N_f$ and $N_b$ are the numbers of external lines of fermions and bosons, respectively, whereas $N_{Aa}$ is the number of internal lines associated to the mixed propagator $\Delta_{Aa}$. Also, $N_e$ and $N_g$ are the powers of the coupling constants, $e$ and $g$, in the integral corresponding to the graph $\gamma$.

The 1-loop vacuum-polarization tensors, self energies and vertex functions diagrams are identified in Fig. 6.1, whereas their respectives UV and IR superficial degrees of divergence are displayed in Table 6.2. At this time, it should be mentioned that for any graph $\gamma_{i\pm}$ the subscript $\pm$ refers to external legs or internal lines of either $\psi_+$ or $\psi_-$.

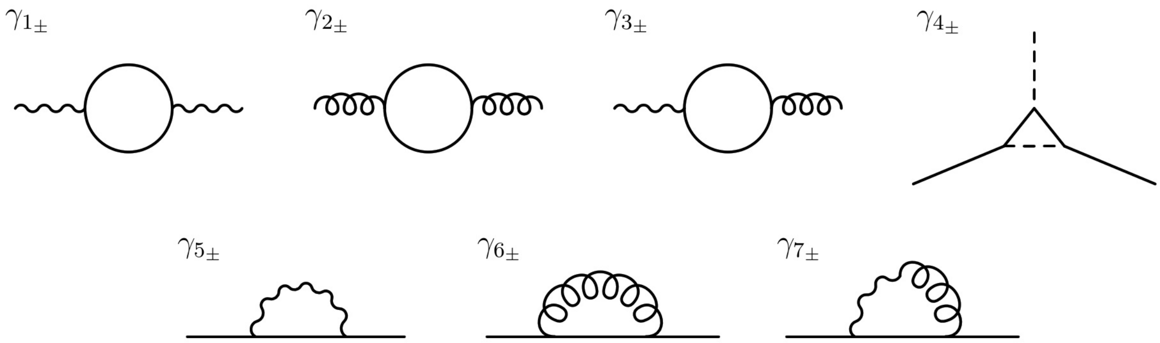



Figure 6.1: The 1-loop diagrams $\gamma_{1\pm}$, $\gamma_{2\pm}$ and $\gamma_{3\pm}$ are the vacuum-polarization tensors, $\gamma_{4\pm}$ is the vertex functions and $\gamma_{5\pm}$, $\gamma_{6\pm}$ and $\gamma_{7\pm}$ are the self-energies. The continuous line represents external legs or propagators of either $\psi_+$ or $\psi_-$, whereas the dashed lines in $\gamma_{4\pm}$ denote the propagator of $A_\mu$, $a_\mu$ or the mixed one, and the external leg of either $A_\mu$ or $a_\mu$.

| | $\gamma_{1\pm}$ | $\gamma_{2\pm}$ | $\gamma_{3\pm}$ | $\gamma_{4\pm}^{(a)}$ | $\gamma_{4\pm}^{(b)}$ | $\gamma_{5\pm}$ | $\gamma_{6\pm}$ | $\gamma_{7\pm}$ |
|---|---|---|---|---|---|---|---|---|
| $d$ | 1 | 1 | 1 | $-1$ | $-2$ | 0 | 0 | $-1$ |
| $r$ | 1 | 1 | 1 | 1 | 0 | 2 | 2 | 1 |

Table 6.2: UV ($d$) and the IR ($r$) superficial degrees of divergence of a 1-particle irreducible Feynman diagrams of Fig. 6.1, where for the vertex functions $\gamma_{4\pm}^{(a)}$ the dashed internal line represents either $\Delta_{AA}$ or $\Delta_{aa}$ propagators, while for $\gamma_{4\pm}^{(b)}$ it symbolizes the mixed propagator $\Delta_{Aa}$.

Accordingly to the power counting theorem, in view of the fact that the 1-loop diagrams $\gamma_{1\pm}$, $\gamma_{2\pm}$, $\gamma_{3\pm}$, $\gamma_{5\pm}$ and $\gamma_{6\pm}$ (see Fig. 6.1 and Table 6.2) are superficially UV divergent, they have to be UV and IR subtracted. Whenever a graph $\gamma$ is possibly UV divergent, *i.e.* $d(\gamma) \geq 0$, the BPHZL renormalization method is followed so as to make the graph convergent [95,96] by also subtracting the IR divergences induced by the UV subtractions. The BPHZL subtraction program consists of performing UV and IR subtraction operations upon a UV divergent Feynman graph integrand, $I_\gamma(p,k,s)$:

$$R_\gamma(p,k,s) = \left(1 - t^{\rho(\gamma)-1}_{p,s-1}\right)\left(1 - t^{\delta(\gamma)}_{p,s}\right) I_\gamma(p,k,s)\ , \tag{6.10}$$

where $R_\gamma(p,k,s)$ is the renormalized integrand, which is UV convergent. Moreover, $\delta(\gamma)$ and $\rho(\gamma)$ are the UV and IR degrees of subtraction, respectively, given by [95,96]:

$$\delta(\gamma) = d(\gamma) + b(\gamma) \quad \text{and} \quad \rho(\gamma) = r(\gamma) - c(\gamma)\ , \tag{6.11}$$

where at 1-loop $b(\gamma)$ and $c(\gamma)$ are non-negative integers constrained as follows:

$$\rho(\gamma) \leq \delta(\gamma) + 1\ , \tag{6.12}$$

with $t^\tau_{x,y}$ being the Taylor expansion operator about $x = y = 0$ to order $\tau$, provided $\tau \geq 0$.

### 6.2.3 The vacuum-polarization tensor

The BPHZL renormalization procedures of all 1-particle irreducible vacuum-polarization tensor divergent diagrams (see Fig. 6.1 and Table 6.2) are rather similar, since they possess the same loop structure, their integrands are equal up to coupling constants dependent factors, $\pm e^2$, $\pm g^2$ and $\mp eg$, corresponding to the 1-loop graphs, $\gamma_{1\pm}$, $\gamma_{2\pm}$ and $\gamma_{3\pm}$, respectively. Initially, the analysis is carried out for the $\gamma_{1\pm}$ Feynman graphs, where the 1-loop vacuum-polarization tensor, $\Pi^{\mu\nu}_{\gamma_{1\pm}}(p,s)$, reads

$$\Pi^{\mu\nu}_{\gamma_{1\pm}}(p,s) = \int \frac{d^3k}{(2\pi)^3} \underbrace{\left\{-e^2 \mathrm{Tr}\left[\gamma^\mu \frac{\not{k} \mp m(s-1)}{k^2 - m^2(s-1)^2}\gamma^\nu \frac{\not{k} - \not{p} \mp m(s-1)}{(k-p)^2 - m^2(s-1)^2}\right]\right\}}_{I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)}\ . \tag{6.13}$$

Bearing in mind the conditions (6.11) and the inequality (6.12), by taking $b(\gamma_{1\pm}) = c(\gamma_{1\pm}) = 0$, the UV and IR subtraction degrees are such that $\delta(\gamma_{1\pm}) = \rho(\gamma_{1\pm}) = 1$. Consequently, the 1-loop BPHZL subtracted (renormalized) integrand, $R^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)$, is written in terms of the unsubtracted one, $I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)$, in the following way:

$$\begin{aligned} R^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) &= (1 - t^0_{p,s-1})(1 - t^1_{p,s}) I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) \\ &= (1 - t^0_{p,s-1} - t^1_{p,s} + t^0_{p,s-1} t^1_{p,s}) I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)\ . \end{aligned} \tag{6.14}$$

However, as previously mentioned by setting $s = 1$ at the end of all Taylor expansion operations, to retrieve the massless condition, the subtracted integrand, $R^{\mu\nu}_{\gamma_{1\pm}}(p,k,1)$, results:

$$R^{\mu\nu}_{\gamma_{1\pm}}(p,k,1) = \underbrace{I^{\mu\nu}_{\gamma_{1\pm}}(p,k,1)}_{\text{parity-even}} - \underbrace{I^{\mu\nu}_{\gamma_{1\pm}}(0,k,1)}_{\text{parity-even}} - \underbrace{p^\rho \frac{\partial}{\partial p^\rho} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)}_{\text{parity-odd}}\Bigg|_{p=s=0} , \tag{6.15}$$

where

$$\begin{aligned} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,1) &= -e^2 \mathrm{Tr}\left\{\gamma^\mu \frac{\not k}{k^2}\gamma^\nu \frac{\not k - \not p}{(k-p)^2}\right\} , \\ I^{\mu\nu}_{\gamma_{1\pm}}(0,k,1) &= -e^2 \mathrm{Tr}\left\{\gamma^\mu \frac{\not k}{k^2}\gamma^\nu \frac{\not k}{k^2}\right\} , \\ p^\rho \frac{\partial}{\partial p^\rho} I^{\mu\nu}_{\gamma_{1\pm}}(p,k,s)\Bigg|_{p=s=0} &= -e^2 \mathrm{Tr}\left\{\gamma^\mu \frac{\not k \mp m}{k^2-m^2}\gamma^\nu \left[-\frac{\not p}{k^2-m^2} + 2p\cdot k \frac{\not k \mp m}{(k^2-m^2)^2}\right]\right\} \end{aligned} \tag{6.16}$$

In addition to that, since the renormalized vacuum-polarization tensor, $\Pi^{(R)\mu\nu}_{\gamma_{1\pm}}(p,s)$, is defined by

$$\Pi^{(R)\mu\nu}_{\gamma_{1\pm}}(p,s) = \int \frac{d^3k}{(2\pi)^3}\, R^{\mu\nu}_{\gamma_{1\pm}}(p,k,s) , \tag{6.17}$$

and recalling to the fact that the issue here is to verify if Levi-Civita symbol $\epsilon^{\mu\nu\rho}$ dependent terms might be induced by UV and IR subtractions, only parity-odd pieces of the subtracted integrand, $R^{\mu\nu}_{\text{odd}\gamma_{1\pm}}(p,k,1)$ (6.15), shall be taken into account, then from the Eqs.(6.15)–(6.16), leads to

$$\Pi^{(R)\mu\nu}_{\text{odd}\gamma_{1\pm}} = \pm\frac{e^2 m}{4\pi|m|}\epsilon^{\mu\alpha\nu}p_\alpha , \tag{6.18}$$

with $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_{1\pm}} \equiv \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{1\pm}}(p,1)$.

Analogously to the previous case, $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_{1\pm}}$, the renormalized parity-odd vacuum-polarization tensors $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_{2\pm}}$ and $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_{3\pm}}$, corresponding to $\gamma_{2\pm}$ and $\gamma_{3\pm}$ diagrams, are respectively given by

$$\Pi^{(R)\mu\nu}_{\text{odd}\gamma_{2\pm}} = \pm\frac{g^2 m}{4\pi|m|}\epsilon^{\mu\alpha\nu}p_\alpha \quad \text{and} \quad \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{3\pm}} = \mp\frac{egm}{4\pi|m|}\epsilon^{\mu\alpha\nu}p_\alpha . \tag{6.19}$$

Finally, the 1-loop renormalized parity-odd vacuum polarization tensors, $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_1}$, $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_2}$ and $\Pi^{(R)\mu\nu}_{\text{odd}\gamma_3}$ :

$$\begin{aligned} \Pi^{(R)\mu\nu}_{\text{odd}\gamma_1} &= \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{1+}} + \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{1-}} \equiv 0 , \\ \Pi^{(R)\mu\nu}_{\text{odd}\gamma_2} &= \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{2+}} + \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{2-}} \equiv 0 , \\ \Pi^{(R)\mu\nu}_{\text{odd}\gamma_3} &= \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{3+}} + \Pi^{(R)\mu\nu}_{\text{odd}\gamma_{3-}} \equiv 0 , \end{aligned} \tag{6.20}$$

vanishes identically. In conclusion, besides there is no 1-loop counterterm for the mixed Chern-Simons term, $\varepsilon^{\mu\alpha\nu}A_\mu\partial_\alpha a_\nu$ – which sets out that the 1-loop $\beta$-function associated to the Chern-Simons mass parameter ($\mu$) vanishes – the BPHZL subtraction scheme applied to the 1-loop vacuum-polarization tensor preserves parity, being the opposite to what takes place in ordinary massless $U(1)$ QED$_3$ [77].

### 6.2.4 The self-energy

Among the six self-energy diagrams (see Fig. 6.1 and Table 6.2), two are UV finite, $\gamma_{7\pm}$, while the four remaining, $\gamma_{5\pm}$ and $\gamma_{6\pm}$, are UV divergent, thus those which have to be renormalized. However, the BPHZL subtraction procedures for the 1-particle irreducible self-energy divergent diagrams are analogous, differing only by coupling constants dependent factors, $\pm e^2$ and $\pm g^2$, corresponding to the 1-loop graphs, $\gamma_{5\pm}$ and $\gamma_{6\pm}$, respectively. Starting the analysis with $\gamma_{5\pm}$ Feynman graphs, the 1-loop self-energy, $\Sigma(\gamma_{5\pm})$, reads

$$\Sigma(\gamma_{5\pm}) = \int \frac{d^3k}{(2\pi)^3} \underbrace{\left\{ -e^2\gamma^\mu \left[ \frac{1}{k^2-\mu^2} \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \right] \left[ \frac{(\not{k}-\not{p}) \mp m(s-1)}{(k-p)^2 - m^2(s-1)^2} \right] \gamma^\nu \right\}}_{I_{\gamma_{5\pm}}(p,k,s)} . \tag{6.21}$$

Keeping in mind one more time the conditions (6.11) and the inequality (6.12), the UV and IR subtraction degrees are $\delta(\gamma_{5\pm}) = \delta(\gamma_{6\pm}) = 0$ and $\rho(\gamma_{6\pm}) = \rho(\gamma_{5\pm}) = 1$, where it has been fixed $b(\gamma_{5\pm}) = b(\gamma_{6\pm}) = 0$ and $c(\gamma_{5\pm}) = c(\gamma_{6\pm}) = 1$. The 1-loop BPHZL subtracted (renormalized) integrand, $R_{\gamma_{5\pm}}(p,k,s)$, can be expressed in terms of the unsubtracted one, $I_{\gamma_{5\pm}}(p,k,s)$, as follows:

$$\begin{aligned} R_{\gamma_{5\pm}}(p,k,s) &= (1 - t^0_{p,s-1})(1 - t^0_{p,s}) I_{\gamma_{5\pm}}(p,k,s) \\ &= (1 - t^0_{p,s-1} - t^0_{p,s} + t^0_{p,s-1}t^0_{p,s}) I_{\gamma_{5\pm}}(p,k,s) \; . \end{aligned} \tag{6.22}$$

Yet again, setting $s = 1$ at the end of the Taylor expansion operations, restoring the massless condition, the subtracted integrand, $R_{\gamma_{5\pm}}(p,k,1)$, results:

$$R_{\gamma_{5\pm}}(p,k,1) = I_{\gamma_{5\pm}}(p,k,1) - I_{\gamma_{5\pm}}(0,k,1) \; , \tag{6.23}$$

where,

$$\begin{aligned} I_{\gamma_{5\pm}}(p,k,1) &= 2e^2 \left\{ \frac{1}{k^2-\mu^2} \frac{1}{(k-p)^2} \left[ \not{k} - \frac{\not{k}(k\cdot p)}{k^2} \right] \right\} \; , \\ I_{\gamma_{5\pm}}(0,k,1) &= 2e^2 \frac{\not{k}}{k^2(k^2-\mu^2)} \; . \end{aligned} \tag{6.24}$$

Additionally, once the renormalized self-energy, $\Sigma^{(R)}_{\gamma_{5\pm}}(p,s)$, is defined by

$$\Sigma^{(R)}_{\gamma_{5\pm}}(p,s) = \int \frac{d^3k}{(2\pi)^3}\, R_{\gamma_{5\pm}}(p,k,s), \tag{6.25}$$

such that, from the Eqs.(6.23)–(6.24), leads to

$$\Sigma^{(R)}_{\gamma_{5\pm}} = -\frac{ie^2\not{p}}{4\pi}\left[\frac{1}{4\sqrt{p^2}}\left(\frac{p^2}{\mu^2}+\frac{3\mu^2}{p^2}+2\right)\ln\left(\frac{\mu^2-p^2}{(\sqrt{\mu^2}-\sqrt{p^2})^2}\right)-\frac{|\mu|}{2}\left(\frac{1}{\mu^2}+\frac{3}{p^2}\right)+i\pi\frac{p^2}{4\mu^2\sqrt{p^2}}\right], \tag{6.26}$$

with $\Sigma^{(R)}_{\gamma_{5\pm}} \equiv \Sigma^{(R)}_{\gamma_{5\pm}}(p,1)$.

