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Exponential stability in thermoviscoelastic mixtures of solids

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In this paper we investigate the asymptotic behavior of solutions to the initial boundary value problem for a one-dimensional mixture of thermoviscoelastic solids. Our main result is to establish the exponential stability of the corresponding semigroup and the lack of exponential stability of the corresponding semigroup.

1. Introduction

Thermoviscoelastic mixtures of solids is a subject which has deserved much attention in recent years. The first works on the continuum theory of mixture were the contribution by Truesdell and Toupin (1960), Green and Naghdi (1965, 1968) or Bowen and Wiese (1969). Presentations of these theories can be found in the articles Atkin and Craine. (1976), Bedford and Drumheller (1983) or the books by Bowen (1976), Rajagopal and Tao (1995).

The theory of viscoelastic mixtures has been investigated by several authors (see, e.g., Ieşan and Quintanilla, 2002; Ieşan and Quintanilla, 2007; Ieşan and Nappa, 2008 and references therein). In (Ieşan and Quintanilla, 2007; Ieşan and Nappa, 2008), the authors derive the basic equations of a nonlinear theory of heat conducting viscoelastic mixtures in Lagrangian description. They assume that the constituents have a common temperature and that every thermodynamical process which takes place in the mixture satisfies the Clausius-Duhem inequality. In this paper we want to emphasize the study of the decay of solutions to the case of a one-dimensional beam composed by a mixture of two thermoviscoelastic solids and we want to know when we can expect exponential stability for our system. The model considered has been treated by Ieşan and Quintanilla (2007), Ieşan and Nappa (2008) and Alves et al. (2009b). In what follows, we briefly describe this model.

We consider a mixture of two interacting continua that occupies the interval $(0, L)$. The displacements of typical particles at time t are u and w, where $u = u(x,t), w = w(y,t), x, y \in (0, L)$. We assume that the particles under consideration occupy the same position at time $t = 0$, such that $x = y$ (see, e.g., Bedford and Stern, 1972). The temperature deviation(difference to a fixed constant reference temperature) in each point x and the time t is given by $\theta = \theta(x,t)$. We denote by ρ_1 and ρ_2 the mass densities of the two constituents at time $t = 0$. T, S the partial stresses associated with the constituents, P the internal diffusive force, Θ the entropy density, Q the heat flux vector and T_0 is the absolute temperature in the reference configuration. In the absence of body forces and heat sources the system of equations which governs the linear theory consists of the equations of motion

$$\rho_1 u_{tt} = T_x - P, \quad \rho_2 w_{tt} = S_x + P,$$

the energy equation

$$\rho T_0 \Theta_t = Q_x,$$

where $\rho = \rho_1 + \rho_2$, and the constitutive equations. From the one-dimensional linear theory established in (Ieşan and Quintanilla, 2007), it results that in the absence of porosity, the constitutive equations are the following

$$T = a_{11}u_x + a_{12}w_x + b_{11}u_{xt} + b_{12}w_{xt} - b_1\theta,$$
$$S = a_{12}u_x + a_{22}w_x + b_{21}u_{xt} + b_{22}w_{xt} - b_2\theta,$$
$$P = \alpha(u - w) + \alpha_1(u_t - w_t) + \alpha_2\theta_x,$$
$$\rho\Theta = b_1u_x + b_2w_x + c\theta,$$
$$Q = K\theta_x + K_1(u_t - w_t),$$

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where $c, \alpha, K, K_1, \alpha_i, \beta_i, a_{ij}, b_{ij}$ $(i,j = 1,2)$ are constitutive coefficients. The Clausius-Duhem inequality reduces to

$$b_{11}x^2 + (b_{12} + b_{21})xy + b_{22}y^2 + \alpha_1 z^2 + (\alpha_2 + K_1 T_0^{-1})z\ell + KT_0^{-1}\ell^2 \geqslant 0,$$

for all x, y, z and ℓ. This inequality and the above constitutive equations can be found also in (Ieşan and Nappa, 2008). If we assume that $b_{12} = b_{21}$ and $\alpha_2 + K_1 T_0^{-1} = 0$ and substitute the constitutive equations into the motion equations and the energy equation, we obtain the system of field equations

$$\begin{aligned}
&\rho_1 u_{tt} - a_{11}u_{xx} - a_{12}w_{xx} - b_{11}u_{xxt} - b_{12}w_{xxt} + \alpha(u-w)\\
&\quad + \alpha_1(u_t - w_t) + \beta_1\theta_x = 0,\\
&\rho_2 w_{tt} - a_{12}u_{xx} - a_{22}w_{xx} - b_{12}u_{xxt} - b_{22}w_{xxt} - \alpha(u-w)\\
&\quad - \alpha_1(u_t - w_t) + \beta_2\theta_x = 0,\\
&c\theta_t - \kappa\theta_{xx} + \beta_1 u_{xt} + \beta_2 w_{xt} = 0,
\end{aligned} \tag{1}$$

with $0 < x < L, t > 0, \kappa = KT_0^{-1}, \beta_1 = b_1 + \alpha_2$, and $\beta_2 = -b_2 + \alpha_2$. We assume that $\rho_1, \rho_2, c, \kappa$, and α are positive constants, $\alpha_1 \geqslant 0$ and $\beta_1^2 + \beta_2^2 \neq 0$. The matrix $A = (a_{ij})$ is symmetric and positive definite and $B = (b_{ij}) \neq 0$ is symmetric and nonnegative definite, that is,

$$a_{11} > 0, \quad a_{11}a_{22} - a_{12}^2 > 0,$$
$$b_{11} \geqslant 0, \quad b_{11}b_{22} - b_{12}^2 \geqslant 0.$$

We study the system (1) with the following initial conditions:

$$\begin{aligned}
&u(\,.\,,0) = u_0, \quad u_t(\,.\,,0) = u_1, \quad w(\,.\,,0) = w_0,\\
&w_t(\,.\,,0) = w_1, \quad \theta(\,.\,,0) = \theta_0,
\end{aligned} \tag{2}$$

and the boundary conditions:

$$u(0,t) = u(L,t) = w(0,t) = w(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0 \quad \text{in} \quad (0,\infty). \tag{3}$$

Our purpose in this work is to investigate the stability of the solutions to system (1)–(3). The asymptotic behavior as $t \to \infty$ of solutions to the equations of linear thermoelasticity has been studied by many authors. We refer to the book of Liu and Zheng (1999) for a general survey on these topics.

We recall that very few contributions have been addressed to study the time behavior of the solutions of nonclassical elastic theories. In this direction we mention the works (Quintanilla, 2005; Quintanilla, 2005; Martínez and Quintanilla, 1995; Alves et al., 2009a). In (Quintanilla, 2005), the author deal with the theory of elastic mixtures. He proved the exponential decay of solutions of the equations of motion of a mixture of two linear isotropic one-dimensional elastic materials when the diffusive force is a function which depends on the point and can be localized. The paper (Quintanilla, 2005) deal with the theory of mixtures of viscoelastic solids. The author states the linear equations of the thermomechanical deformations and studies several suitable conditions to guarantee the exponential stability of solutions. On the other hand, the exponential stability for the case of the thermoelastic mixtures (when $B = 0, \alpha_1 = 0$ in (1)) has been studied at (Martínez and Quintanilla, 1995; Alves et al., 2009a). In (Martínez and Quintanilla, 1995), the authors prove (generically) the asymptotic stability. In (Alves et al., 2009a), the authors prove that the semigroup associated is exponentially stable if and only if

$$\beta_2(\beta_1\rho_2 a_{11} + \beta_2\rho_1 a_{12}) \neq \beta_1(\beta_2\rho_1 a_{22} + \beta_1\rho_2 a_{12}) \tag{4}$$

and

$$\frac{n^2\pi^2}{L^2} \neq \frac{\alpha\big((\rho_1\beta_2^2 - \rho_2\beta_1^2) + \beta_1\beta_2(\rho_1 - \rho_2)\big)}{\beta_1\beta_2(\rho_2 a_{11} - a_{22}\rho_1) - a_{12}(\beta_1^2\rho_2 - \beta_2^2\rho_1)}, \tag{5}$$

holds for all $n \in \mathbb{N}$.

We note that we can not expect that this system always decays in a exponential way. For instance, in case when $\beta_1 + \beta_2 = 0, \rho_2(a_{11} + a_{12}) = \rho_1(a_{12} + a_{22})$ and $b_{11} + b_{12} = b_{12} + b_{22} = 0$ we can obtain solutions of the form $u = w$ and $\theta = 0$. These solutions are undamped and do not decay to zero. These are very particular cases, but we will see that there are some other cases where the solutions decay, but the decay is not so fast to be controlled by an exponential.

Our main result is to establish conditions to guarantee the exponential stability of the semigroup associated with (1)–(3). For $\alpha_1 > 0$, the semigroup associated is exponentially stable if and only if

$$b_{11} \neq -b_{12} \quad \text{or} \quad b_{22} \neq -b_{12} \quad \text{or} \quad \beta_1 \neq -\beta_2, \tag{6}$$

or

$$\rho_2(a_{11} + a_{12}) \neq \rho_1(a_{22} + a_{12}). \tag{7}$$

Moreover, for $\alpha_1 = 0$, the semigroup associated is exponentially stable if and only if

$$\{(b_{11}, b_{12}), (\beta_1, \beta_2)\} \quad \text{or} \quad \{(b_{12}, b_{22}), (\beta_1, \beta_2)\} \quad \text{is linearly independent}, \tag{8}$$

or (4) and (5) hold.

We want to emphasize that in a certain way, due to papers (Quintanilla, 2005; Alves et al., 2009a), the conditions (4), (5) and (7) are expected to guarantee the exponentially stable of the semigroup. However, we do not know any paper that indicates the same for conditions (6) and (8).

This paper is organized as follows: Section 2, outlines briefly the notation and the well-posedness of the system is established. In Section 3, we show the exponential stability of the corresponding semigroup. In Section 4 we show the lack of stability exponential of the semigroup provided $\alpha_1 = 0, (b_{11}, b_{12}), (b_{12}, b_{22})$ and (β_1, β_2) are collinear and (4) does not hold; or $\alpha_1 > 0, b_{11} = b_{22} = -b_{12}, \beta_2 = -\beta_1$ and (7) does not hold. Finally, in Section 5, we give some numerical examples to show the asymptotic behavior of the solution.

Our main tool is the theorem given by Gearhart (1978), Huang (1985), Pruss (1984) as well as the spectral arguments.

Finally, throughout this paper C is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

2. Existence and uniqueness of the solutions

In this section we study the setting of the semigroup and we establish the well-posedness of the system. To study the initial boundary value problem by the semigroup theory, we introduce new variables

$$u_t = v \quad \text{and} \quad w_t = \eta, \quad \forall t > 0. \tag{9}$$

Replacing (9) in (1) we have

$$\begin{aligned}
&u_t = v,\\
&w_t = \eta,\\
&v_t = \frac{1}{\rho_1}(a_{11}u + a_{12}w + b_{11}v + b_{12}\eta)_{xx} - \frac{\alpha}{\rho_1}(u-w)\\
&\qquad - \frac{\alpha_1}{\rho_1}(v - \eta) - \frac{\beta_1}{\rho_1}\theta_x,\\
&\eta_t = \frac{1}{\rho_2}(a_{12}u + a_{22}w + b_{12}v + b_{22}\eta)_{xx} + \frac{\alpha}{\rho_2}(u-w)\\
&\qquad + \frac{\alpha_1}{\rho_2}(v - \eta) - \frac{\beta_2}{\rho_2}\theta_x,\\
&\theta_t = -\frac{\beta_1}{c}v_x - \frac{\beta_2}{c}\eta_x + \frac{\kappa}{c}\theta_{xx}.
\end{aligned} \tag{10}$$

Then, the initial boundary value problem (1)–(3) is reduced to the following abstract initial value problem for a first-order evolution equation

$$\frac{d}{dt}U(t) = \mathscr{A}U(t), \quad U(0) = U_0, \quad \forall t > 0 \tag{11}$$

where $U(t) = (u, w, u_t, w_t, \theta)^T, U_0 = (u_0, w_0, u_1, w_1, \theta_0)^T$ and

$$\mathscr{A}\begin{pmatrix} u \\ w \\ v \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \eta \\ \frac{1}{\rho_1}(a_{11}u+a_{12}w+b_{11}v+b_{12}\eta)_{xx}-\frac{\alpha}{\rho_1}(u-w)-\frac{\alpha_1}{\rho_1}(v-\eta)-\frac{\beta_1}{\rho_1}\theta_x \\ \frac{1}{\rho_2}(a_{12}u+a_{22}w+b_{12}v+b_{22}\eta)_{xx}+\frac{\alpha}{\rho_2}(u-w)+\frac{\alpha_1}{\rho_2}(v-\eta)-\frac{\beta_2}{\rho_2}\theta_x \\ -\frac{\beta_1}{c}v_x-\frac{\beta_2}{c}\eta_x+\frac{\kappa}{c}\theta_{xx} \end{pmatrix}. \quad (12)$$