Similarly to the previous case, $\Sigma^{(R)}_{\gamma_{5\pm}}$, the renormalized self-energies $\Sigma^{(R)}_{\gamma_{6\pm}}$, corresponding to $\gamma_{6\pm}$ diagram, read

$$\Sigma^{(R)}_{\gamma_{6\pm}} = -\frac{ig^2\not{p}}{4\pi}\left[\frac{1}{4\sqrt{p^2}}\left(\frac{p^2}{\mu^2}+\frac{3\mu^2}{p^2}+2\right)\ln\left(\frac{\mu^2-p^2}{(\sqrt{\mu^2}-\sqrt{p^2})^2}\right)-\frac{|\mu|}{2}\left(\frac{1}{\mu^2}+\frac{3}{p^2}\right)+i\pi\frac{p^2}{4\mu^2\sqrt{p^2}}\right]. \tag{6.27}$$

Accordingly, the 1-loop renormalized self-energies, $\Sigma^{(R)}_{+} = \Sigma^{(R)}_{\gamma_{5+}} + \Sigma^{(R)}_{\gamma_{6+}}$ and $\Sigma^{(R)}_{-} = \Sigma^{(R)}_{\gamma_{5-}} + \Sigma^{(R)}_{\gamma_{6-}}$, associated respectively to $\psi_+$ and $\psi_-$:

$$\begin{aligned}\Sigma^{(R)}_{+} &= -\frac{i(e^2+g^2)\not{p}}{4\pi}\left[\frac{1}{4\sqrt{p^2}}\left(\frac{p^2}{\mu^2}+\frac{3\mu^2}{p^2}+2\right)ln\left(\frac{\mu^2-p^2}{(\sqrt{\mu^2}-\sqrt{p^2})^2}\right)+\right.\\ &\left.-\frac{|\mu|}{2}\left(\frac{1}{\mu^2}+\frac{3}{p^2}\right)+i\pi\frac{p^2}{4\mu^2\sqrt{p^2}}\right],\\ &= \frac{(e^2+g^2)}{4\pi}\not{p}\ \mathcal{O}(p^2,\mu)\ ,\end{aligned} \tag{6.28}$$

In the same way we get $\Sigma^{(R)}_{+} = \Sigma^{(R)}_{-}$, that is:

$$\Sigma^{(R)}_{-} = \frac{(e^2+g^2)}{4\pi}\not{p}\ \mathcal{O}(p^2,\mu)\ , \tag{6.29}$$

contribute to the 1-loop effective action (in momenta space) with the following term:

$$\overline{\psi}_+\Sigma^{(R)}_+\psi_+ + \overline{\psi}_-\Sigma^{(R)}_-\psi_- \quad \xmapsto{P} \quad \overline{\psi}_-\Sigma^{(R)}_-\psi_- + \overline{\psi}_+\Sigma^{(R)}_+\psi_+\ , \tag{6.30}$$

that shows to be invariant under parity, thereby the BPHZL subtraction scheme does not break parity in the case of the 1-loop self-energy either. Finally, it has been finished the proof on the BPHZL parity invariance at 1-loop for the massless parity-even $U(1)\times U(1)$ Maxwell-Chern-Simons QED$_3$ model [109]. Nevertheless, thanks to divergent 2-loops vacuum polarization tensor diagrams (see Fig. 6.2), it remains to verify if whether or not parity still be preserved in the course of the 2-loops BPHZL ultraviolet and infrared subtractions.

## 6.3 BPHZL: 2-loops

In order to complete the proof if parity is broken or not by the BPHZL renormalization method, once the model into consideration here is superrenormalizable and the ultraviolet divergences are bounded up to 2-loops (6.9), it still remains to identify and investigate the potential UV divergent 2-loops diagrams in what concerns parity breakdown. By power-counting inspection (6.9), exclusively twenty four of the thirty six vacuum-polarization tensor Feynman graphs[5] show to be divergent at 2-loops (see Fig. 6.2), furthermore, it shall be verified if parity-odd local counterterms, with UV dimension 2, of the type $\epsilon^{\mu\alpha\nu}A_\mu p_\alpha A_\nu$ or $\epsilon^{\mu\alpha\nu}a_\mu p_\alpha a_\nu$ – local counterterm of the type $\epsilon^{\mu\alpha\nu}A_\mu p_\alpha a_\nu$ shall be discarded throughout this analysis because it is parity-even – might be generated by the UV and IR subtractions. However, power-counting (6.9) dimensional analysis reveals that even though parity-odd Levi-Civita symbol dependent counterterms could appear, they would be nonlocal since their coupling constant order should be of mass dimension 2, namely, $e^4$, $e^2g^2$ or $g^4$.

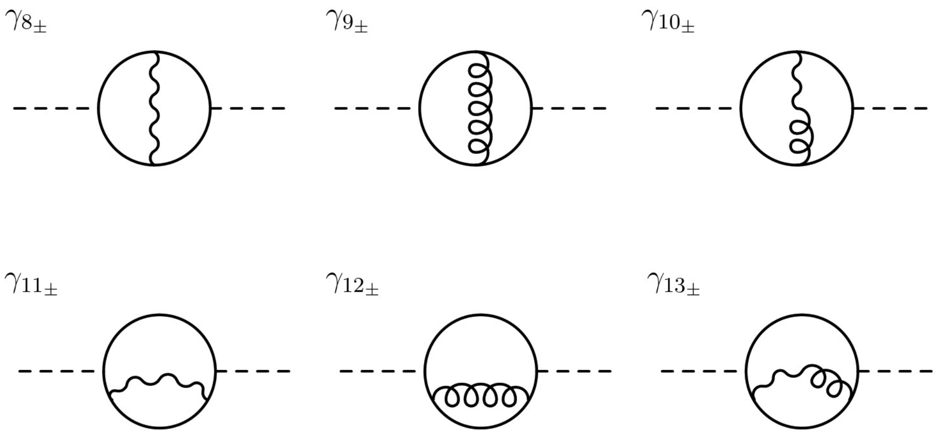



Figure 6.2: The 2-loops vacuum-polarization tensor graphs, which the continuous lines represent propagators of either $\psi_+$ or $\psi_-$, and the dashed lines the external legs of either $A_\mu$ or $a_\mu$.

Complementary to the previous dimensional discussion, a tensor structure analysis of the 2-loops vacuum-polarization tensor integrands is opportune. First of all, the thirty six 2-loops vacuum-polarization tensors diagrams ($\gamma_{i\pm}$, $i = 8 \ldots 13$) are displayed in Fig. 6.2, and their UV superficial degrees of divergence ($d(\gamma_{i\pm})$) are $d(\gamma_{8\pm}) = d(\gamma_{9\pm}) = d(\gamma_{11\pm}) = d(\gamma_{12\pm}) = 0$ and $d(\gamma_{10\pm}) = d(\gamma_{13\pm}) = -1$, thus from the former UV degree of divergences, the graphs $\gamma_{8\pm}$, $\gamma_{9\pm}$, $\gamma_{11\pm}$ and $\gamma_{12\pm}$ have to be renormalized, on the other hand the graphs $\gamma_{10\pm}$ and $\gamma_{13\pm}$ are already UV finite. Also, prior to the proof on the non generation of possible parity-odd Levi-Civita symbol dependent counterterms, it is suitable to write

[5] It should be pointed that, for the sake of subsequent renormalization, the symmetrical diagrams corresponding to $\gamma_{11\pm}$, $\gamma_{12\pm}$ and $\gamma_{13\pm}$ – those with the propagators $\Delta^{\mu\nu}_{AA}$, $\Delta^{\mu\nu}_{aa}$ or $\Delta^{\mu\nu}_{Aa}$ inside the loop in its upper part – have to be taken into consideration.

down explicitly the divergent vacuum-polarization tensors corresponding to the diagrams[6] $\gamma_{8\pm}$, $\gamma_{9\pm}$, $\gamma_{11\pm}$ and $\gamma_{12\pm}$:

$$\Pi^{\mu\nu}_{\gamma_{8\pm}}(p,s) = \lambda_8^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\, e^2\, \widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ , \tag{6.31}$$

$$\Pi^{\mu\nu}_{\gamma_{9\pm}}(p,s) = \lambda_9^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\, g^2\, \widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ ; \tag{6.32}$$

and

$$\Pi^{\mu\nu}_{\gamma_{11\pm}}(p,s) = \lambda_{11}^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\, e^2\, \widetilde{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ , \tag{6.33}$$

$$\Pi^{\mu\nu}_{\gamma_{12\pm}}(p,s) = \lambda_{12}^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\, g^2\, \widetilde{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s)\ ; \tag{6.34}$$

such that $\lambda_i = e$ $(i = 8, 9, 11, 12)$ if the two external legs are of $A_\mu$, otherwise, if $a_\mu$ as the two external legs, $\lambda_i = g$, and

$$\begin{aligned}\widehat{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s) =& -\mathrm{Tr}\Bigg\{\gamma^\mu \underbrace{\left[i\frac{\not{k}_1 \mp m(s-1)}{k_1^2 - m^2(s-1)^2}\right]}\gamma_\alpha \left[-i\frac{1}{(k_1-k_2)^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{(k_1^\alpha - k_2^\alpha)(k_1^\beta - k_2^\beta)}{(k_1-k_2)^2}\right)\right]\times \\ &\times \underbrace{\left[i\frac{\not{k}_2 \mp m(s-1)}{k_2^2 - m^2(s-1)^2}\right]}\gamma^\nu \underbrace{\left[i\frac{(\not{k}_2 - \not{p}) \mp m(s-1)}{(k_2-p)^2 - m^2(s-1)^2}\right]}\gamma_\beta \underbrace{\left[i\frac{(\not{k}_1 - \not{p}) \mp m(s-1)}{(k_1-p)^2 - m^2(s-1)^2}\right]}\Bigg\}\ ,\end{aligned} \tag{6.35}$$

$$\begin{aligned}\widetilde{I}^{\mu\nu}_{\pm}(k_1,k_2,p,s) =& -\mathrm{Tr}\Bigg\{\gamma^\mu \underbrace{\left[i\frac{\not{k}_1 \mp m(s-1)}{k_1^2 - m^2(s-1)^2}\right]}\gamma^\nu \underbrace{\left[i\frac{(\not{k}_1 - \not{p}) \mp m(s-1)}{(k_1-p)^2 - m^2(s-1)^2}\right]}\gamma_\alpha \times \\ &\times\left[-i\frac{1}{k_2^2-\mu^2}\left(\eta^{\alpha\beta} - \frac{k_2^\alpha k_2^\beta}{k_2^2}\right)\right] \underbrace{\left[i\frac{(\not{k}_1 - \not{k}_2 - \not{p}) \mp m(s-1)}{(k_1-k_2-p)^2 - m^2(s-1)^2}\right]}\gamma_\beta \underbrace{\left[i\frac{(\not{k}_1 - \not{p}) \mp m(s-1)}{(k_1-p)^2 - m^2(s-1)^2}\right]}\Bigg\}\ ,\end{aligned} \tag{6.36}$$

where $p$ is the external momentum and the subscripts $+$ and $-$ refer to the internal lines of $\psi_+$ and $\psi_-$, respectively.

Drawing attention to the integrands $\widetilde{I}^{\mu\nu}_{\pm}$ (6.35) and $\widetilde{I}^{\mu\nu}_{\pm}$ (6.36), it can be seen that trace of the product of four to eight gamma matrices is generated, notwithstanding that solely trace of five and seven gamma matrices produces the Levi-Civita symbol $\epsilon^{\mu\nu\rho}$ (6.2). Also, it shall be noticed from the terms of the integrands, $\widetilde{I}^{\mu\nu}_{\pm}$ (6.35) and $\widetilde{I}^{\mu\nu}_{\pm}$ (6.36),

[6]As already mentioned, since possible parity-even local counterterm of the type $\epsilon^{\mu\alpha\nu}A_\mu p_\alpha a_\nu$ has not been taken into consideration, it remains sixteen graphs that could generate parity-odd-like counterterms $\epsilon^{\mu\alpha\nu}A_\mu p_\alpha A_\nu$ and $\epsilon^{\mu\alpha\nu}a_\mu p_\alpha a_\nu$.

identified by under braces that they contribute each one to the trace product with at most one gamma matrix. Furthermore, as an example, by picking out from the integrand $\widetilde{I}^{\mu\nu}_{\pm}$ (6.35) a piece of trace product of five gamma matrices, *e.g.*: $\mathcal{Z}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = -\mathrm{Tr}\{\gamma^\mu[\mp im(s-1)]\gamma_\alpha[\Delta^{\alpha\beta}(k_1,k_2)][\mp im(s-1)]\gamma^\nu[\mp im(s-1)]\gamma_\beta[i(\not{k}_1-\not{p})]\}$, it can be written as $\mathcal{Z}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = \pm\,\mathrm{e}^{\mu\nu\rho}\,\mathcal{X}_{5\rho}(k_1,k_2,p,s) + \mathcal{Y}^{\mu\nu}_{5\pm}(k_1,k_2,p,s)$, where the first term is parity-odd whereas the second one is parity-even. In the sequence, using the same strategy applied to all five gamma matrices dependent terms, of the integrands (6.35) and (6.36), they can be rewritten as:

$$\widehat{I}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = \pm\,\mathrm{e}^{\mu\nu\rho}\,\widehat{\mathcal{A}}_{5\rho}(k_1,k_2,p,s) + \widehat{\mathcal{S}}^{\mu\nu}_{5\pm}(k_1,k_2,p,s)\ , \tag{6.37}$$

$$\widetilde{I}^{\mu\nu}_{5\pm}(k_1,k_2,p,s) = \pm\,\mathrm{e}^{\mu\nu\rho}\,\widetilde{\mathcal{A}}_{5\rho}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}^{\mu\nu}_{5\pm}(k_1,k_2,p,s)\ , \tag{6.38}$$

where $\widehat{\mathcal{S}}^{\mu\nu}_{5\pm}$ and $\widetilde{\mathcal{S}}^{\mu\nu}_{5\pm}$ are parity-even tensors, and their subscripts $+$ and $-$ refer to the internal lines of $\psi_+$ and $\psi_-$, respectively. Consequently, the total integrands stemming from the trace of five gamma matrices, $\widehat{I}^{\mu\nu}_{5} = \widehat{I}^{\mu\nu}_{5+} + \widehat{I}^{\mu\nu}_{5-}$ and $\widetilde{I}^{\mu\nu}_{5} = \widetilde{I}^{\mu\nu}_{5+} + \widetilde{I}^{\mu\nu}_{5-}$, read:

$$\widehat{I}^{\mu\nu}_{5}(k_1,k_2,p,s) = \widehat{\mathcal{S}}^{\mu\nu}_{5+}(k_1,k_2,p,s) + \widehat{\mathcal{S}}^{\mu\nu}_{5-}(k_1,k_2,p,s)\ , \tag{6.39}$$

$$\widetilde{I}^{\mu\nu}_{5}(k_1,k_2,p,s) = \widetilde{\mathcal{S}}^{\mu\nu}_{5+}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}^{\mu\nu}_{5-}(k_1,k_2,p,s)\ , \tag{6.40}$$

thence there is no Levi-Civita symbol $\epsilon^{\mu\nu\rho}$ dependent terms emerged from the trace of five gamma matrices contributing to the total divergent integrand of vacuum-polarization tensor, remaining therefore only parity-even terms. Beyond that, it lacks to discuss the issue of non generation of possible parity-odd Levi-Civita symbol dependent counterterms for the case of the trace product of seven gamma matrices. Analogously to the preceding discussion, from the integrands $\widetilde{I}^{\mu\nu}_{\pm}$ (6.35) and $\widetilde{I}^{\mu\nu}_{\pm}$ (6.36), considering the terms highlighted by under braces, and for instance, by picking out from the integrand $\widetilde{I}^{\mu\nu}_{\pm}$ (6.35) a piece of trace product of seven gamma matrices, *e.g.*: $\mathcal{Z}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) = -\mathrm{Tr}\{\gamma^\mu[i(\not{k}_1)]\gamma_\alpha[\Delta^{\alpha\beta}(k_1,k_2)][i(\not{k}_2)]\gamma^\nu[i(\not{k}_2-\not{p})]\gamma_\beta[\mp im(s-1)]\}$, it follows that $\mathcal{Z}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) = \pm\,\mathrm{e}^{\mu\nu\rho}\,\mathcal{X}_{7\rho}(k_1,k_2,p,s) + \mathcal{Y}^{\mu\nu}_{7\pm}(k_1,k_2,p,s)$, with the first term being parity-odd whereas the second one being parity-even. In addition to, doing similarly to all seven gamma matrices dependent terms of (6.35) and (6.36), it can be shown that:

$$\widehat{I}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) = \pm\,\mathrm{e}^{\mu\nu\rho}\,\widehat{\mathcal{A}}_{7\rho}(k_1,k_2,p,s) + \widehat{\mathcal{S}}^{\mu\nu}_{7\pm}(k_1,k_2,p,s)\ , \tag{6.41}$$

$$\widetilde{I}^{\mu\nu}_{7\pm}(k_1,k_2,p,s) = \pm\,\mathrm{e}^{\mu\nu\rho}\,\widetilde{\mathcal{A}}_{7\rho}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}^{\mu\nu}_{7\pm}(k_1,k_2,p,s)\ , \tag{6.42}$$

where $\widehat{\mathcal{S}}^{\mu\nu}_{7\pm}$ and $\widetilde{\mathcal{S}}^{\mu\nu}_{7\pm}$ are parity-even tensors, and the internal lines of $\psi_+$ and $\psi_-$ in the corresponding graphs are respectively represented by the subscripts $+$ and $-$. Morover, from the trace of seven gamma matrices, the total integrands, $\widehat{I}^{\mu\nu}_{7} = \widehat{I}^{\mu\nu}_{7+} + \widehat{I}^{\mu\nu}_{7-}$ and

$\widetilde{I}_7^{\mu\nu} = \widetilde{I}_{7+}^{\mu\nu} + \widetilde{I}_{7-}^{\mu\nu}$, are given by:

$$\widehat{I}_7^{\mu\nu}(k_1,k_2,p,s) = \widehat{\mathcal{S}}_{7+}^{\mu\nu}(k_1,k_2,p,s) + \widehat{\mathcal{S}}_{7-}^{\mu\nu}(k_1,k_2,p,s)\ , \tag{6.43}$$

$$\widetilde{I}_7^{\mu\nu}(k_1,k_2,p,s) = \widetilde{\mathcal{S}}_{7+}^{\mu\nu}(k_1,k_2,p,s) + \widetilde{\mathcal{S}}_{7-}^{\mu\nu}(k_1,k_2,p,s)\ , \tag{6.44}$$

thus likewise the five gamma matrices case, there is no Levi-Civita symbol $\epsilon^{\mu\nu\rho}$ dependent terms yielded from the trace of seven gamma matrices, surviving only parity-even terms which contribute to the total divergent vacuum-polarization tensor.

Ultimately, based on the argumentations above, the 2-loops unsubtracted integrands associated to the vacuum-polarization tensors $\Pi_{AA}^{\mu\nu}$ and $\Pi_{aa}^{\mu\nu}$ do not produce parity-violating counterterms of the type, $\epsilon^{\mu\alpha\nu}A_\mu\partial_\alpha A_\nu$ and $\epsilon^{\mu\alpha\nu}a_\mu\partial_\alpha a_\nu$, therefore it is concluded that parity is still preserved at 2-loops under the BPHZL renormalization procedures. Besides, due to the fact that the UV divergences are restricted up to 2-loops, thus for higher perturbative orders greater than two there is no need of UV subtractions, consequently it is definitely proved that the BPHZL renormalization method preserves parity for the massless parity-even $U(1)\times U(1)$ Maxwell-Chern-Simons QED$_3$ model [109].

However the presence of counterterms breaking the parity [109] do not showns up we still need the complete renormalizability of the model.

## 6.4 Conclusion

The massless parity-even $U(1)\times U(1)$ planar quantum electrodynamics (QED$_3$) model [109] exhibits quantum parity conservation at all orders in perturbation theory. The proof has been performed using the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization method, however owing to the presence of two massless fermions in the spectrum, infrared divergences might emerge in the course of the ultraviolet divergences subtractions and must be subtracted as well, for this reason, the Lowenstein-Zimmermann (LZ) subtraction scheme has been adopted. The power-counting – the ultraviolet and infrared superficial degrees of divergence (6.9) of any 1-particle irreducible Feynman diagram – reveals that ultraviolet divergences are bounded at most to two loops. At one loop all six vacuum-polarization tensor diagrams are linear ultraviolet divergent, four of the six self-energy diagrams are logarithm ultraviolet divergent, while all the vertex-function diagrams are ultraviolet finite, beyond that at two loops, twenty four of the thirty six vacuum-polarization tensor Feynman graphs are ultraviolet divergent (Fig. 6.1 and Table 6.2). Although there are counterterms[7] at one and two loops, none of them violate parity symmetry and together to the fact that the model is superrenormalizable, it stems

[7]The explicitly BPHZL renormalization and the calculations of all counterterms at 1- and 2-loops, whether parity-even or -odd, are left to another work [145], since the purpose of this one was to verify if the LZ subtraction scheme in the framework of the BPHZ renormalization method would preserve or not parity symmetry.

as a byproduct that parity is guaranteed at any radiative order. As a final conclusion, for the model presented in this work, opposite to the case of the ordinary massless parity-even $U(1)$ $\mathrm{QED}_3$ [77], the BPHZL subtraction scheme with the Lowenstein's adaptation of the Zimmermann's forest formula [95, 96] preserves parity symmetry at all perturbative order.

# Chapter 7

# On the infrared and ultraviolet finiteness of parity-preserving $U(1) \times U(1)$ massless QED$_3$

As discussed in Chapter 4, here we will extend to the quantum level the model presented in [109], already analyzed up to two loops by the BPHZL method [130], but now using the Algebraic Renormalization method [1] .

## 7.1 Introduction

The study of models in different space-time dimensions, beyond the traditional $1+3$ dimensions, has been considered as a prominent tool in the search for the comprehension of physical problems. In the branch of condensed matter, special in the theoretical framework, the quantum electrodynamics in tree space-time dimensions (QED$_3$) has been considered an outstanding improvement for our attempts in the comprehension of phenomena like high-temperature superconductivity [26–28], quantum Hall effect [29], graphene [16, 22, 38–40, 42, 44, 141, 142], topological insulators [32–34] and topological superconductors [35–37].

The massless parity-even $U(1) \times U(1)$ planar quantum electrodynamics (QED$_3$) model [109], exhibits, by using the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization method quantum parity conservation at all orders in perturbation theory [130]. Owing to the presence of two massless fermions in the spectrum, infrared divergences might emerge in the course of the ultraviolet divergences subtractions and must be subtracted as well, for this reason, the Lowenstein-Zimmermann (LZ) subtraction scheme was adopted [130]. Nevertheless the algebraic method is independently of any

[1] O.M. Del Cima, D.H.T. Franco, L.S. Lima and E.S. Miranda, "On the infrared and ultraviolet finiteness of parity-preserving $U(1) \times U(1)$ massless QED$_3$", in preparation.

regularization scheme. In this chapter we shall verify the issue of anomalies and stability of the model ensuring its renormalizability.

Guided by this purpose, initially we define the model and its classical symmetries, continuous and discrete, Sections 7.2 and 7.3, respectively. Almost concomitant to that, still in Section 7.3 the action for the gauge-fixing and the action of external sources, coupling antifields to the nonlinear BRS transformations of the fields, are established. Regarding the perturbative quantization, that is, the extension of parity-even $U_A(1) \times U_a(1)$ hybrid QED$_3$ at the classical level to all orders in perturbation theory, first is analyzed the stability of the classical action – if the radiative corrections can be reabsorbed by a redefinition of the initial parameters of the model – which is done by taking in account the ultraviolet and infrared bounds required by the process in Section 7.4. Next, in Section 7.5, all potential anomalies are identified by means of the analysis of the Wess-Zumino consistency condition, in other words, solving the Slavnov-Taylor cohomology problem in the sector of ghost number one. Besides that, it is checked if the radiatively induced breakings might be fine-tuned by an appropriate choice of local non-invariant counterterms as wellas if the non-invariant counterterms do not generate possible infrared anomalies. The conclusion is left to Section 7.6.

## 7.2 Some model statements

We start by presenting the parity-preserving $U(1) \times U(1)$ hybrid QED$_3$ proposed in [109],

$$\begin{aligned}
\Sigma^{(s-1)}_{inv} \;=\; & \int d^3x \Big\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\alpha\nu}A_\mu\partial_\alpha a_\nu + i\overline{\psi}_+ \not{D}\psi_+ + i\overline{\psi}_- \not{D}\psi_- + \\
& \underbrace{-\, m(s-1)\overline{\psi}_+\psi_+ + m(s-1)\overline{\psi}_-\psi_-}_{\text{Lowenstein-Zimmermann mass term}} \;, \qquad (7.1)
\end{aligned}$$

here $\not{D}\psi_\pm \equiv (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm$ with gamma matrices $\gamma^\mu = (\sigma_z, -i\sigma_x, i\sigma_y)$ and the field strengths, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, are related to the electromagnetic field ($A_\mu$) and the pseudo chiral gauge field ($a_\mu$), respectively. The Dirac spinors $\psi_+$ and $\psi_-$ are two kinds of fermions where the $\pm$ subscripts are related to their pseudo-spin sign [17, 109]. In the processes of renormalization is imperative to known that $m$ and $\mu$ are mass parameters with mass dimension 1 and the coupling constants $e$ (electric charge) and $g$ (pseudo chiral charge) are dimensionful with mass dimension $\frac{1}{2}$. Besides that, we have the Lowenstein-Zimmermann parameter $s$ which lies in the interval $0 \leq s \leq 1$ and has the same status of an additional subtraction variable (as the external momentum) in the BPHZL renormalization scheme, it allows us to recovery the massless model [109] by taking $s = 1$ at the end of calculations.

## 7.3 The symmetries

It is important to elucidate that the action (7.1) is $U_A(1) \times U_a(1)$ gauge invariant (1.9),

$$
\begin{aligned}
\delta_{\rm g} A_\mu(x) &= -\frac{1}{e}\,\partial_\mu \rho(x) \ , \\
\delta_{\rm g} a_\mu(x) &= -\frac{1}{g}\,\partial_\mu \lambda(x) \ , \\
\delta_{\rm g} \psi_\pm(x) &= i[\rho(x) \pm \lambda(x)]\psi_\pm(x) \ , \\
\delta_{\rm g} \overline{\psi}_\pm(x) &= -i[\rho(x) \pm \lambda(x)]\overline{\psi}_\pm(x) \ .
\end{aligned} \tag{7.2}
$$

Its gauge invariance constitutes a necessary requirement in the processes of construction of the Becchi-Rouet-Stora (BRS) symmetry and the action is parity invariant which shall be fixed posteriorly (See eq. 7.37).

The quantization of the model requires a gauge fixing :

$$
\Sigma_{\rm gf} = \int d^3x \left\{ b\partial^\mu A_\mu + \frac{\alpha}{2} b^2 + \overline{c}\Box c + \pi \partial^\mu a_\mu + \frac{\beta}{2}\pi^2 + \overline{\xi}\Box\xi \right\} . \tag{7.3}
$$

where $(c, \xi)$ are called ghosts and $(\overline{c}, \overline{\xi})$ the anti-ghosts of Faddeev-Popov. Being the ($b$ and $\pi$) the Lautrup-Nakanish fields [97–99, 146] acting like Lagrange multipliers.

The BRS transformations are given by,

$$
\begin{aligned}
s\psi_\pm &= i(c \pm \xi)\psi_\pm \ , & s\overline{\psi}_\pm &= -i(c \pm \xi)\overline{\psi}_\pm \\
sA_\mu &= -\frac{1}{e}\partial_\mu c \ , & sa_\mu &= -\frac{1}{g}\partial_\mu \xi \ , \\
s\overline{c} &= \frac{b}{e} \ , & s\overline{\xi} &= \frac{\pi}{g} \\
sc &= 0 \ , sb = 0 & s\xi &= 0 \ , s\pi = 0 \ .
\end{aligned} \tag{7.4}
$$

The operator $s$ is nilpotent $s^2 = 0$ and all Grassmann variables thereby obeying the algebra $\{c, c\} = \{\xi, \xi\} = 0$.

For the non-linear BRS transformations we couple external sources ($\Sigma_{ext}$) to avoid renormalization of the symmetries

$$
\Sigma_{\rm ext} = \int d^3x \left\{ i\overline{\Omega}_+ c_+ \psi_+ - i\overline{\Omega}_- c_- \psi_- + ic_+\overline{\psi}_+\Omega_+ - ic_-\overline{\psi}_-\Omega_- \right\} . \tag{7.5}
$$

where we defined $c_\pm = c \pm \xi$ and due to the algebra of ($c$ and $\xi$) we obtain $c_+^2 = c_-^2 = 0$.

The complete classical action $\Sigma^{(s-1)}$ is:

$$
\Sigma^{(s-1)} = \Sigma^{(s-1)}_{\rm inv} + \Sigma_{\rm gf} + \Sigma_{\rm ext} \ , \tag{7.6}
$$

The calculation of the propagators, or the second order Green functions, given us the following result:

$$\Delta_{++}(k) = i\frac{\not{k} - m(s-1)}{k^2 - m^2(s-1)^2} \; , \; \Delta_{--}(k) = i\frac{\not{k} + m(s-1)}{k^2 - m^2(s-1)^2} \; . \tag{7.7}$$

$$\Delta^{\mu\nu}_{AA}(k) = -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\alpha}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \; , \tag{7.8}$$

$$\Delta^{\mu\nu}_{aa}(k) = -i\left\{\frac{1}{k^2-\mu^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\beta}{k^2}\frac{k^\mu k^\nu}{k^2}\right\} \; , \tag{7.9}$$

$$\Delta^{\mu\nu}_{Aa}(k) = \frac{\mu}{k^2(k^2-\mu^2)}\,\mathrm{e}^{\mu\alpha\nu}\, k_\alpha \; , \tag{7.10}$$

$$\Delta^{\mu}_{Ab}(k) = \Delta^{\mu}_{a\pi}(k) = \frac{k^\mu}{k^2} \; , \tag{7.11}$$

$$\Delta_{bb}(k) = \Delta_{\pi\pi}(k) = 0 \; , \tag{7.12}$$

$$\Delta_{\bar{c}c}(k) = \Delta_{\bar{\xi}\xi}(k) = -\frac{i}{k^2} \; , \tag{7.13}$$

As we know, the propagators carry the bases for the unitarity and spectral consistency analyses at the tree-level of a model. In [60] there is detailed spectral analysis together with the two-particle scattering potentials.