We define $L^2_*(0,L) = \{\theta \in L^2(0,L) : \int_0^L \theta dx = 0\}$ the Hilbert space with the usual inner product and norm of $L^2(0,L)$ and consider

$$\mathscr{H} = H^1_0(0,L) \times H^1_0(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2_*(0,L)$$

equipped with the inner product given by

$$\left\langle (u,w,v,\eta,\theta), or(\widetilde{u},\widetilde{w},\widetilde{v},\widetilde{\eta},\widetilde{\theta}) \right\rangle_{\mathscr{H}} = \int_0^L \left(a_{11}u_x\overline{\widetilde{u}}_x + a_{12}(u_x\overline{\widetilde{w}}_x + w_x\overline{\widetilde{u}}_x) + a_{22}w_x\overline{\widetilde{w}}_x\right)dx + \alpha\int_0^L (u-w) \times (\overline{\widetilde{u}-\widetilde{w}})dx + \rho_1\int_0^L v\overline{\widetilde{v}}dx + \rho_2\int_0^L \eta\overline{\widetilde{\eta}}dx + c\int_0^L \theta\overline{\widetilde{\theta}}dx.$$

and the norm induced $\|\cdot\|_{\mathscr{H}}$. We can show that the norm $\|\cdot\|_{\mathscr{H}}$ is equivalent to usual norm of $\mathscr{H}$. We also consider the Hilbert space $V = \{\varphi \in H^2(0,L) \cap L^2_*(0,L) : \varphi_x \in H^1_0(0,L)\}$ with norm $\|\varphi\|_V = \|\varphi_{xx}\|_{L^2(0,L)}$.

Instead of dealing with (10) we will consider (11) with the operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, with domain

$$\mathscr{D}(\mathscr{A}) = \Big\{U = (u,w,v,\eta,\theta) \in \mathscr{H} : v,\eta \in H^1_0(0,L), a_{11}u + a_{12}w + b_{11}v + b_{12}\eta \in H^2(0,L), a_{12}u + a_{22}w + b_{12}v + b_{22}\eta \in H^2(0,L), \theta \in V\Big\}$$

dense in $\mathscr{H}$.

Firstly, we show that the operator $\mathscr{A}$ generates a C_0-semigroup of contractions on the space $\mathscr{H}$.

Proposition 2.1. *The operator $\mathscr{A}$ generates a C_0-semigroup $S_{\mathscr{A}}(t)$ of contractions on the space $\mathscr{H}$.*

Proof:. We will show that $\mathscr{A}$ is a dissipative operator and $0 \in \rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$. Then our conclusion will follow using the well known the Lumer-Phillips theorem. We observe that if $U = (u,w,v,\eta,\theta) \in \mathscr{D}(\mathscr{A})$ then

$$\begin{aligned}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = {} & a_{11}\int_0^L v_x\overline{u}_x dx + a_{12}\int_0^L v_x\overline{w}_x dx + \alpha\int_0^L v\overline{u}dx \\ & - \alpha\int_0^L v\overline{w}dx + a_{12}\int_0^L \eta_x\overline{u}_x dx + a_{22}\int_0^L \eta_x\overline{w}_x dx \\ & - \alpha\int_0^L \eta\overline{u}dx + \alpha\int_0^L \eta\overline{w}dx - \int_0^L [a_{11}u_x + a_{12}w_x + b_{11}v_x \\ & + b_{12}\eta_x]\overline{v}_x dx - \int_0^L \beta_1\theta_x\overline{v}dx - \int_0^L [a_{12}u_x + a_{22}w_x \\ & + b_{12}v_x + b_{22}\eta_x]\overline{\eta}_x dx - \int_0^L \beta_2\theta_x\overline{\eta}dx - \alpha\int_0^L u\overline{v}dx \\ & + \alpha\int_0^L w\overline{v}dx + \alpha\int_0^L u\overline{\eta}dx - \alpha\int_0^L w\overline{\eta}dx - \int_0^L (\beta_1 v_x \\ & + \beta_2\eta_x - \kappa\theta_{xx})\overline{\theta}dx - \alpha_1\int_0^L |v-\eta|^2 dx.\end{aligned}$$

Taking the real part we obtain

$$Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa\|\theta_x\|^2_{L^2(0,L)} - b_{11}\|v_x\|^2_{L^2(0,L)} - b_{22}\|\eta_x\|^2_{L^2(0,L)} - 2b_{12}Re\int_0^L v_x\overline{\eta}_x dx - \alpha_1\|v-\eta\|^2_{L^2(0,L)}. \quad (13)$$

Case I. $b_{11} > 0$ implies $b_{22} = b^2_{12}/b_{11}$. Then in (13) we have

$$Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa\|\theta_x\|^2_{L^2(0,L)} - \frac{1}{b_{11}}\|b_{11}v_x + b_{12}\eta_x\|^2_{L^2(0,L)} - \alpha_1\|v-\eta\|^2_{L^2(0,L)} \leqslant 0. \quad (14)$$

Thus the operator $\mathscr{A}$ is dissipative.

Case II. $b_{11} = 0$ implies $b_{12} = 0$. Then in (13) we have

$$Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa\|\theta_x\|^2_{L^2(0,L)} - b_{22}\|\eta_x\|^2_{L^2(0,L)} - \alpha_1\|v-\eta\|^2_{L^2(0,L)} \leqslant 0.$$

Thus $\mathscr{A}$ is also dissipative.

On the other hand, we show that $0 \in \rho(\mathscr{A})$. In fact, given $F = (f,g,h,p,q) \in \mathscr{H}$, we must show that there exists a unique $U = (u,w,v,\eta,\theta)$ in $\mathscr{D}(\mathscr{A})$ such that $\mathscr{A}U = F$, that is,

$$v = f \quad \text{in} \quad H^1_0(0,L), \quad (15)$$

$$\eta = g \quad \text{in} \quad H^1_0(0,L), \quad (16)$$

$$(a_{11}u + a_{12}w + b_{11}v + b_{12}\eta)_{xx} - \alpha(u-w) - \alpha_1(v-\eta) - \beta_1\theta_x = \rho_1 h \quad \text{in} \quad L^2(0,L), \quad (17)$$

$$(a_{12}u + a_{22}w + b_{12}v + b_{22}\eta)_{xx} + \alpha(u-w) + \alpha_1(v-\eta) - \beta_2\theta_x = \rho_2 p \quad \text{in} \quad L^2(0,L), \quad (18)$$

$$-\beta_1 v_x - \beta_2\eta_x + \kappa\theta_{xx} = cq \quad \text{in} \quad L^2_*(0,L). \quad (19)$$

Replacing (15) and (16) in (19) we have

$$\kappa\theta_{xx} = cq + \beta_1 f_x + \beta_2 g_x \in L^2_*(0,L). \quad (20)$$

It is known that there is a unique $\theta \in V$, satisfying (20). Moreover, given the continuous and coercive sesquilinear form

$$\mathscr{B}((u,w),(\varphi,\psi)) = a_{11}\int_0^L u_x\overline{\varphi}_x dx + a_{12}\int_0^L u_x\overline{\psi}_x dx + a_{12}\int_0^L w_x\overline{\varphi}_x + a_{22}\int_0^L w_x\overline{\psi}_x dx + \alpha\int_0^L (u-w)(\overline{\varphi-\psi})dx,$$

for $(u,w),(\varphi,\psi) \in H^1_0(0,L) \times H^1_0(0,L)$ and the functional $\mathscr{G} : H^1_0(0,L) \times H^1_0(0,L) \to \mathbb{C}$,

$$\begin{aligned}\mathscr{G}(\varphi,\psi) = {} & -\int_0^L (b_{11}v + b_{12}\eta)_x\overline{\varphi}_x dx - \rho_1\int_0^L h\overline{\varphi}dx - \beta_1\int_0^L \theta_x\overline{\varphi}dx \\ & - \int_0^L (b_{12}v + b_{22}\eta)_x\overline{\psi}_x dx - \rho_2\int_0^L p\overline{\psi}dx - \beta_2\int_0^L \theta_x\overline{\psi}dx \\ & - \alpha_1\int_0^L (v-\eta)(\overline{\varphi-\psi})dx,\end{aligned}$$

we have using the Lax-Milgram theorem that there exists a unique vector function (u,w) in $H^1_0(0,L) \times H^1_0(0,L)$ such that

$$\mathscr{B}((u,w),(\varphi,\psi)) = \mathscr{G}(\varphi,\psi), \quad \forall\, (\varphi,\psi) \in H^1_0(0,L) \times H^1_0(0,L).$$

Thus

$$\int_0^L (a_{11}u_x + a_{12}w_x + b_{11}v_x + b_{12}\eta_x)\overline{\varphi}_x dx + \alpha\int_0^L (u-w)\overline{\varphi}dx + \beta_1\int_0^L \theta_x\overline{\varphi}dx + \alpha_1\int_0^L (v-\eta)\overline{\varphi}dx = -\rho_1\int_0^L h\overline{\varphi}dx,$$

$$\int_0^L (a_{12}u_x + a_{22}w_x + b_{12}v_x + b_{22}\eta_x)\overline{\psi}_x dx - \alpha\int_0^L (u-w)\overline{\psi}dx + \beta_2\int_0^L \theta_x\overline{\psi}dx - \alpha_1\int_0^L (v-\eta)\overline{\psi}dx = -\rho_2\int_0^L p\overline{\psi}dx,$$

$\forall\varphi,\psi\in H_0^1(0,L)$. It follows that $a_{11}u+a_{12}w+b_{11}v+b_{12}\eta$ and $a_{12}u+a_{22}w+b_{12}v+b_{22}\eta$ belong to $H_0^1(0,L)\cap H^2(0,L)$ and the Eqs. (17) and (18) are verified. Moreover, it is easy to show that $\|U\|_{\mathscr{H}}\leqslant C\|F\|_{\mathscr{H}}$, for a positive constant$C$. Therefore, we conclude that $0\in\rho(\mathscr{A})$.

From the Proposition 2.1 we can state the following result (see Pazy, 1983). □

Theorem 2.2. *For any $U_0\in\mathscr{H}$ there exists a unique solution $U(t)=(u,w,u_t,w_t,\theta)$ of (1)–(3) satisfying*

$$u,w\in C([0,\infty[:\ H_0^1(0,L))\cap C^1([0,\infty[:\ L^2(0,L)),$$
$$\theta\in C([0,\infty[:\ L_*^2(0,L))\cap L^2(]0,\infty[:\ H^1(0,L)).$$

However, if $U_0\in\mathscr{D}(\mathscr{A})$ we have

$$u,w\in C^1([0,\infty[:H_0^1(0,L))\cap C^2([0,\infty[:L^2(0,L)),$$
$$a_{11}u+a_{12}w+b_{11}u_t+b_{12}w_t\in C([0,\infty[:H^2(0,L)),$$
$$a_{12}u+a_{22}w+b_{12}u_t+b_{22}w_t\in C([0,\infty[:H^2(0,L)),$$
$$\theta\in C([0,\infty[:V)\cap C^1([0,\infty[:L_*^2(0,L)).$$

3. Exponential stability

In this section the first result that we are going to present is about the necessary and sufficient conditions of exponential stability of a C_0-semigroup on a Hilbert space. This result was obtained by Gearhart (1978), and Huang (1985), independently (see also Pruss, 1984).

Theorem 3.1. *Let $S(t)$ be a C_0-semigroup of contractions of linear operators on Hilbert space $\mathscr{H}$ with infinitesimal generator $\mathscr{B}$. Then $S(t)$ is exponentially stable if and only if*

(i) $i\mathbb{R}\subset\rho(\mathscr{B})$.
(ii) $\limsup_{|\beta|\to+\infty}\|(i\beta I-\mathscr{B})^{-1}\|_{\mathscr{L}(\mathscr{H})}<\infty$.