To determine the ultraviolet (UV) and infrared (IR) dimensions of any fields, $X$ and $Y$, we use the asymptotical behaviour of their propagator $\Delta_{XY}(k)$, $d_{XY}$ and $r_{XY}$, respectively, as follows:

$$d_{XY} = \overline{\deg}_k \Delta_{XY}(k) \quad \text{and} \quad r_{XY} = \underline{\deg}_k \Delta_{XY}(k) \; , \tag{7.14}$$

such that $\overline{\deg}_k$ provides the asymptotic power when $k \to \infty$, while $\underline{\deg}_k$ provides the asymptotic power when $k \to 0$. The UV ($d$) and IR ($r$) dimensions of the fields, $X$ and $Y$, satisfy inequalities below [19]:

$$d_X + d_Y \geq 3 + d_{XY} \quad \text{and} \quad r_X + r_Y \leq 3 + r_{XY} \; . \tag{7.15}$$

Thereby, making use of the propagators (7.7) with the conditions (7.15), specifically aiming to determinate the UV and IR dimensions of the spinor fields, $\psi_+$ and $\psi_-$, we have:

$$d_{++} = -1 \;\Rightarrow\; 2d_+ \geq 2 \;\Rightarrow\; d_+ = 1 \; ; \quad r_{++} = -1 \Rightarrow 2r_+ \leq 2 \;\Rightarrow\; r_+ = 1 \tag{7.16}$$

$$d_{--} = -1 \;\Rightarrow\; 2d_- \geq 2 \;\Rightarrow\; d_- = 1 \; ; \quad r_{--} = -1 \Rightarrow 2r_- \leq 2 \;\Rightarrow\; r_- = 1 \tag{7.17}$$

and for the vector fields, $A_\mu$ and $a_\mu$, we make use of the propagators (7.8)–(7.10):

$$\begin{aligned} d_{AA} = -2 \;\Rightarrow\; 2d_A \geq 1 \;; &\qquad r_{AA} = 0 \Rightarrow\; 2r_A \leq 3 \\ d_{aa} = -2 \;\Rightarrow\; 2d_a \geq 1 \;; &\qquad r_{aa} = 0 \Rightarrow\; 2r_a \leq 3 \\ d_{Aa} = -3 \;\Rightarrow\; d_A + d_a \geq 0 \;; &\qquad r_{Aa} = -1 \Rightarrow\; r_A + r_a \leq 2 \\ d_A = d_a = \frac{1}{2} \;; &\qquad r_A = r_a = 1 \end{aligned} \tag{7.18}$$

It is important to mention here though the mixed propagator (7.10) does not carry any degrees of freedom on-shell [22, 60]. However, since $d_{Aa} < d_{AA} = d_{aa}$, internal lines containing the mixed propagator (7.10), instead of propagators (7.8) and (7.9), reduces the UV degree of divergence of Feynman diagrams, as indicated by the power-counting formula (7.53).

From the propagators (7.11) and the conditions (7.15) and (7.18), the UV and IR dimensions of the Lautrup-Nakanishi fields, $b$ and $\pi$, are:

$$d_{Ab} = -1 \;;\; d_A + d_b \geq 2 \;\Rightarrow\; d_b = \frac{3}{2} \;, \quad r_{Ab} = -1 \;;\; r_A + r_b \leq 2 \;\Rightarrow\; r_b = 1 \;, \tag{7.19}$$

$$d_{a\pi} = -1 \;;\; d_a + d_\pi \geq 2 \;\Rightarrow\; d_\pi = \frac{3}{2} \;, \quad r_{a\pi} = -1 \;;\; r_a + r_\pi \leq 2 \;\Rightarrow\; r_\pi = 1 \;. \tag{7.20}$$

To complete the UV and IR dimensions we consider the propagators (7.13), that is , the dimensions of the Faddeev-Popov ghosts, $c$ and $\xi$, and antighosts, $\bar{c}$ and $\bar{\xi}$, are constrained by:

$$d_{\bar{c}c} = -2 \;\Rightarrow\; d_c + d_{\bar{c}} \geq 1 \;, \quad r_{\bar{c}c} = -2 \;\Rightarrow\; r_c + r_{\bar{c}} \leq 1 \;, \tag{7.21}$$

$$d_{\bar{\xi}\xi} = -2 \;\Rightarrow\; d_\xi + d_{\bar{\xi}} \geq 1 \;, \quad r_{\bar{\xi}\xi} = -2 \;\Rightarrow\; r_\xi + r_{\bar{\xi}} \leq 1 \;. \tag{7.22}$$

Furthermore, by fixing the BRS operator ($s$) as dimensionless and knowing that the coupling constants $e$ and $g$ have mass dimension $\frac{1}{2}$, from the conditions (7.22), the UV dimensions of the Faddeev-Popov ghosts and anti-ghosts result:

$$d_c = 0 \quad \text{and} \quad d_{\bar{c}} = 1 \;, \qquad r_c = 0 \quad \text{and} \quad r_{\bar{c}} = 1 \;, \tag{7.23}$$

$$d_\xi = 0 \quad \text{and} \quad d_{\bar{\xi}} = 1 \;, \qquad r_\xi = 0 \quad \text{and} \quad r_{\bar{\xi}} = 1 \;. \tag{7.24}$$

After all, from the antifields action ($\Sigma_{\text{ext}}$) together with the UV and IR dimensions of all the quantum fields previously computed, it follows that:

$$d_{\Omega_+} = 2 \quad \text{and} \quad d_{\Omega_-} = 2 \;;\; r_{\Omega_+} = 2 \quad \text{and} \quad r_{\Omega_-} = 2 \;. \tag{7.25}$$

In summary, the UV dimension ($d$), the IR dimension ($r$), the ghost number ($\Phi\Pi$) and the Grassmann parity ($GP$) of all fields are displayed in Table 7.1. The statistics is defined in

| | $A_\mu$ | $a_\mu$ | $\psi_+$ | $\psi_-$ | $c$ | $\overline{c}$ | $b$ | $\xi$ | $\overline{\xi}$ | $\pi$ | $s$ | $s-1$ | $\Omega_+$ | $\Omega_-$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | $1/2$ | $1/2$ | 1 | 1 | 0 | 1 | $\frac{3}{2}$ | 0 | 1 | $\frac{3}{2}$ | 1 | 1 | 2 | 2 |
| $r$ | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 2 | 2 |
| $\Phi\Pi$ | 0 | 0 | 0 | 0 | 1 | $-1$ | 0 | 1 | $-1$ | 0 | 0 | 0 | $-1$ | $-1$ |
| $GP$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |

Table 7.1: The UV dimension ($d$), the infrared dimension ($r$), ghost number ($\Phi\Pi$) and Grassmann parity ($GP$).

such a way that, the integer spin fields with odd ghost number and the half integer spin fields with even ghost number anticommute among themselves, in any other case the fields commute among themselves.

Representing symmetries in a functional way, namely the rigid and the BRS symmetries, is a powerful and elegant manner to deal with them. Therefore, the Slavnov-Taylor operator, a functional representation of the BRS symmetry, has its invariance for this model given by:

$$\mathcal{S}(\Sigma^{(s-1)}) = 0 \;, \tag{7.26}$$

where the explicit form of the Slavnov-Taylor operator $\mathcal{S}$ acting in an arbitrary functional ($\mathcal{F}$) is given by:

$$\begin{aligned}\mathcal{S}(\mathcal{F}) = \int d^3x \Big\{ & -\frac{1}{e}\partial^\mu c \frac{\delta \mathcal{F}}{\delta A^\mu} + \frac{b}{e}\frac{\delta \mathcal{F}}{\delta \overline{c}} - \frac{1}{g}\partial^\mu \xi \frac{\delta \mathcal{F}}{\delta a^\mu} + \frac{\pi}{g}\frac{\delta \mathcal{F}}{\delta \overline{\xi}} + \\ & + \frac{\delta \mathcal{F}}{\delta \overline{\Omega}_+}\frac{\delta \mathcal{F}}{\delta \psi_+} - \frac{\delta \mathcal{F}}{\delta \Omega_+}\frac{\delta \mathcal{F}}{\delta \overline{\psi}_+} - \frac{\delta \mathcal{F}}{\delta \overline{\Omega}_-}\frac{\delta \mathcal{F}}{\delta \psi_-} + \frac{\delta \mathcal{F}}{\delta \Omega_-}\frac{\delta \mathcal{F}}{\delta \overline{\psi}_-} \Big\} \;.\end{aligned} \tag{7.27}$$

Also the corresponding linearized Slavnov-Taylor operator ($\mathcal{S}_\mathcal{F}$) is:

$$\begin{aligned}\mathcal{S}_\mathcal{F} \;=\; & \int d^3x \Big\{ -\frac{1}{e}\partial^\mu c \frac{\delta}{\delta A^\mu} + \frac{b}{e}\frac{\delta}{\delta \overline{c}} - \frac{1}{g}\partial^\mu \xi \frac{\delta}{\delta a^\mu} + \frac{\pi}{g}\frac{\delta}{\delta \overline{\xi}} + \frac{\delta \mathcal{F}}{\delta \overline{\Omega}_+}\frac{\delta}{\delta \psi_+} + \frac{\delta \mathcal{F}}{\delta \psi_+}\frac{\delta}{\delta \overline{\Omega}_+} + \\ & - \frac{\delta \mathcal{F}}{\delta \Omega_+}\frac{\delta}{\delta \overline{\psi}_+} - \frac{\delta \mathcal{F}}{\delta \overline{\psi}_+}\frac{\delta}{\delta \Omega_+} - \frac{\delta \mathcal{F}}{\delta \overline{\Omega}_-}\frac{\delta}{\delta \psi_-} - \frac{\delta \mathcal{F}}{\delta \psi_-}\frac{\delta}{\delta \overline{\Omega}_-} + \frac{\delta \mathcal{F}}{\delta \Omega_-}\frac{\delta}{\delta \overline{\psi}_-} + \frac{\delta \mathcal{F}}{\delta \overline{\psi}_-}\frac{\delta}{\delta \Omega_-} \Big\}\end{aligned} \tag{7.28}$$

can be shown that the following nilpotence identities are true:

$$\mathcal{S}_\mathcal{F}\mathcal{S}(\mathcal{F}) = 0 \;, \quad \forall \mathcal{F} \;, \tag{7.29}$$

$$\mathcal{S}_\mathcal{F}\mathcal{S}_\mathcal{F} = 0 \;\; \text{if} \;\; \mathcal{S}(\mathcal{F}) = 0 \;. \tag{7.30}$$

In the particular case, $(\mathcal{S}_\Sigma)^2 = 0$, it was expected by the constructed relation of the BRS action invariance (7.26).

### 7.3.1 Some others symmetries of the classical action

Still dealing with the Slavnov-Taylor operator upon the classical action we can find some important relations for the renormalization process. Some of then are:

$$\begin{aligned}
&\mathcal{S}_\Sigma \phi = s\phi \ , \quad \phi = \{\psi_\pm, \overline{\psi}_\pm, A_\mu, a_\mu, c, \overline{c}, b, \pi, \overline{\xi}, \xi\} \ , \\
&\mathcal{S}_\Sigma \Omega_+ = -\frac{\delta\Sigma}{\delta\overline{\psi}_+} \ , \quad \mathcal{S}_\Sigma \overline{\Omega}_+ = \frac{\delta\Sigma}{\delta\psi_+} \ , \\
&\mathcal{S}_\Sigma \Omega_- = \frac{\delta\Sigma}{\delta\overline{\psi}_-} \ , \quad \mathcal{S}_\Sigma \overline{\Omega}_- = -\frac{\delta\Sigma}{\delta\psi_-} \ .
\end{aligned} \tag{7.31}$$

Notwithstanding, others relations satisfied by our classical model, namely ghost equations (7.32)–(7.33), gauge conditions (7.34) and antighost equations (7.35), read:

$$-i\frac{\delta\Sigma^{(s-1)}}{\delta c} = i\Box\overline{c} + \overline{\Omega}_+\psi_+ - \overline{\Omega}_-\psi_- + \overline{\psi}_+\Omega_+ - \overline{\psi}_-\Omega_- = \Delta^{(1)}_{\rm class} \ , \tag{7.32}$$

$$-i\frac{\delta\Sigma^{(s-1)}}{\delta \xi} = i\Box\overline{\xi} + \overline{\Omega}_+\psi_+ + \overline{\Omega}_-\psi_- + \overline{\psi}_+\Omega_+ + \overline{\psi}_-\Omega_- = \Delta^{(2)}_{\rm class} \ . \tag{7.33}$$

$$\frac{\delta\Sigma^{(s-1)}}{\delta b} = \partial^\mu A_\mu + \alpha b \quad \text{and} \quad \frac{\delta\Sigma^{(s-1)}}{\delta \pi} = \partial^\mu a_\mu + \beta\pi \ , \tag{7.34}$$

$$\frac{\delta\Sigma^{(s-1)}}{\delta \overline{c}} = \Box c \quad \text{and} \quad \frac{\delta\Sigma^{(s-1)}}{\delta \overline{\xi}} = \Box\xi \ , \tag{7.35}$$

Moreover, the classical action is invariant under the rigid symmetries:

$$W^{(e)}_{\rm rigid}\Sigma^{(s-1)} = 0 \ , \quad \text{and} \quad W^{(g)}_{\rm rigid}\Sigma^{(s-1)} = 0 \ , \tag{7.36}$$

where the Ward operators are defined by:

$$W^{(e)}_{\rm rigid} = \int d^3x \left\{ \psi_+\frac{\delta}{\delta\psi_+} - \overline{\psi}_+\frac{\delta}{\delta\overline{\psi}_+} + \Omega_+\frac{\delta}{\delta\Omega_+} - \overline{\Omega}_+\frac{\delta}{\delta\overline{\Omega}_+} + \psi_-\frac{\delta}{\delta\psi_-} - \overline{\psi}_-\frac{\delta}{\delta\overline{\psi}_-} + \Omega_-\frac{\delta}{\delta\Omega_-} - \overline{\Omega}_-\frac{\delta}{\delta\overline{\Omega}_-} \right\} \ ,$$

and

$$W^{(g)}_{\rm rigid} = \int d^3x \left\{ \psi_+\frac{\delta}{\delta\psi_+} - \overline{\psi}_+\frac{\delta}{\delta\overline{\psi}_+} + \Omega_+\frac{\delta}{\delta\Omega_+} - \overline{\Omega}_+\frac{\delta}{\delta\overline{\Omega}_+} - \psi_-\frac{\delta}{\delta\psi_-} + \overline{\psi}_-\frac{\delta}{\delta\overline{\psi}_-} - \Omega_-\frac{\delta}{\delta\Omega_-} + \overline{\Omega}_-\frac{\delta}{\delta\overline{\Omega}_-} \right\} \ .$$

Finally, the action $\Sigma$ is invariant under the discrete symmetry, parity ($P$), described bellow:

$$\begin{aligned}
&\psi_\pm \xrightarrow{P} \psi_\pm^P = -i\gamma^1\psi_\mp \ , \ \overline{\psi}_\pm \xrightarrow{P} \overline{\psi}_\pm^P = i\overline{\psi}_\mp\gamma^1 \ , \\
&\Omega_\pm \xrightarrow{P} \Omega_\pm^P = -i\gamma^1\psi_\mp \ , \ \overline{\Omega}_\pm \xrightarrow{P} \overline{\Omega}_\pm^P = i\overline{\Omega}\mp\gamma^1 \ , \\
&A_\mu \xrightarrow{P} A_\mu^P = (A_0, -A_1, A_2) \ , \\
&a_\mu \xrightarrow{P} a_\mu^P = (-a_0, a_1, -a_2) \ , \\
&\chi \xrightarrow{P} \chi^P = -\chi \ , \\
&\phi \xrightarrow{P} \phi^P = \phi \ .
\end{aligned} \tag{7.37}$$

where $\phi$ and $\chi$ are the fields $\phi = \{b, \overline{c}, c\}$ and $\chi = \{\pi, \overline{\xi}, \xi\}$.