Lemma 3.2. *Let $\mathscr{A}$ be defined in (12). Assume that*

(a) $\alpha_1>0$ and
  (a.1) *condition (6) holds.*
  (a.2) $b_{11}=b_{22}=-b_{12},\beta_1=-\beta_2$ and *(7) holds.*

(b) $\alpha_1=0$ and
  (b.1) *condition (8) holds.*
  (b.2) $(b_{11},b_{12}),(b_{12},b_{22})$ and(β_1,β_2) *are collinear and (5) holds.*

Then the set $i\mathbb{R}=\{i\lambda:\lambda\in\mathbb{R}\}$ is contained in $\rho(\mathscr{A})$.

Proof. Following the arguments given in Liu and Zheng (1999), the proof consists of the following steps:

Step 1. Since $0\in\rho(\mathscr{A})$, for any real number λ with $\|\lambda\mathscr{A}^{-1}\|<1$, the linear bounded operator $i\lambda\mathscr{A}^{-1}-I$ is invertible, therefore $i\lambda I-\mathscr{A}=\mathscr{A}(i\lambda\mathscr{A}^{-1}-I)$ is invertible and its inverse belongs to $\mathscr{L}(\mathscr{H})$, that is, $i\lambda\in\rho(\mathscr{A})$. Moreover, $\|(i\lambda I-\mathscr{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathscr{A}^{-1}\|^{-1},\|\mathscr{A}^{-1}\|^{-1})$.

Step 2. If $\sup\{\|(i\lambda I-\mathscr{A})^{-1}\|:|\lambda|<\|\mathscr{A}^{-1}\|^{-1}\}=M<\infty$, then for $|\lambda_0|<\|\mathscr{A}^{-1}\|^{-1}$ and $\lambda\in\mathbb{R}$ such that $|\lambda-\lambda_0|<M^{-1}$, we have $\|(\lambda-\lambda_0)(i\lambda_0 I>-\mathscr{A})^{-1}\|<1$, therefore the operator

$$i\lambda I-\mathscr{A}=(i\lambda_0 I-\mathscr{A})(I+i(\lambda-\lambda_0)(i\lambda_0 I-\mathscr{A})^{-1})$$

is invertible with its inverse in $\mathscr{L}(\mathscr{H})$, that is, $i\lambda\in\rho(\mathscr{A})$. Since λ_0 is arbitrary we can conclude that $\{i\lambda:|\lambda|<\|\mathscr{A}^{-1}\|^{-1}+M^{-1}\}\subset\rho(\mathscr{A})$ and the function $\|(i\lambda I-\mathscr{A})^{-1}\|$ is continuous in the interval $(-\|\mathscr{A}^{-1}\|^{-1}-M^{-1},\|\mathscr{A}^{-1}\|^{-1}+M^{-1})$.

Step 3. Thus, it follows by item (ii) that if $i\mathbb{R}\subset\rho(A)$ is not true, then there exists $\omega\in\mathbb{R}$ with $\|\mathscr{A}^{-1}\|^{-1}\leqslant|\omega|$ such that $\{i\lambda:|\lambda|<|\omega|\}\subset\rho(\mathscr{A})$ and $\sup\{\|(i\lambda I-\mathscr{A})^{-1}\|:|\lambda|<|\omega|\}=\infty$. Therefore, there exists a sequence of real numbers (λ_n) with $\lambda_n\to\omega$, $|\lambda_n|<|\omega|$ and sequences of vector functions $U_n=(u_n,w_n,v_n,\eta_n,\theta_n)\in\mathscr{D}(\mathscr{A}),F_n=(f_n,g_n,h_n,p_n,q_n)\in\mathscr{H}$, such that $(i\lambda_n I-\mathscr{A})U_n=F_n$ and $\|U_n\|_{\mathscr{H}}=1$ and $F_n\to 0$ in $\mathscr{H}$ when $n\to\infty$, that is,

$$i\lambda_n u_n-v_n=f_n\to 0\quad\text{in}\quad H_0^1(0,L),\tag{21}$$
$$i\lambda_n w_n-\eta_n=g_n\to 0\quad\text{in}\quad H_0^1(0,L),\tag{22}$$
$$i\lambda_n\rho_1 v_n-(a_{11}u_n+a_{12}w_n+b_{11}v_n+b_{12}\eta_n)_{xx}+\alpha(u_n-w_n)+\alpha_1(v_n-\eta_n)+\beta_1\theta_{nx}=\rho_1 h_n\to 0\quad\text{in}\quad L^2(0,L),\tag{23}$$
$$i\lambda_n\rho_2\eta_n-(a_{12}u_n+a_{22}w_n+b_{12}v_n+b_{22}\eta_n)_{xx}-\alpha(u_n-w_n)-\alpha_1(v_n-\eta_n)+\beta_2\theta_{nx}=\rho_2 p_n\to 0\quad\text{in}\quad L^2(0,L),\tag{24}$$
$$i\lambda_n c\theta_n+\beta_1 v_{nx}+\beta_2\eta_{nx}-\kappa\theta_{nxx}=cq_n\to 0\quad\text{in}\quad L_*^2(0,L).\tag{25}$$

We observe that

$$Re\langle(i\lambda_n I-\mathscr{A})U_n,U_n\rangle_{\mathscr{H}}\to 0\quad\text{as}\quad n\to\infty.\tag{26}$$

Suppose that $b_{11}>0$. Then, using the same idea as (15) we obtain

$$\kappa\int_0^L|\theta_{nx}|^2dx+\alpha_1\int_0^L|v_n-\eta_n|^2dx+\frac{1}{b_{11}}\int_0^L|b_{11}v_{nx}+b_{12}\eta_{nx}|^2dx\to 0$$
as $n\to\infty$.

It follows by the Poincaré inequality that $\theta_n\to 0$ in $H^1(0,L)$ and $b_{11}v_n+b_{12}\eta_n\to 0$ in $H_0^1(0,L)$. Since $(u_n)_{n\in\mathbb{N}},(w_n)_{n\in\mathbb{N}}$ are bounded sequences in $H_0^1(0,L)$, by the compactness of the embedding of $H^1(0,L)$ in $L^2(0,L)$, there are subsequences, still denoted by $(u_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$, such that $u_n\to u$ and $w_n\to w$ in $L^2(0,L)$. Using (21) and (22) we deduce that $v_n\to v$ and $\eta_n\to\eta$ in $L^2(0,L)$.

Integrating (25) from 0 to x we deduce

$$\|\beta_1 v_n+\beta_2\eta_n\|_{L^2(0,L)}\to 0\quad\text{as}\quad n\to\infty.\tag{28}$$

$(a)-(a.1)$. From (27) and (28) we conclude that $v=\eta=0$ and by (21) and (22) we have $u=w=0$. On the other hand, it follows by (23) and (24) that the sequences

$$(a_{11}u_n+a_{12}w_n+b_{11}v_n+b_{12}w_n)_{n\in\mathbb{N}}\quad\text{and}\quad(a_{12}u_n+a_{22}w_n+b_{12}v_n+b_{22}w_n)_{n\in\mathbb{N}}\tag{29}$$

converge to zero in $H_0^1(0,L)$. Therefore, by (27) we obtain that $a_{11}u_n+a_{12}w_n\to 0$ and $a_{12}u_n+a_{22}w_n\to 0$ in $H_0^1(0,L)$. Since (a_{ij}) is positive definite, it follows that $u_n\to 0,w_n\to 0$ in $H_0^1(0,L)$. Thus we have a contradiction and the result follows. If $b_{11}=0$ then $b_{22}>0$ and we can use similar manipulations.

$(a)-(a.2)$. It results by (27) that $v=\eta$ and then $u=w$. Consequently, from (23) and (24) we obtain that the sequences given in (29) converge in $H_0^1(0,L)$. Using again (27) we obtain that $(a_{11}u_n+a_{12}w_n)_{n\in\mathbb{N}}$ and $(a_{12}u_n+a_{22}w_n)_{n\in\mathbb{N}}$ converge in $H_0^1(0,L)$. We can conclude that $u_n\to u$, $w_n\to w$, and $v_n\to v$ and $\eta_n\to\eta$ in $H_0^1(0,L)$. It results by (23) and (24) that

$$-\omega^2u-\frac{a_{11}+a_{12}}{\rho_1}u_{xx}=0\quad\text{and}\quad-\omega^2u-\frac{a_{12}+a_{22}}{\rho_2}u_{xx}=0.$$

By hypothesis (7) we conclude that $u=0$ and hence $w=v=\eta=0$. The result follows.

$(b)-(b.1)$. Suppose that $\{(b_{11},b_{12}),(\beta_1,\beta_2)\}$ is linearly independent. Hence (27) yields

$$\kappa\|\theta_{nx}\|^2_{L^2(0,L)}+\frac{1}{b_{11}}\|b_{11}v_{nx}+b_{12}\eta_{nx}\|^2_{L^2(0,L)}\to 0 \quad \text{as} \quad n\to\infty. \tag{30}$$

Using (28) and (30), we conclude that $v_n\to 0$ and $\eta_n\to 0$ in $L^2(0,L)$. As in $(a.1)$, it follows that $u_n\to 0, w_n\to 0$ in $H^1_0(0,L)$. Thus we have a contradiction. If $\{(b_{12},b_{22}),(\beta_1,\beta_2)\}$ is linearly independent the procedure is similar.

$(b)-(b.2)$. Suppose $b_{11}b_{12}\neq 0$. By (30) we conclude again that $u_n\to u, w_n\to w, v_n\to v$ and $\eta_n\to\eta$ in $H^1_0(0,L)$ and $b_{11}v_x+b_{12}\eta_x=0$. Therefore $\beta_1 v+\beta_2\eta=0$ and $\beta_1 u+\beta_2 w=0$. It results by (23) and (24) that

$$-\omega^2\rho_1 u-(a_{11}u_{xx}+a_{12}w_{xx})+\alpha(u-w)=0,$$
$$-\omega^2\rho_2 w-(a_{12}u_{xx}+a_{22}w_{xx})-\alpha(u-w)=0.$$

Since $w=-\frac{\beta_1}{\beta_2}u$, it results that

$$-(\beta_2 a_{11}-\beta_1 a_{12})u_{xx}=(\rho_1\beta_2\omega^2-\alpha(\beta_1+\beta_2))u$$
$$-(\beta_1 a_{22}-\beta_2 a_{12})u_{xx}=(\rho_2\beta_1\omega^2-\alpha(\beta_1+\beta_2))u.$$

By hypothesis (5), we conclude that $u=0$ and hence $w=v=\eta=0$.

Now, we suppose that $b_{11}>0$ and $b_{12}=b_{22}=0$. Then $\beta_2=0$ and by (30) we get that $v_n\to 0$ in $H^1_0(0,L)$ and by (21), $u_n\to 0$ in $H^1_0(0,L)$. By (22)–(24) it results that the sequences $(a_{11}u_n+a_{12}w_n+b_{11}v_n)$ and $(a_{12}u_n+a_{22}w_n)$ converge in $H^1_0(0,L)$, and therefore (w_n) converge to w in $H^1_0(0,L)$. Taking the inner product of (23) with $\varphi\in C^\infty_0(0,L)$ we obtain

$$i\lambda_n\rho_1\int_0^L v_n\overline{\varphi}dx+a_{11}\int_0^L u_{nx}\overline{\varphi}_x dx+a_{12}\int_0^L w_{nx}\overline{\varphi}_x dx+b_{11}\times\int_0^L v_{nx}\overline{\varphi}_x dx+\alpha\int_0^L u_n\overline{\varphi}dx-\alpha\int_0^L w_n\overline{\varphi}dx+\beta_1\int_0^L\theta_{nx}\overline{\varphi}dx=\rho_1\int_0^L h_n\overline{\varphi}dx,$$

Taking $n\to\infty$ in the above equations we get

$$a_{12}\int_0^L w_x\overline{\varphi}_x dx-\alpha\int_0^L w\overline{\varphi}dx=0,\quad \forall\varphi\in C^\infty_0(0,L). \tag{31}$$

By hypothesis (5) and using (31) we obtain $w=0$ and $\eta=0$.