## 7.4 Searching for counterterms

At this stage, we will analyze the stability of the model. In other words, we want to guarantee that perturbative corrections do not generate any local counterterms corresponding to renormalization of fields and parameters which are not already present in the classical theory, that is, the radiative corrections can be reabsorbed order by order through redefinitions of the initial physical quantities – fields, coupling constants and masses – of the model.

Our starting point is the expression:

$$\Sigma^{(s-1)}[\Phi_i, \rho_i, \lambda_i] + \varepsilon\Sigma^c[\Phi_i, \rho_i, \lambda_i] = \Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0] + \mathcal{O}(\varepsilon^2) \tag{7.38}$$

where $\Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0]$ is the perturbed classical action. We call perturbed action the classical action $\Sigma^{(s-1)}[\Phi_i, \rho_i, \lambda_i]$ written as a functional of the fields $\Phi_i$, external sources $\rho_i$ and the parameters $\lambda_i$ replaced by the respective expanded versions $\Phi_i^0$, $\rho_i^0$,$\lambda_i^0$, given by

$$\Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0], \quad \text{with } \Phi_i^0 = \Phi_i(1 + \varepsilon Z_{\Phi_i}) \ , \ \rho_i^0 = \rho_i(1 + \varepsilon Z_{\rho_i}) \text{ e } \lambda_i^0 = \lambda_i(1 + \varepsilon Z_{\lambda_i}) \tag{7.39}$$

The $\Sigma^c[\Phi_i, \rho_i, \lambda_i]$ is the counterterm action, such counterterm is find out by a set of symmetries and conditions imposed upon the classical action $\Sigma^{(s-1)}[\Phi_i, \rho_i, \lambda_i]$.

The perturbed action $\Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0]$, we normally just write $\Sigma^0$, obeys the same conditions that the classical action $\Sigma^{(s-1)}$. In this way we verify that:

$$\frac{\delta\Sigma^0}{\delta b} = \frac{\delta\Sigma^{(s-1)}}{\delta b} + \varepsilon\frac{\delta\Sigma^c}{\delta b} = \partial^\mu A_\mu + \alpha b \Longrightarrow \frac{\delta\Sigma^c}{\delta b} = 0 \ , \tag{7.40}$$

in a similar way it can be concluded that:

$$\frac{\delta\Sigma^c}{\delta\bar{c}} = \frac{\delta\Sigma^c}{\delta c} = \frac{\delta\Sigma^c}{\delta\pi} = \frac{\delta\Sigma^c}{\delta\bar{\xi}} = \frac{\delta\Sigma^c}{\delta\xi} \ , \tag{7.41}$$

$$W^{(e)}_{\text{rigid}}\Sigma^c = 0 \ , \quad \text{and} \quad W^{(g)}_{\text{rigid}}\Sigma^c = 0 \ . \tag{7.42}$$

The $\Sigma^c$ independence of the ghost fields $(c, \bar{c}, \xi, \bar{\xi})$ together with the fact that the action has $\Phi\Pi[\Sigma] = 0$ implying that the counterterms are external sources ( $\Phi\Pi[\Omega] = -1$) independent,

$$\frac{\delta\Sigma^c}{\delta\Omega_+} = \frac{\delta\Sigma^c}{\delta\Omega_-} = \frac{\delta\Sigma^c}{\delta\bar{\Omega}_+} = \frac{\delta\Sigma^c}{\delta\bar{\Omega}_-} = 0 \ . \tag{7.43}$$

The counterterm invariance under the Slavnov-Taylor operator is another important feature related to the UV and IR dimensions. Analysing the action of the operator on the counterterm, we can establish the UV and IR limit of field polynomials.

$$\begin{aligned}
\mathcal{S}_{\mathcal{F}}\Sigma^c = \int d^3x \Big\{ & \underbrace{-\frac{1}{e}\partial^\mu\Big|^{+1}_{+1} c\Big|^{0}_{0} \frac{\delta\Sigma^c}{\delta A^\mu}\Big|^{3-\frac{1}{2}}_{3-1}}_{UV\leq\frac{7}{2},\ IR\geq 3} + \underbrace{\frac{b}{e}\Big|^{+\frac{3}{2}}_{+1} \frac{\delta\Sigma^c}{\delta\bar{c}}\Big|^{3-1}_{3-1}}_{UV\leq\frac{7}{2},\ IR\geq 3} - \underbrace{\frac{1}{g}\partial^\mu\Big|^{+1}_{+1} \xi\Big|^{0}_{0} \frac{\delta\Sigma^c}{\delta a^\mu}\Big|^{3-\frac{1}{2}}_{3-1}}_{UV\leq\frac{7}{2},\ IR\geq 3} + \\
& + \underbrace{\frac{\pi}{g}\Big|^{+\frac{3}{2}}_{+1} \frac{\delta\Sigma^c}{\delta\bar{\xi}}\Big|^{3-1}_{3-1}}_{UV\leq\frac{7}{2},\ IR\geq 3} + \underbrace{\frac{\delta\Sigma^c}{\delta\bar{\Omega}_+}\Big|^{3-2}_{3-2} \frac{\delta\Sigma^c}{\delta\psi_+}\Big|^{3-1}_{3-1}}_{UV\leq 3,\ IR\geq 3} + \underbrace{\frac{\delta\Sigma^c}{\delta\psi_+}\Big|^{3-2}_{3-2} \frac{\delta\Sigma^c}{\delta\bar{\Omega}_+}\Big|^{3-2}_{3-2}}_{UV\leq 3,\ IR\geq 3} + \cdots \Big\} \ .
\end{aligned} \tag{7.44}$$

In (7.44) the superscripts refer to the UV dimensions and the subscripts to the IR dimensions, where we obtain the following UV and IR limits the counterterm has to satisfy $d \leq \frac{7}{2}$ and $r \geq 3$, respectively. Therefore, the Lorentz invariant counterterm, $\Sigma^c$, is formed by polynomials satisfying the following properties.

$$\Sigma^c = \Sigma^c[\bar{\psi}_+, \bar{\psi}_-, \psi_+, \psi_-, A_\mu, a_\mu] \ , \tag{7.45}$$

$$W^{(e)}_{\text{rigid}}\Sigma^c = 0 \quad \text{and} \quad W^{(g)}_{\text{rigid}}\ \Sigma^c = 0 \ , \tag{7.46}$$

$$\Sigma^c \xrightarrow{P} \Sigma^c \ , \tag{7.47}$$

$$\mathcal{S}_\Sigma\Sigma^c = 0 \ , \tag{7.48}$$

$$\Sigma^c\Big|^{d\leq\frac{7}{2}}_{r\geq 3} \ , \tag{7.49}$$

At this point we shall remember that the condition (7.47) is assumed having in mind the result presented in Chapter 6 and [130] where was showed that the BPHZL renormalization scheme does not break parity.

The most general Lorentz invariant polynomial observing the requirement (7.45) reads:

$$\begin{aligned}
\Sigma^c \;=\; & \int d^3x \; \{a_1 A^\mu A_\mu + a_2 a^\mu a_\mu + a_3 A^\mu a_\mu + \\
& + b_1 A^\mu A_\mu A^\nu A_\nu + b_2 a^\mu a_\mu a^\nu a_\nu + b_3 A^\mu A_\mu a^\nu a_\nu + b_4 A^\mu a_\mu A^\nu a_\nu + \\
& + c_1 \partial_\mu A^\mu a^\nu a_\nu + c_2 \partial_\mu a^\mu A^\nu A_\nu + c_3 \partial_\mu A^\mu A^\nu A_\nu + c_4 \partial_\mu a^\mu a^\nu a_\nu + \\
& + d_1 \partial_\mu A^\mu \partial_\nu A^\nu + d_2 \partial^\mu A^\nu \partial_\mu A_\nu + d_3 \partial^\mu A^\nu \partial_\nu A_\mu + \\
& + d_4 \partial_\mu a^\mu \partial_\nu a^\nu + d_5 \partial^\mu a^\nu \partial_\mu a_\nu + d_6 \partial^\mu A^\nu \partial_\nu a_\mu + \\
& + d_7 \partial_\mu A^\mu \partial_\nu a^\nu + d_8 \partial^\mu a^\nu \partial_\mu A_\nu + d_9 \partial^\mu a^\nu \partial_\nu a_\mu + \\
& + e_1 \epsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu + \epsilon^{\mu\rho\nu} a_\mu \partial_\rho a_\nu + \epsilon^{\mu\rho\nu} A_\mu \partial_\rho a_\nu + \\
& + f_1 \overline{\psi}_+ \psi_+ + f_2 \overline{\psi}_- \psi_- + f_3 \overline{\psi}_+ \psi_- + f_4 \overline{\psi}_- \psi_+ + f_5 \overline{\psi}_+ \gamma^\mu \partial_\mu \psi_+ \\
& + f_6 \overline{\psi}_- \gamma^\mu \partial_\mu \psi_- + f_7 \overline{\psi}_+ \gamma^\mu \partial_\mu \psi_- + f_8 \overline{\psi}_- \gamma^\mu \partial_\mu \psi_+ + \\
& + g_1 \overline{\psi}_+ \gamma^\mu A_\mu \psi_+ + g_2 \overline{\psi}_+ \gamma^\mu a_\mu \psi_+ + g_3 \overline{\psi}_- \gamma^\mu A_\mu \psi_- + g_4 \overline{\psi}_- \gamma^\mu a_\mu \psi_- + \\
& + g_5 \overline{\psi}_+ \psi_+ A^\mu A_\mu + g_6 \overline{\psi}_+ \psi_+ a^\mu a_\mu + g_7 \overline{\psi}_- \psi_- A^\mu A_\mu + g_8 \overline{\psi}_- \psi_- a^\mu a_\mu + \\
& + h_1 \overline{\psi}_+ \gamma^\mu A_\mu \psi_- + h_2 \overline{\psi}_+ \gamma^\mu a_\mu \psi_- + h_3 \overline{\psi}_+ A^\mu A_\mu \psi_- + h_4 \overline{\psi}_+ a^\mu a_\mu \psi_- + \\
& + h_5 \overline{\psi}_+ A^\mu a_\mu \psi_- + h_6 \overline{\psi}_- \gamma^\mu A_\mu \psi_+ + h_7 \overline{\psi}_- \gamma^\mu a_\mu \psi_+ + h_8 \overline{\psi}_- A^\mu A_\mu \psi_+ + \\
& + h_9 \overline{\psi}_- a^\mu a_\mu \psi_+ + h_{10} \overline{\psi}_- A^\mu a_\mu \psi_+\}. 
\end{aligned} \tag{7.50}$$

Applying now (7.46)-(7.49) we obtain:

$$\begin{aligned}
\Sigma^c \;=\; & \int d^3x \{f_6 (\overline{\psi}_+ \gamma^\mu \partial_\mu \psi_+ + ie \overline{\psi}_+ \gamma^\mu A_\mu \psi_+ + ig \overline{\psi}_+ \gamma^\mu a_\mu \psi_+) + \\
& + f_6 (\overline{\psi}_- \gamma^\mu \partial_\mu \psi_- + ie \overline{\psi}_- \gamma^\mu A_\mu \psi_- - ig \overline{\psi}_- \gamma^\mu a_\mu \psi_-) + \\
& + d_2 F^{\mu\nu} F_{\mu\nu} + d_5 f^{\mu\nu} f_{\mu\nu} + e_3 \epsilon^{\mu\rho\nu} A_\mu \partial_\rho a_\nu\} \; .
\end{aligned} \tag{7.51}$$

Adopting $f_6 = i\alpha_1$, $d_2 = \alpha_2$, $d_5 = \alpha_3$, $e_3 = \alpha_4$ and rewriting the expression (7.51) using the covariant derivatives definition, we get:

$$\Sigma^c = \int d^3x \{\alpha_1 i \overline{\psi}_+ \not{D} \psi_+ + \alpha_1 i \overline{\psi}_- \not{D} \psi_- + \alpha_2 F^{\mu\nu} F_{\mu\nu} + \alpha_3 f^{\mu\nu} f_{\mu\nu} + \alpha_4 \epsilon^{\mu\rho\nu} A_\mu \partial_\rho a_\nu\} \; . \tag{7.52}$$

Furthermore, taking into consideration the super-renormalizability of the model, its power counting [2] coupling constant dependency [105] is given by the formula:

$$\begin{pmatrix} d(\gamma) \\ r(\gamma) \end{pmatrix} = 3 - \sum_f \begin{pmatrix} d_f \\ r_f \end{pmatrix} N_f - \sum_b \begin{pmatrix} d_b \\ \frac{3}{2} r_b \end{pmatrix} N_b + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_e + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_g - N_{Aa} \; , \tag{7.53}$$

expressing the UV ($d(\gamma)$) and IR ($r(\gamma)$) degree of divergence of some 1-PI (one particle irreducible ) Feynman diagram ($\gamma$) having $N_f$ and $N_b$ as the number of fermions and

[2] See more in the Appendix B.

bosons external legs, respectively. The ultraviolet and infrared dimensions of the fermion fields ($d_f$, $r_f$) as well as the dimensions associated to the bosons fields ($d_b$, $r_b$) are displayed in Table **7.1**. $N_e$ and $N_g$ are the power of the coupling constants, $e$ and $g$, in $\gamma$ and $N_{Aa}$ counts the number of internal lines associated to the mixed propagator (7.10). Any loop is at least of order two in the coupling constant,$e^2$, $g^2$ or $eg$, consequently the counterterms are restricted by the following UV and IR constraints, $d \leq \frac{3}{2}$ and $r \geq 4$, respectively, implying that:

$$\Sigma^c = 0 \ . \tag{7.54}$$

Looking the stability condition (7.38) and replacing all respective fields $\Phi_i^0, \rho_i^0$ and parameters $\lambda_i^0$, using the following definitions:

$$\begin{aligned}
\psi_\pm^0(x) &= \psi_\pm(x)(1+\varepsilon Z_\pm) \ , \quad \Omega_\pm^0(x) = \Omega_\pm(x)(1+\varepsilon Z_{\Omega_\pm}) \ , \\
A^0(x) &= A(x)(1+\varepsilon Z_A) \ , \qquad b^0(x) = b(x)(1+\varepsilon Z_b) \ , \\
a^0(x) &= a(x)(1+\varepsilon Z_a) \ , \qquad \pi^0(x) = \pi(x)(1+\varepsilon Z_\pi) \ , \\
m^0 &= m(1+\varepsilon Z_m) \ , \qquad \xi^0(x) = \xi(x)(1+\varepsilon Z_\xi) \ , \\
\mu^0 &= \mu(1+\varepsilon Z_\mu) \ , \qquad c^0(x) = c(x)(1+\varepsilon Z_c) \ , \\
e^0 &= e(1+\varepsilon Z_e) \ , \qquad \alpha^0 = \alpha(1+\varepsilon Z_\alpha) \ , \\
g^0 &= g(1+\varepsilon Z_g) \ , \qquad \beta^0 = \beta(1+\varepsilon Z_\beta) \ ,
\end{aligned} \tag{7.55}$$

from the expanded action, $\Sigma^0[\Phi_i^0, \rho_i^0, \lambda_i^0]$, and the equation (7.38), we get

$$Z_\pm = Z_A = Z_a = Z_m = Z_\mu = Z_e = Z_g = Z_{\Omega_\pm} = Z_b = Z_\pi = Z_\xi = Z_c = Z_\alpha = Z_\beta = 0 \ , \tag{7.56}$$

We can assert an important result related to the coupling constants $e$, $g$ and the parameter $\mu$, its associated beta-functions $\beta_e = 0$, $\beta_g = 0$ and $\beta_\mu = 0$ vanish at all orders in perturbation theory, as well as, the same appointment about the anomalous dimensions of all the fields ($\gamma_{\phi_i} = 0$).