If $b_{11}=0$, then $\beta_1=b_{12}=0$ and $b_{22}>0$. Using (26) we have that $\eta_n\to 0$ in $H^1_0(0,L)$ and by (22), $w_n\to 0$ in $H^1_0(0,L)$. We can use the same procedure as before to get $u=v=0$. Thus we have a contradiction and the proof of the lemma is complete. □

Theorem 3.3. *Let $\mathscr{A}$ be defined in (12). Assume that*

(a) $\alpha_1>0$ and
 (a.1) *condition (6) holds.*
 (a.2) $b_{11}=b_{22}=-b_{12}, \beta_1=-\beta_2$ and *(7) holds.*

(b) $\alpha_1=0$ and
 (b.1) *condition (8) holds.*
 (b.2) $(b_{11},b_{12}),(b_{12},b_{22})$ and (β_1,β_2) *are collinear and (4) and (5) hold.*

Then $S_{\mathscr{A}}(t)$ is exponentially stable, that is, there exist positive constants M and μ such that

$$\|S_{\mathscr{A}}(t)\|_{\mathscr{L}(\mathscr{H})}\leqslant M\exp(-\mu t).$$

Proof. By Theorem 3.1 and Lemma 3.2, we must prove that (ii) is true. Given $\lambda\in\mathbb{R}$ and $F=(f,g,h,p,q)\in\mathscr{H}$, let us $U=(u,w,v,\eta,\theta)\in\mathscr{D}(\mathscr{A})$ be the unique solution of $(i\lambda I-\mathscr{A})U=F$, that is,

$$i\lambda u-v=f \quad \text{in} \quad H^1_0(0,L), \tag{32}$$
$$i\lambda w-\eta=g \quad \text{in} \quad H^1_0(0,L), \tag{33}$$
$$i\lambda\rho_1 v-(a_{11}u+a_{12}w+b_{11}v+b_{12}\eta)_{xx}+\alpha(u-w)+\alpha_1(v-\eta)+\beta_1\theta_x=\rho_1 h \quad \text{in} \quad L^2(0,L), \tag{34}$$
$$i\lambda\rho_2\eta-(a_{12}u+a_{22}w+b_{12}v+b_{22}\eta)_{xx}-\alpha(u-w)-\alpha_1(v-\eta)+\beta_2\theta_x=\rho_2 p \quad \text{in} \quad L^2(0,L), \tag{35}$$
$$i\lambda c\theta+\beta_1 v_x+\beta_2\eta_x-\kappa\theta_{xx}=cq \quad \text{in} \quad L^2_*(0,L). \tag{36}$$

Note that if $b_{11}>0$ (the other case is similarly analyzed)

$$Re\langle(i\lambda I-\mathscr{A})U,U\rangle_{\mathscr{H}}=\kappa\|\theta_x\|^2_{L^2(0,L)}+\frac{1}{b_{11}}\|b_{11}v_x+b_{12}\eta_x\|^2_{L^2(0,L)}+\alpha_1\|v-\eta\|^2_{L^2(0,L)}=Re\langle F,U\rangle_{\mathscr{H}}.$$

Thus

$$\kappa\|\theta_x\|^2_{L^2(0,L)}+\frac{1}{b_{11}}\|b_{11}v_x+b_{12}\eta_x\|^2_{L^2(0,L)}+\alpha_1\|v-\eta\|^2_{L^2(0,L)}\leqslant\|F\|_{\mathscr{H}}\|U\|_{\mathscr{H}}. \tag{37}$$

Taking the inner product in $L^2(0,L)$ of (34) with u, (35) with w and using (32) and (33) we obtain

$$\begin{aligned}&\frac{(a_{11}a_{22}-a_{12}^2)}{2a_{22}}\int_0^L|u_x|^2dx+\frac{(a_{11}a_{22}-a_{12}^2)}{2a_{11}}\int_0^L|w_x|^2dx\\&\leqslant\rho_1\int_0^L|v|^2dx+\rho_2\int_0^L|\eta|^2dx+|\beta_1|\int_0^L|\theta_x||u|dx\\&+|\beta_2|\int_0^L|\theta_x||w|dx+\alpha_1\int_0^L|v-\eta||u-w|dx+\rho_1\int_0^L|v||f|dx\\&+\rho_1\int_0^L|h||u|dx+\int_0^L|b_{11}v_x+b_{12}\eta_x||u_x|dx+\rho_2\int_0^L|\eta||g|dx\\&+\rho_2\int_0^L|p||w|dx+\int_0^L|b_{12}v_x+b_{22}\eta_x||w_x|dx.\end{aligned} \tag{38}$$

We define $\varphi(x)=\int_0^x(\beta_1 v+\beta_2\eta)(y)dy$. Taking the inner product in $L^2(0,L)$ of (36) with φ and integrating by parts we obtain

$$i\lambda c\int_0^L\theta\overline{\varphi}dx-\int_0^L|\beta_1 v+\beta_2\eta|^2dx+\kappa\int_0^L\theta_x(\overline{\beta_1 v+\beta_2\eta})dx=c\int_0^L q\overline{\varphi}dx. \tag{39}$$

Consider $\xi\in H^2(0,L)$ solution of the problem $\xi_{xx}=\theta, \xi_x(0)=0$, $\xi_x(L)=0$. It follows from (39) that

$$-i\lambda c\int_0^L\xi_x(\overline{\beta_1 v+\beta_2\eta})dx-\int_0^L|\beta_1 v+\beta_2\eta|^2dx+\kappa\int_0^L\theta_x(\overline{\beta_1 v+\beta_2\eta})dx=c\int_0^L q\overline{\varphi}dx. \tag{40}$$

On the other hand, multiplying the Eq. (34) by $\frac{\beta_1}{\rho_1}$, (35) by $\frac{\beta_2}{\rho_2}$ and adding the result we obtain

$$\begin{aligned}i\lambda(\beta_1 v+\beta_2\eta)=&-\alpha\left(\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right)(u-w)-\alpha_1\left(\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right)(v-\eta)\\&-\left(\frac{\beta_1^2}{\rho_1}+\frac{\beta_2^2}{\rho_2}\right)\theta_x+\beta_1 h+\beta_2 p\\&+\frac{\beta_1}{\rho_1}(a_{11}u+a_{12}w+b_{11}v+b_{12}\eta)_{xx}\\&+\frac{\beta_2}{\rho_2}(a_{12}u+a_{22}w+b_{12}v+b_{22}\eta)_{xx}.\end{aligned} \tag{41}$$