## 7.5 Searching for anomalies

The classical stability does not imply necessarily the possibility to extend the theory to the quantum level, it is still necessary to show that the model is absent from anomalies. This result combined to the previous one, namely the absence of counterterms (7.54), ensures the nullity of all $\beta$-functions and anomalous dimensions of all fields and absence of anomalies at all orders in perturbation theory.

At the quantum level the vertex functional, $\Gamma$, coincide with the classical action, $\Sigma$

(7.6), at zeroth order in $\hbar$,

$$\Gamma = \Sigma^{(s-1)} + \mathcal{O}(\hbar) \ , \tag{7.57}$$

and shall obey the same conditions of the classical action, see Eqs.(7.32)-(7.36), that is:

$$\begin{aligned}
-i\frac{\delta\Gamma}{\delta c} &= \Delta^{(1)}_{\text{class}} \ , & -\, i\frac{\delta\Gamma}{\delta\xi} &= \Delta^{(2)}_{\text{class}} \ . \\
\frac{\delta\Gamma}{\delta b} &= \partial^{\mu}A_{\mu} + \alpha b \ , & \frac{\delta\Gamma}{\delta\pi} &= \partial^{\mu}a_{\mu} + \beta\pi \ , \\
\frac{\delta\Gamma}{\delta\bar{c}} &= \Box c \ , & \frac{\delta\Gamma}{\delta\bar{\xi}} &= \Box\xi \ , \\
W^{(e)}_{\text{rigid}}\Gamma &= 0 \qquad \text{and} & W^{(g)}_{\text{rigid}}\Gamma &= 0 \ .
\end{aligned} \tag{7.58}$$

In accordance with the Quantum Action Principle [19, 91, 92], the Slavnov-Taylor identity (7.26) gets the following quantum break:

$$\mathcal{S}(\Gamma) = \Delta \cdot \Gamma = \Delta + \mathcal{O}(\hbar\Delta) \ , \tag{7.59}$$

where $\Delta$ is a local integrable functional,

$$\Delta = \int d^3x\mathcal{A} \tag{7.60}$$

with ghost number 1, UV limited by $d \leq \frac{7}{2}$ and IR by $r \geq 3$. To understand that, see the relation (7.59) which $\Gamma$ has $\Phi\Pi = 0$ whereas $\Phi\Pi[\mathcal{S}] = 1$.

The nilpotent identity (7.29) together with the fact we expand the functional $\mathcal{S}_\Gamma$ in terms $\mathcal{S}_\Sigma$, we find in first order:

$$\mathcal{S}_\Gamma = \mathcal{S}_\Sigma + \mathcal{O}(\hbar) \ , \tag{7.61}$$

and therefore imply,

$$\mathcal{S}_\Sigma\Delta = 0 \ . \tag{7.62}$$

Besides that, other restrictions on $\Delta$ resulting from the algebra involving $\mathcal{S}$ and $\mathcal{S}_\mathcal{F}$ acting upon some functional $\mathcal{F}$:

$$-i\int d^3x\frac{\delta\mathcal{S}(\mathcal{F})}{\delta c}+\mathcal{S}_{\mathcal{F}}\int d^3x\left(-i\frac{\delta\mathcal{F}}{\delta c}-\Delta^{(1)}_{\rm class}\right)=W^e_{\rm rig}\mathcal{F}\ , \tag{7.63}$$

$$-i\int d^3x\frac{\delta\mathcal{S}(\mathcal{F})}{\delta \xi}+\mathcal{S}_{\mathcal{F}}\int d^3x\left(-i\frac{\delta\mathcal{F}}{\delta \xi}-\Delta^{(2)}_{\rm class}\right)=W^g_{\rm rig}\mathcal{F}\ , \tag{7.64}$$

$$\frac{\delta\mathcal{S}(\mathcal{F})}{\delta b}-\mathcal{S}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta b}-\partial^\mu A_\mu-\alpha b\right)=\frac{1}{e}\left(\frac{\delta\mathcal{F}}{\delta\bar{c}}-\Box c\right)\ , \tag{7.65}$$

$$\frac{\delta\mathcal{S}(\mathcal{F})}{\delta \pi}-\mathcal{S}_{\mathcal{F}}\left(\frac{\delta\mathcal{F}}{\delta \pi}-\partial^\mu a_\mu-\beta \pi\right)=\frac{1}{g}\left(\frac{\delta\mathcal{F}}{\delta\bar{\xi}}-\Box \xi\right)\ , \tag{7.66}$$

$$\frac{\delta\mathcal{S}(\mathcal{F})}{\delta\bar{c}}-\mathcal{S}_{\mathcal{F}}\frac{\delta\mathcal{F}}{\delta\bar{c}}=0 \qquad \text{and} \qquad \frac{\delta\mathcal{S}(\mathcal{F})}{\delta\bar{\xi}}-\mathcal{S}_{\mathcal{F}}\frac{\delta\mathcal{F}}{\delta\bar{\xi}}=0\ , \tag{7.67}$$

$$W^{(e)}_{\rm rig}\mathcal{S}(\mathcal{F})-\mathcal{S}_{\mathcal{F}}W^{(e)}\mathcal{F}=0 \quad \text{and} \quad W^{(g)}_{\rm rig}\mathcal{S}(\mathcal{F})-\mathcal{S}_{\mathcal{F}}W^{(g)}\mathcal{F}=0\ . \tag{7.68}$$

Therefore, with the relations (7.63)-(7.68) now acting on $\Gamma$ and observing the Quantum Action Principle (7.59) it provides us the following conditions on $\Delta$:

$$\begin{aligned}
&\frac{\delta\Delta}{\delta b}=\frac{\delta\Delta}{\delta\bar{c}}=0\ ,\quad \int d^3x\frac{\delta\Delta}{\delta c}=0 \quad \text{and} \quad W^{(e)}_{\rm rigid}\Delta=0\ ,\\
&\frac{\delta\Delta}{\delta \pi}=\frac{\delta\Delta}{\delta\bar{\xi}}=0\ ,\quad \int d^3x\frac{\delta\Delta}{\delta \xi}=0 \quad \text{and} \quad W^{(g)}_{\rm rigid}\Delta=0\ .
\end{aligned} \tag{7.69}$$

This ensures that $\Delta$ is not an explicit function of the fields $\{b,\pi,\bar{c}$ and $\bar{\xi}\}$ as well as invariant under the rigid symmetries, $W^{(e)}_{\rm rigid}$ and $W^{(g)}_{\rm rigid}$. The last conditions is a start point to find the explicit form of the quantum break.

The Wess-Zumino condition (7.62) constitutes a cohomology problem in the sector ghost number one. The solution can always be splitted in two parts, $\mathcal{S}_\Sigma\widehat{\Delta}^{(0)}$ (where $\widehat{\Delta}^{(0)}$ has the ghost number zero) and nontrivial elements belonging to the cohomology of $\mathcal{S}_\Sigma$ (7.28) in the sector of ghost number one:

$$\Delta^{(1)}=\widehat{\Delta}^{(1)}+\mathcal{S}_\Sigma\widehat{\Delta}^{(0)}\ . \tag{7.70}$$

Thereby $\Delta^{(1)}$ satisfies (7.62) and (7.69) and the $\mathcal{S}_\Sigma\widehat{\Delta}^{(0)}$ can always be absorbed into the vertex functional $\Gamma$ like a integrable local non-invariant counterterm $-\widehat{\Delta}^{(0)}$.

$$\mathcal{S}_\Sigma(\Gamma-\widehat{\Delta}^{(0)})=\widehat{\Delta}^{(1)}\ . \tag{7.71}$$

Taking in account the Slavnov-Taylor operator $\mathcal{S}_\Sigma$ (7.28) and the quantum breaking (7.59) we can see that $\Delta^{(1)}$ must have UV dimensions bounded by $d\leq\frac{7}{2}$ and IR dimension bounded by $r \geq 3$. Again, the radiative corrections are at least of one loop or equivalent order two in the coupling constants $e^2$, $g^2$ and $eg$, so by looking at the power counting formula (7.53) we easily conclude that the UV and IR dimensions are bounded by $d \leq \frac{5}{2}$ and $r \geq 4$.

Returning to the conditions (7.69) we find:

$$\int d^3x \frac{\delta\Delta^{(1)}}{\delta c} = 0 \quad \text{and} \quad \int d^3x \frac{\delta\Delta^{(1)}}{\delta\xi} = 0 \; . \tag{7.72}$$

We conclude that the terms should be linear in derivative of the ghost fields, $c$ e $\xi$, that is:

$$\Delta^{(1)} = \int d^3x \, \left(\mathcal{K}^{(0)}_\mu \partial^\mu c + \mathcal{X}^{(0)}_\mu \partial^\mu \xi\right) \; , \tag{7.73}$$

where $\mathcal{K}^{(0)}_\mu$ and $\mathcal{X}^{(0)}_\mu$ are rank one tensors with zero ghost number ($\Phi\Pi = 0$), with UV and IR limited by $d \leq \frac{3}{2}$ and $r \geq 3$. We still can write the breaking $\Delta^{(1)}$ in two parts, one odd and the other even under parity symmetry:

$$\mathcal{K}^{(0)}_\mu = \sum_{i=1} v_{k,i}\mathcal{V}^i_\mu + \sum_{i=1} p_{k,i}\mathcal{P}^i_\mu \quad \text{and} \quad \mathcal{X}^{(0)}_\mu = \sum_{i=1} v_{x,i}\Upsilon^i_\mu + \sum_{i=1} p_{x,i}\Pi^i_\mu \; , \tag{7.74}$$

where $v_{k,i}$, $p_{k,i}$, $v_{x,i}$ and $p_{x,i}$ being coefficients to be determined. Moreover, $\mathcal{V}^i_\mu$ and $\Upsilon^i_\mu$ was defined as vectors, whereas $\mathcal{P}^i_\mu$ and $\Pi^i_\mu$ as pseudo-vectors, in such way $\mathcal{V}^i_\mu\partial^\mu c$ and $\Pi^i_\mu\partial^\mu\xi$ are even, and $\mathcal{P}^i_\mu\partial^\mu c$ and $\Upsilon^i_\mu\partial^\mu\xi$ odd under parity.

In this case, for the anomaly analysis here, we get $\{\mathcal{P}^i_\mu\} = \emptyset$ and $\{\Upsilon^i_\mu\} = \emptyset$, lefting behind just even terms for the quantum Slavnov-Taylor $\Delta^{(1)}$ (7.73).

$$\Delta^{(1)}_{\text{even}} = \int d^3x \Big\{ \sum_{i=1} v_{k,i} \; \mathcal{V}^i_\mu \partial^\mu c + \sum_{i=1} p_{x,i} \; \Pi^i_\mu \partial^\mu \xi \Big\} \; . \tag{7.75}$$

It still remains to analyze the parity even breaking $\Delta^{(1)}_{\text{even}}$ (7.75) if it is a genuine gauge anomaly or some non invariant counterterm absorbable into the quantum action. However, it is necessary to find out the possible terms for $\mathcal{V}_\mu$ (vectors) and $\Pi_\mu$ (pseudovectors) limited by $d \leq \frac{3}{2}$ and $r \geq 3$ which left $\Delta^{(1)}_{\text{even}}$ (7.75) satisfying (7.62) and (7.69). The possible candidates are:

$$\begin{aligned} &\mathcal{V}^1_\mu = A_\mu A^\nu A_\nu \; , \quad \mathcal{V}^2_\mu = A_\mu a^\nu a_\nu \; , \quad \mathcal{V}^3_\mu = A_\nu a^\nu a_\mu \; , \\ &\Pi^1_\mu = a_\mu a^\nu a_\nu \; , \quad \Pi^2_\mu = a_\mu A^\nu A_\nu \; , \quad \Pi^3_\mu = a_\nu A^\nu A_\mu \; , \end{aligned} \tag{7.76}$$

We can write the quantum breaking $\Delta^{(1)}_{\text{even}}$ (7.75) as:

$$\begin{aligned} \Delta^{(1)}_{\text{even}} = \int d^3x \Big\{ \; & v_{k,1} A_\mu A^\nu A_\nu \partial^\mu c + v_{k,2} A_\mu a^\nu a_\nu \partial^\mu c + v_{k,3} A_\nu a^\nu a_\mu \partial^\mu c + \\ & + p_{x,1} a_\mu a^\nu a_\nu \partial^\mu \xi + p_{x,2} a_\mu A^\nu A_\nu \partial^\mu \xi + p_{x,3} a_\nu A^\nu A_\mu \partial^\mu \xi \Big\} \; . \end{aligned} \tag{7.77}$$

If a gauge anomaly generated by radiative corrections exists or not should be proved by analysing the quantum breaking that is what says the Wess-Zumino condition (7.62). Consequently, if the breaking can be expressed as a trivial cocycle, $\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)}$ where $\widehat{\Delta}^{(0)}$ being a local integrable monomial with ghost number zero,i.e, a non-invariant counterterm, thereby we do not have gauge anomaly and the unitarity is ensured. Otherwise, if exist just one nontrivial element $\widehat{\Delta}^{(1)}$ belonging to the cohomology $\mathcal{S}_{\Gamma^{(0)}}$ (7.28) in the sector of ghost number one, the gauge symmetry is anomalous and the unitarity is compromised. Following this argument to solve the gauge anomaly problem we can check that:

$$
\begin{aligned}
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)1} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ A^\mu A_\mu A^\nu A_\nu = -\frac{4}{e}\int d^3x\ A_\mu A^\nu A_\nu \partial^\mu c\ ,\\
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)2} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ a^\mu a_\mu a^\nu a_\nu = -\frac{4}{g}\int d^3x\ a_\mu a^\nu a_\nu \partial^\mu \xi\ ,\\
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)3} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ A^\mu A_\mu a^\nu a_\nu = -\frac{2}{e}\int d^3x\ A_\mu a^\nu a_\nu \partial^\mu c - \frac{2}{g}\int d^3x\ a_\mu A^\nu A_\nu \partial^\mu \xi\ ,\\
\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)4} &= \mathcal{S}_{\Gamma^{(0)}}\int d^3x\ A^\mu A_\nu a^\nu a_\mu = -\frac{2}{e}\int d^3x\ A_\nu a^\nu a_\mu \partial^\mu c - \frac{2}{g}\int d^3x\ a_\nu A^\nu A_\mu \partial^\mu \xi\ .
\end{aligned}
\tag{7.78}
$$

Moreover, considering the four cocycle above (7.78), it follows that the quantum break (7.75) assume the form:

$$
\begin{aligned}
\Delta^{(1)}_{\text{even}} &= \mathcal{S}_{\Gamma^{(0)}}\left\{\lambda_1\widehat{\Delta}^{(0)1} + \lambda_2\widehat{\Delta}^{(0)2} + \lambda_3\widehat{\Delta}^{(0)3} + \lambda_4\widehat{\Delta}^{(0)4}\right\}\\
&= \mathcal{S}_{\Gamma^{(0)}}\left\{\lambda_1\!\int d^3x\ A^\mu A_\mu A^\nu A_\nu + \lambda_2\!\int d^3x\ a^\mu a_\mu a^\nu a_\nu + \lambda_3\!\int d^3x\ A^\mu A_\mu a^\nu a_\nu + \lambda_4\!\int d^3x\ A^\mu A_\nu a^\nu a_\mu\right\}\\
&= \int d^3x\Big\{-\frac{4}{e}\lambda_1 A_\mu A^\nu A_\nu \partial^\mu c - \frac{2}{e}\lambda_3 A_\mu a^\nu a_\nu \partial^\mu c - \frac{2}{e}\lambda_4 A_\nu a^\nu a_\mu \partial^\mu c +\\
&\qquad - \frac{4}{g}\lambda_2 a_\mu a^\nu a_\nu \partial^\mu \xi - \frac{2}{g}\lambda_3 a_\mu A^\nu A_\nu \partial^\mu \xi - \frac{2}{g}\lambda_4 a_\nu A^\nu A_\mu \partial^\mu \xi\Big\}\ ,
\end{aligned}
\tag{7.79}
$$

where it is verified:

$$
\begin{aligned}
v_{k,1} &= -\frac{4}{e}\lambda_1\ , \quad v_{k,2} = -\frac{2}{e}\lambda_3\ , \quad v_{k,3} = -\frac{2}{e}\lambda_4\ ,\\
p_{x,1} &= -\frac{4}{g}\lambda_2\ , \quad p_{x,2} = -\frac{2}{g}\lambda_3\ , \quad p_{x,3} = -\frac{2}{g}\lambda_4\ .
\end{aligned}
\tag{7.80}
$$

In this way, we finally show that the quantum breaking $\Delta^{(1)}_{\text{even}}$ (7.75) is really a trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)}$:

$$
\Delta^{(1)}_{\text{even}} = \mathcal{S}_{\Gamma^{(0)}}\widehat{\Delta}^{(0)} = \mathcal{S}_{\Gamma^{(0)}}\left\{\lambda_1\widehat{\Delta}^{(0)1} + \lambda_2\widehat{\Delta}^{(0)2} + \lambda_3\widehat{\Delta}^{(0)3} + \lambda_4\widehat{\Delta}^{(0)4}\right\}\ . \tag{7.81}
$$

Therefore, as a final result, the local integrable monomials $\widehat{\Delta}^{(0)1}$, $\widehat{\Delta}^{(0)2}$, $\widehat{\Delta}^{(0)3}$ e $\widehat{\Delta}^{(0)4}$ can be absorbed as non-invariant counterterms order by order in the quantum action, that is, in order $n$ of $\hbar$:

$$\mathcal{S}_{\Gamma^{(0)}}(\Gamma-\hbar^n\widehat{\Delta}^{(0)})\equiv\mathcal{S}_{\Gamma^{(0)}}\left(\Gamma-\hbar^n\lambda_1\widehat{\Delta}^{(0)1}-\hbar^n\lambda_2\widehat{\Delta}^{(0)2}-\hbar^n\lambda_3\widehat{\Delta}^{(0)3}-\hbar^n\lambda_4\widehat{\Delta}^{(0)4}\right)=0\hbar^n+\mathcal{O}(\hbar^{n+1})\,. \tag{7.82}$$

Here, we have an important result, the proof of the gauge anomaly absence independently of the loop order, that is, meaning that the local symmetry $U_A(1) \times U_a(1)$ is not an anomaly at the quantum level.

Notwithstanding the absence of the gauge anomaly does not show the complete consistency of the model at the quantum level, a more serious question, now a physical one, can arise when massless fields are present and propagate any freedom degree, the infrared anomaly. To finalize this question it is necessary to check the UV and IR dimensions of the noninvariant counterterms.

$$\begin{aligned}
\widehat{\Delta}^{(0)1} &= \int d^3x\ A^\mu A_\mu A^\nu A_\nu\ ,\quad d=2\ ,\ r=4\ ,\\
\widehat{\Delta}^{(0)2} &= \int d^3x\ a^\mu a_\mu a^\nu a_\nu\ ,\quad d=2\ ,\ r=4\ ,\\
\widehat{\Delta}^{(0)3} &= \int d^3x\ A^\mu A_\mu a^\nu a_\nu\ ,\quad d=2\ ,\ r=4\ ,\\
\widehat{\Delta}^{(0)4} &= \int d^3x\ A^\mu A_\nu a^\nu a_\mu\ ,\quad d=2\ ,\ r=4\ ,
\end{aligned} \tag{7.83}$$

At each loop order, the non invariant counterterms are absorbed into quantum action $\Gamma$ in order to restore the gauge symmetry order by order. However, due to the presence of massless fields, the non invariant counterterms could introduce IR divergences, namely the infrared anomaly, spoiling the renormalizability of the model. Finally, we can sse that the noinvariant counterterms (7.83) satisfy the IR constraint, $r \leq 2$, consequently no infrared anomaly shows up.

## 7.6 Conclusion

In summary, the last result (7.82), about the consistency of the Wess-Zumino condition, combined with the previous one (7.54), the stability analysis, completes the proof about the vanishing of the $\beta$-functions associated to the coupling constants ($\beta_e = 0$ and $\beta_g = 0$), the mass parameter ($\beta_\mu$) and the anomalous dimensions of all fields ($\gamma_{\phi_i} = 0$), as well as the absence of parity and gauge anomalies at all orders of perturbation theory. As a final conclusion, the $U_A(1) \times U_a(1)$ hybrid QED$_3$ is ultraviolet and infrared finite at all orders in perturbation theory [130].

# Chapter 8

# Final conclusions

In Chapter 3, the Lorentz invariant parity-preserving $U(1) \times U(1)$ massive $\mathrm{QED}_3$, a mass-gap graphene-like planar quantum electrodynamics model, at low-energy limit exhibits electron-polaron–electron-polaron scattering short range non confining potentials, similarly it can be concluded that the same behaviour takes place for hole-polaron–hole-polaron scatterings. The interactions among electron-polarons and hole-polarons are mediated by two massive vector mesons, the photon (electric charge source) and the Néel quasiparticle (chiral charge source), both stemming from the $U(1)_A \times U(1)_a$ gauge symmetry. It should be noticed that it was disclosed the correlation among the electron-polaron (hole-polaron) spin polarization and correspondent chiral charge. At the tree-level, the absence of tachyons ($k^2 < 0$) and ghosts ($\langle\psi|\psi\rangle < 0$) in the model spectrum guarantees causality and unitarity, respectively, at this level. Notwithstanding, in order to complete the tree-level unitarity analysis, it remains to finish the proof that the scattering cross sections in the limit of high center of mass energies respect the Froissart-Martin bound [53, 61, 62], but since ultraviolet problems are less critical in lower dimensional quantum field models, together with the fact that the four space-time dimensional QED ($\mathrm{QED}_4$) [59] satisfies the Froissart-Martin bound, consequently this fulfilment shall be foreseen for parity-even $U(1) \times U(1)$ massive $\mathrm{QED}_3$. Also, it shall be pointed out that for condensed matter systems like graphene, the quasiparticles (electron-polaron and hole-polaron) dynamics is in non relativistic regime, so ultraviolet unitarity upper bound violations should not be expected.

Bearing in mind hypothetical applications of the model presented here to graphene, or any other two dimensional system, the orders of magnitude of some theoretical parameters need to be established firstly, namely, a typical mass-gap in graphene is around meV [45–49] whereas the low-energy limit for a condensed matter system is of eV order. In addition to that, the characteristic range of the two interactions, mediated by the both massive photon and the Néel quantum, shall be associated to the pair-coherence length measured in graphene, orders of magnitude in nm [65]. The mass-gap in graphene [45–49], besides of being more realistic, can be either achieved when pure graphene monolayer is settled on substrates [11], increasing its application range and improving device developments.

At the low-energy limit, the non relativistic electron-polaron–electron-polaron (or hole-polaron–hole-polaron) scattering potential, owing to photon and Néel quasiparticle short range exchanges, shows to be always repulsive (5.46) for parallel ($p$-wave) electron-polaron (hole-polaron) spin polarizations ($|\uparrow\rangle + |\uparrow\rangle$ or $|\downarrow\rangle + |\downarrow\rangle$). Nevertheless, for electron-polaron–electron-polaron (or hole-polaron–hole-polaron) scatterings with antiparallel ($s$-wave) spin polarizations ($|\uparrow\rangle + |\downarrow\rangle$), the $s$-wave interaction potential might be attractive provided $e^-(e^+)$-polaron–Néel-quasiparticle coupling strength ($|g|$) be stronger than the strength of $e^-(e^+)$-polaron–photon coupling ($|e|$), $g^2 > e^2$. Moreover, the $s$-wave attractive scattering potential satisfies the Kato condition [55], the Newton-Setô and the Bargmann upper bounds [56, 58], indicating that $s$-wave bipolarons [50–52] might stem from these electron-polaron–electron-polaron quasiparticles bound states [64]. The possible emergence of such a Cooper-type $e^-$-polaron–$e^-$-polaron condensate (bipolaron) directly calls the issue of superconductivity in graphene [66–68], thus a deep investigation on that deserves special attention.

The model presented in Chapter 3, the parity-even $U_A(1) \times U_a(1)$ massive QED$_3$ [22], has been shown in Chapter 4 to be free from any gauge anomaly and parity anomaly at all orders in perturbation theory. Beyond that, it exhibits vanishing $\beta$-functions associated to the gauge coupling constants ($e$ and $g$) and the Chern-Simons mass parameter ($\mu$), and all the anomalous dimensions ($\gamma$) of the fields as well. The proof is independent of particular diagrammatic calculations or regularization schemes, since the BRS (Becchi-Rouet-Stora) algebraic renormalization method together with the BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) subtraction scheme [19, 86–96] is grounded in the general theorems of perturbative quantum field theory. Furthermore, once the quantum perturbative physical consistency of the mass-gap graphene-like planar quantum electrodynamics has been proven from the results demonstrated here together with those presented in [22], it should be newsworthy to deepen its analysis so as to apply in graphene-like electronic systems [108]. As a final comment, the vanishing of all $\beta$-functions associated to the electric charge ($e$), the pseudochiral charge ($g$) and the Chern-Simons mass parameter ($\mu$) – with the exception of that associated to the fermions mass parameter ($m$) – foresees the independence of those system parameters with respect to the temperature, on the other hand for instance the mass-gap in graphene, which can be described by the fermions mass parameter, shall be temperature dependent.

The model proposed in Chapter 5, a gapless pristine graphene-like planar quantum electrodynamics model, the parity-preserving $U(1) \times U(1)$ massless QED$_3$, exhibits two-fermion scattering short range non confining potentials originated by two massive vector-mediated quanta, the photon (electric charge) and the Néel (chiral charge) quasiparticle, both stemming from the gauging of the $U(1) \times U(1)$ global symmetry. At the tree-level, the absence in the spectrum of tachyons ($k^2 < 0$) and ghosts ($\langle\psi|\psi\rangle < 0$) assures respectively, causality and unitarity, at this level. Additionally, the charges of the quasiparticles (electron-polaron, hole-polaron, photon and Néel quasiparticles), their masses, degrees of freedom and (pseudo)spin are determined and discussed. As a by-product, it is obtained the four-fold broken degeneracy of the Landau levels, reminding those experimentally observed in pristine graphene subjected to high external magnetic fields [114–129], moreover, the system presents zero-energy Landau level suggesting a kind of anomalous quantum Hall effect – detailing results and discussions shall appear further [131].

The $p$-wave state fermion–fermion (or antifermion–antifermion) scattering potential shows to be repulsive (5.46) whatever the values of the electric ($e$) and chiral ($g$) charges. Nevertheless, for $s$-wave scattering of fermion–fermion (or antifermion–antifermion), the interaction potential (5.45) might be attractive provided $g^2 > e^2$. In summary, if two electron-polarons (or hole-polarons) lie in the inequivalent $\mathbf{K}$ and $\mathbf{K}'$ points in the Brillouin zone the interaction might be attractive, otherwise the interaction is always repulsive if those two electron-polarons (or hole-polarons) rest both either in $\mathbf{K}$ or in $\mathbf{K}'$ points.

In view of possible applications of this quantum electrodynamics three space-time dimensional model to pristine (gapless) graphene, or any other planar system, the orders of magnitude of some theoretical parameters have to be estimated. The typical energy-scale in graphene – for instance $E = v_F|\vec{p}|$, $C_s = \frac{1}{2\pi}(e^2 - g^2)$ or $C_p = \frac{1}{2\pi}(e^2 + g^2)$ – is around meV [16, 38–44, 111], while the length-scale interaction $\lambda\!\!\!/ = \frac{2\pi\hbar}{\mu c}$ – the reduced Compton wavelength of the quantum-mediated photon and Néel massive quasiparticles – is orders of magnitude in nm [65].

To end this conclusions, it is in progress the proof, analogously to the relativistic massive case [140], if whether the attractive $s$-wave scattering potential can lead to bound states, that is, if the potential (5.45), provided $g^2 > e^2$, could favour $s$-wave massless bipolarons. The possible emergence of such a kind of Cooper-type electron-polaron–electron-polaron (hole-polaron–hole-polaron) condensate draws attention to superconductivity in graphene [66–68].

In Chapter 6, the model presented in Chapter 5, the massless parity-even $U(1) \times U(1)$ planar quantum electrodynamics ($\mathrm{QED}_3$) model [109] exhibits quantum parity conservation at all orders in perturbation theory. The proof has been performed using the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) renormalization method, however owing to the presence of two massless fermions in the spectrum, infrared divergences might emerge in the course of the ultraviolet divergences subtractions and must be subtracted as well, for this reason, the Lowenstein-Zimmermann (LZ) subtraction scheme has been adopted. The power-counting – the ultraviolet and infrared superficial degrees of divergence (6.9) of any 1-particle irreducible Feynman diagram – reveals that ultraviolet divergences are bounded at most to two loops. At one loop all six vacuum-polarization tensor diagrams are linear ultraviolet divergent, four of the six self-energy diagrams are logarithm ultraviolet divergent, while all the vertex-function diagrams are ultraviolet finite, beyond that at two loops, twenty four of the thirty six vacuum-polarization tensor

Feynman graphs are ultraviolet divergent (Fig. 6.1 and Table 6.2). Although there are counterterms at one and two loops, none of them violate parity symmetry and together to the fact that the model is superrenormalizable, it stems as a byproduct that parity is guaranteed at any radiative order. As a final conclusion, for the model presented in this work, opposite to the case of the ordinary massless parity-even $U(1)$ QED$_3$ [77], the BPHZL subtraction scheme with the Lowenstein's adaptation of the Zimmermann's forest formula [95, 96] preserves parity symmetry at all perturbative order.

Finishing our conclusions, still with the model introduced in Chapter 5, there in the Chapter 7 some results were presented. The parity-even $U_A(1) \times U_a(1)$ hybrid QED$_3$ [109] is free from any gauge anomaly and parity anomaly at all orders in perturbation theory. Besides that, it exhibits vanishing $\beta$-functions associated to the gauge coupling constants ($e$ and $g$) and the Chern-Simons mass parameter ($\mu$), and all the anomalous dimensions ($\gamma$) of the fields as well, foresseing by the way, the independence of those system parameters with respect to the temperature.It shall be stressed that the proof is independent of particular diagrammatic calculations or regularization schemes, since the BRS (Becchi-Rouet-Stora) algebraic renormalization method together with the BPHZL (Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein) subtraction scheme [19, 86–96] is grounded on the general theorems of perturbative quantum field theory. Addionaly, indispensable for models with the presence of massless fields is the analysis about infrared anomalies spoiling the renormalizability of the model. However, as proved, the $U_A(1) \times U_a(1)$ hybrid QED$_3$ is infrared anomaly free and ultraviolet and infrared finite at all orders in perturbation theory [130].

# Appendix A

# The terms of the amplitude scattering

In this short appendix is displayed the terms used to the calculus of the amplitude scatterings.

Direct scattering spin $(+)$:

$$\begin{aligned}
\overline{u}_{+}(p_1')\gamma^0 u_{+}(p_1) = &\ \tfrac{(E+m)^2+p^2e^{i\phi}}{2m(E+m)} &= \overline{u}_{+}(p_2')\gamma^0 u_{+}(p_2)\ ,\\
\overline{u}_{+}(p_1')\gamma^1 u_{+}(p_1) = &\ \tfrac{p(1+e^{i\phi})}{2m} &= -\overline{u}_{+}(p_2')\gamma^1 u_{+}(p_2)\ ,\\
\overline{u}_{+}(p_1')\gamma^2 u_{+}(p_1) = &\ \tfrac{ip(1-e^{i\phi})}{2m} &= -\overline{u}_{+}(p_2')\ .\gamma^2 u_{+}(p_2)
\end{aligned} \tag{A.1}$$

Direct scattering spin $(-)$:

$$\begin{aligned}
\overline{u}_{-}(p_1')\gamma^0 u_{-}(p_1) = &\ \tfrac{(E+m)^2+p^2e^{-i\phi}}{2m(E+m)} &= \overline{u}_{-}(p_2')\gamma^0 u_{-}(p_2)\ ,\\
\overline{u}_{-}(p_1')\gamma^1 u_{-}(p_1) = &\ \tfrac{p(1+e^{-i\phi})}{2m} &= -\overline{u}_{-}(p_2')\gamma^1 u_{-}(p_2)\ ,\\
\overline{u}_{-}(p_1')\gamma^2 u_{-}(p_1) = &\ \tfrac{-ip(1-e^{-i\phi})}{2m} &= -\overline{u}_{-}(p_2')\gamma^2 u_{-}(p_2)\ .
\end{aligned} \tag{A.2}$$

Transverse scattering spin $(+)$:

$$\begin{aligned}
\overline{u}_{+}(p_2')\gamma^0 u_{+}(p_1) = &\ \tfrac{(E+m)^2-p^2e^{i\phi}}{2m(E+m)} &= \overline{u}_{+}(p_1')\gamma^0 u_{+}(p_2)\ ,\\
\overline{u}_{+}(p_2')\gamma^1 u_{+}(p_1) = &\ \tfrac{p(1-e^{i\phi})}{2m} &= -\overline{u}_{+}(p_1')\gamma^1 u_{+}(p_2)\ ,\\
\overline{u}_{+}(p_2')\gamma^2 u_{+}(p_1) = &\ \tfrac{ip(1+e^{i\phi})}{2m} &= -\overline{u}_{+}(p_1')\gamma^2 u_{+}(p_2)\ .
\end{aligned} \tag{A.3}$$

Transverse scattering spin $(-)$:

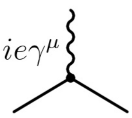
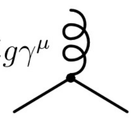
$$\begin{aligned}
\overline{u}_{-}(p_2')\gamma^0 u_{-}(p_1) = &\ \tfrac{(E+m)^2-p^2e^{-i\phi}}{2m(E+m)} &= \overline{u}_{-}(p_1')\gamma^0 u_{-}(p_2)\ ,\\
\overline{u}_{-}(p_2')\gamma^1 u_{-}(p_1) = &\ \tfrac{p(1-e^{-i\phi})}{2m} &= -\overline{u}_{-}(p_1')\gamma^1 u_{-}(p_2)\ ,\\
\overline{u}_{-}(p_2')\gamma^2 u_{-}(p_1) = &\ \tfrac{-ip(1+e^{-i\phi})}{2m} &= -\overline{u}_{-}(p_1')\gamma^2 u_{-}(p_2)\ .
\end{aligned} \tag{A.4}$$

# Appendix B

# Power counting

The power counting theorem establishes asymptotically upper and lower limits for the calculations of possible divergent graphs, in this manner, to known their form is an indispensable tool in the renormalization programs.

In this section, we will establish the power counting relation for the massless model present in the chapters 3,4 and 5 in the asymptotic limits $k \to \infty$ and $k \to 0$ knowing as the UV $(d)$ and IR$(r)$ asymptotic limits for the Feynman diagrams. We start presenting the forms of the vertices interaction together with its Feynman rules,

$V_{\pm A^{\mu} \pm} \equiv$ $ie\gamma^{\mu}$  , $V_{\pm a^{\mu} \pm} \equiv$ $\pm ig\gamma^{\mu}$  . (B.1)

In this way, we have for the fields, the following contributions: The spinorial field,

$$
\begin{aligned}
\psi_{\pm} \quad &: \quad I_{\overline{\psi}_{\pm}\psi_{\pm}}(\text{ 2 spinorial}) \ , \\
&: \quad E_{\overline{\psi}_{\pm}\psi_{\pm}}(\text{ 1 spinorial}) \ .
\end{aligned}
\tag{B.2}
$$

The vectorial field $A_{\mu}$ ,

$$
\begin{aligned}
A_{\mu} \quad &: \quad I_{AA}(\text{ 2 vectorial}) \ , \\
&: \quad I_{Aa}(\text{ 1 vectorial}) \ , \\
&: \quad E_{AA}(\text{ 1 vectorial}) \ .
\end{aligned}
\tag{B.3}
$$

The vectorial field $a_{\mu}$,

$$
\begin{aligned}
a_{\mu} \quad &: \quad I_{aa}(\text{ 2 vectorial}) \ , \\
&: \quad I_{aA}(\text{ 1 vectorial}) \ , \\
&: \quad E_{aa}(\text{ 1 vectorial}) \ .
\end{aligned}
\tag{B.4}
$$

We must to notice is, every vertice with $N$ external legs when connected in some way to form a Feynman graph, the external line $E$ needs two internal lines and it is equal to the number of fields $\phi$ times the number of vertices. A mathematical expression is:

$$2I_\phi + 1E_\phi = Z_\phi V_\phi \ , \tag{B.5}$$

Applying to our model,

$$\begin{aligned}
\psi_+ \quad &\longrightarrow 2I_{\overline{\psi}_+\psi_+} + 1E_{\overline{\psi}_+\psi_+} &&= 2V_{+A+} + 2V_{+a+} \ , \\
\psi_- \quad &\longrightarrow 2I_{\overline{\psi}_-\psi_+} + 1E_{\overline{\psi}_-\psi_-} &&= 2V_{-A-} + 2V_{-a-} \ , \\
A_\mu \quad &\longrightarrow 2I_{AA} + 1E_{AA} + I_{Aa} &&= 1V_{+A+} + 1V_{-A-} \ , \\
a_\mu \quad &\longrightarrow 2I_{aa} + 1E_{aa} + I_{Aa} &&= 1V_{+a+} + 1V_{-a-} \ ,
\end{aligned} \tag{B.6}$$

On the left side of equality (B.6) we wrote how the fields are seeing by the external and internal lines and on the right side by the vertices. Now, using the useful topological relation of the number of loops $L$,

$$L = I - (V - 1) \ , \tag{B.7}$$

Is necessary to emphasize here that $I$ is the total number of internal lines $I = I_{\overline{\psi}_+\psi_+} + I_{\overline{\psi}_-\psi_-} + I_{AA} + I_{aa} + I_{Aa}$ and $V$ the total number of vertices $V = V_{+A+} + 2V_{+a+} + V_{-A-} + 2V_{-a-}$.

When we make a calculus of a graph $\gamma$, the count is for each loop $L$ in a space-time $d$ dimensional we will have $L.d$ power of $k$ in the numerator, for the vertices $V_i$ with derivatives of the order $\tilde{d}_i$ it contributes with $V_i\tilde{d}_i$ in the numerator and finally, the propagators contribution in terms of the internal lines give us in each internal line $I_j$ the inverse of power in $k$ together the UV and IR dimension represented here by the $\tilde{G}$. Resuming all in a expression, we get:

$$D = L.d + \sum_i V_i\tilde{d}_i - \sum_j \tilde{G}_j I_j \ , \tag{B.8}$$

To understand the origin of the relation above (B.8) becomes easier if we express the graphic representation of Feynman diagrams in terms of integrals where we can visualize it in a ludic manner (Chapter 6 has some examples). Let's remember, in the quantum level at each order of loop the Feynman diagrams is equivalent an order of $\hbar$ (See Chapter 2) and an more smart way to count the contribution of momenta in the graphs is to write it in terms of external lines and the number of vertices. This proceeding allow to guess a priori, depending on the power counting results, tendencies about the power limits of the graphs at high orders of loop, be it the ultraviolet or infrared limit, based on the knowledge of lower orders.

The symbol $D$ we use can be the ultraviolet limit $d(\gamma)$ or the infrared limit $r(\gamma)$ depending on the necessity of the model, that is, if all fields of the model are massive we do not need to make the study of the infrared limit, otherwise, it is a necessity.

## B.1 Ultraviolet limit

We still need to known to contribution of the propagators in terms of the internal lines of some graph, represented here by the symbol $\tilde{G}$ which is made calculating the ultraviolet and infrared asymptotic limit of the propagators. We will initiate by the ultraviolet limit, that is, the asymptotic limit of the propagators when the $k \to \infty$.

$$\begin{aligned} \Delta_{++}(k) \quad &\text{and} \ \Delta_{--}(k) \quad \to \tfrac{1}{k}, \\ \Delta_{AA}(k) \quad &\text{and} \quad \Delta_{aa}(k) \quad \to \tfrac{1}{k^2}, \\ \Delta_{Aa}(k) \quad &\text{and} \quad \Delta_{aA}(k) \quad \to \tfrac{1}{k^3}. \end{aligned} \tag{B.9}$$

Following our argument the expression (B.8) in $d = 3$ for the ultraviolet limit of some graph $\gamma$ is $D(\gamma) = d(\gamma)$ taking the form:

$$d(\gamma) = 3L - 1I_{\overline{\psi}_+\psi_+} + 1I_{\overline{\psi}_-\psi_-} + 2I_{AA} + 2I_{aa} + 3I_{Aa} \ , \tag{B.10}$$

Replacing in the equation above (B.10) the term $L$ written in terms of the internal lines and vertices obtained from the expression (B.6) is obtained:

$$d(\gamma) = 3 - 1E_{\overline{\psi}_+\psi_+} - 1E_{\overline{\psi}_-\psi_-} - \frac{1}{2}E_{AA} + \frac{1}{2}E_{Aa} - \frac{1}{2}V_{-A-} - \frac{1}{2}V_{-a-} - \frac{1}{2}V_{+A+} - \frac{1}{2}V_{-A-} - 1I_{Aa} \ , \tag{B.11}$$

Observing the expression above (B.11) the vertices ($V_{-A-}$ and $V_{+A+}$) is one imput of $e$ respectively, in the same way, the vertices ($V_{-a-}$ and $V_{+a+}$) one imput of $g$ and so on we count the power of $e$ and $g$ in the graph. Therefore, we define $N_e$ and $N_g$ are the powers (the number of imputes) of the coupling constants, $e$ and $g$, together with this new definition, rewriting the number of external fields $E_{\phi_i\phi_j}$ by $N_{\phi_j}$ we obtain:

$$d(\gamma) = 3 - 1N_{\psi_+} - 1N_{\psi_-} - \frac{1}{2}N_A - \frac{1}{2}N_a - \frac{1}{2}N_e - \frac{1}{2}N_g - N_{aA} \tag{B.12}$$

also $N_{Aa}$ is the number of internal lines $I_{Aa}$ associated to the mixed propagator $\Delta_{Aa}$in the integral corresponding to the graph $\gamma$.

And finally, for aesthetics reason a last modification can be done, writing the equation of power counting (B.12) introducing the ultraviolet limit of the fields $d_{\phi_i}$ present in the tables, Tab. (6.1) or Tab.(7.1), we finally obtain:

$$d(\gamma) = 3 - d_{\psi_+}N_{\psi_+} - d_{\psi_-}N_{\psi_-} - d_A N_A - d_a N_a - \frac{1}{2}N_e - \frac{1}{2}N_g - N_{Aa} \tag{B.13}$$

## B.2 Infrared limit

Looking for now the power counting in the infrared limit, but first, we need the known the infrared dimension of the propagators, that is, the asymptotic behavior of the propagators when $K \to 0$,

$$\begin{aligned} \Delta_{++}(k) \quad &\text{and } \Delta_{--}(k) \quad \to \tfrac{1}{k}, \\ \Delta_{AA}(k) \quad &\text{and} \quad \Delta_{aa}(k) \quad \to \tfrac{1}{k^0}, \\ \Delta_{Aa}(k) \quad &\text{and} \quad \Delta_{aA}(k) \quad \to \tfrac{1}{k}. \end{aligned} \tag{B.14}$$

In the same way,

$$r(\gamma) = 3L - 1I_{\overline{\psi}_+\psi_+} + 1I_{\overline{\psi}_-\psi_-} + 0I_{AA} + 0I_{aa} + 1I_{Aa} \ , \tag{B.15}$$

Again, substituting the term $L$ (B.7) in terms of vertices and internal lines resulted from the expression (B.6) we get:

$$r(\gamma) = 3 - 1E_{\overline{\psi}_+\psi_+} - 1E_{\overline{\psi}_-\psi_-} - \frac{3}{2}E_{AA} - \frac{3}{2}E_{aa} + \frac{1}{2}V_{+A+} + \frac{1}{2}V_{-A-} + \frac{1}{2}V_{-a-} + \frac{1}{2}V_{-a-} - 1I_{Aa} \ , \tag{B.16}$$

Making a similar substitution like we did in the passage from (B.11) to (B.12) we will obtain for the equation above (B.16) the following result:

$$r(\gamma) = 3 - 1N_{\psi_+} - 1N_{\psi_-} - \frac{3}{2}N_A - \frac{3}{2}N_a + \frac{1}{2}N_e + \frac{1}{2}N_g - N_{aA} \tag{B.17}$$

Now, using the infrared dimension of the fields present in the tables, (6.1) and (7.1), the final result is:

$$r(\gamma) = 3 - r_{\psi_+}N_{\psi_+} - r_{\psi_-}N_{\psi_-} - \frac{3}{2}r_A N_A - \frac{3}{2}r_a N_a + \frac{1}{2}N_e + \frac{1}{2}N_g - N_{aA} \tag{B.18}$$

Putting the expressions (B.13) and (B.18) in a same expression and using the notation present in the Chapters, **Chap. 6** and **Chap. 7**, where we use the notation, $f$ for the spinors fields $f = \{\psi_+, \psi_-\}$ and $b$ for the boson fields $b = \{A, a\}$ just for coincides the notations, the result is that:

$$\begin{pmatrix} d(\gamma) \\ r(\gamma) \end{pmatrix} = 3 - \sum_f \begin{pmatrix} d_f \\ r_f \end{pmatrix} N_f - \sum_b \begin{pmatrix} d_b \\ \frac{3}{2} r_b \end{pmatrix} N_b + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_e + \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{2} N_g - N_{Aa} \;, \quad \text{(B.19)}$$

The results present in this section was choose because it presents the analysis of the power counting in the ultraviolet and infrared limit, in the case of massive models, like the chapters **2** and **3**, the procedure is exactly the same and its content is present in the first part of this appendix when we calculated the ultraviolet limit only.

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