Substituting (41) in (40) and integrating by parts we obtain

$$\int_0^L |\beta_1 v+\beta_2\eta|^2dx = c\int_0^L\left[\theta_x\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)\overline{u}+\theta_x\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)\overline{w}\right]dx$$
$$+c\int_0^L\left[\theta_x\left(\frac{\beta_1}{\rho_1}b_{11}+\frac{\beta_2}{\rho_2}b_{12}\right)\overline{v}+\theta_x\left(\frac{\beta_1}{\rho_1}b_{12}+\frac{\beta_2}{\rho_2}b_{22}\right)\overline{\eta}\right]dx$$
$$+\kappa\int_0^L\theta_x(\overline{\beta_1 v+\beta_2\eta})dx - c\int_0^L q\overline{\varphi}dx + c\beta_1\times\int_0^L\xi_x\overline{h}dx$$
$$+c\beta_2\int_0^L\xi_x\overline{p}dx - c\left(\frac{\beta_1^2}{\rho_1}+\frac{\beta_2^2}{\rho_2}\right)\int_0^L\xi_x\overline{\theta}_xdx$$
$$-c\alpha\left(\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right)\int_0^L\xi_x(\overline{u-w})dx$$
$$-c\alpha_1\left(\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right)\int_0^L\xi_x(\overline{v-\eta})dx.$$

Since that $\int_0^L|\xi_x|^2dx \leqslant L^2\int_0^L|\theta|^2dx$, by the inequalities of Poincaré and Cauchy-Schwartz together with (37) we can verify that there is a positive constant C such that

$$\|\beta_1 v+\beta_2\eta\|^2_{L^2(0,L)}dx \leqslant C\Big(\|\theta_x\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big). \quad (42)$$

In this point, we analyze the hypothesis $(a.1)$ and $(b.1)$.

$(a)-(a.1)$. If $b_{11}\neq -b_{12}$, it results by (37) that there exist a positive constant C such that

$$\|v\|^2_{L^2(0,L)}+\|\eta\|^2_{L^2(0,L)} \leqslant C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}.$$

Last inequality together with the inequalities of Poincaré and Cauchy-Schwartz, (37) and (38) imply

$$\|U\|_{\mathscr{H}} \leqslant C\|F\|_{\mathscr{H}}, \quad \text{for a positive constant } C. \quad (43)$$

If $\beta_1\neq-\beta_2$, it follows by (42) that there is a positive constant C such that

$$\|v\|^2_{L^2(0,L)}+\|\eta\|^2_{L^2(0,L)} \leqslant C\Big(\|\theta_x\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big). \quad (44)$$

Combining of (37), (38) and (44) yields (43). If $b_{22}\neq -b_{12}$ the procedure is similar.

$(b)-(b.1)$. If (b_{11},b_{12}) and (β_1,β_2) are linearly independent, by (37) and (42) we have that (44) holds. Therefore, we can obtain (43). If (b_{12},b_{22}) and (β_1,β_2) are linearly independent the procedure is similar.

$(a)-(a.2)$. By (37) we have

$$\kappa\|\theta_x\|^2_{L^2(0,L)}+b_{11}\|v_x-\eta_x\|^2_{L^2(0,L)}+\alpha_1\|v-\eta\|^2_{L^2(0,L)} \leqslant C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \quad (45)$$

for a positive constant C. Hence by (32) and (33) we obtain

$$\|u_x-w_x\|^2_{L^2(0,L)} \leqslant C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \quad |\lambda|>1, \quad (46)$$

for a positive constant C. Taking the inner product of (41) with $\beta_1\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\beta_1\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w$, using that $i\lambda u=v+f, i\lambda w=\eta$ we get

$$\int_0^L\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u_x+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w_x\right|^2dx$$
$$\leqslant b_{11}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u_x+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w_x\right||v_x-\eta_x|dx$$
$$+\alpha\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L|u-w|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w\right|dx$$
$$+\alpha_1\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L|v-\eta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w\right|dx$$
$$+\int_0^L|v-\eta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)v+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)\eta\right|dx$$
$$+\int_0^L|v-\eta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)f+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)g\right|dx$$
$$+\beta_1\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L|\theta_x|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}+\frac{a_{22}}{\rho_2}\right)w\right|dx$$
$$+\int_0^L|h-p|\left|\left(\frac{a_{11}}{\rho_1}+\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w\right|dx. \quad (47)$$

Using the Cauchy-Schwartz and Young inequalities we obtain

$$\|(\rho_2a_{11}-\rho_1a_{12})u_x+(\rho_2a_{12}-\rho_1a_{22})w_x\|^2_{L^2(0,L)}$$
$$\leqslant C\Big(\|v-\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big), \quad (48)$$

for $|\lambda|>1$ and $C>0$. By (7), (46) and (48) we have

$$\|u_x\|^2_{L^2(0,L)}+\|w_x\|^2_{L^2(0,L)} \leqslant C\Big(\|\beta_1 v+\beta_2\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big), \quad (49)$$

for $|\lambda|>1$ and a positive constant C.

Taking the inner product in $L^2(0,L)$ of (34) with u, (35) with w and using (32) and (33) we obtain

$$\int_0^L|v|^2dx = \frac{a_{11}}{\rho_1}\int_0^L|u_x|^2dx+\frac{\alpha}{\rho_1}\int_0^L|u|^2dx+\frac{a_{12}}{\rho_1}\int_0^L w_x\overline{u}_xdx$$
$$-\frac{\alpha}{\rho_1}\int_0^L w\overline{u}dx+\frac{\beta_1}{\rho_1}\int_0^L\theta_x\overline{u}dx-\int_0^L v\overline{f}dx$$
$$-\int_0^L h\overline{u}dx+\frac{1}{\rho_1}\int_0^L(b_{11}v_x+b_{12}\eta_x)\overline{u}_xdx$$
$$+\frac{\alpha_1}{\rho_1}\int_0^L v\overline{u}dx-\frac{\alpha_1}{\rho_1}\int_0^L\eta\overline{u}dx \quad (50)$$

and

$$\int_0^L|\eta|^2dx = \frac{a_{12}}{\rho_2}\int_0^L u_x\overline{w}_xdx-\frac{\alpha}{\rho_2}\int_0^L u\overline{w}dx+\frac{a_{22}}{\rho_2}\int_0^L|w_x|^2dx$$
$$+\frac{\alpha}{\rho_2}\int_0^L|w|^2dx+\frac{\beta_2}{\rho_2}\int_0^L\theta_x\overline{w}dx-\int_0^L\eta\overline{g}dx$$
$$-\int_0^L p\overline{w}dx+\frac{1}{\rho_2}\int_0^L(b_{12}v_x+b_{22}\eta_x)\overline{w}_xdx$$
$$-\frac{\alpha_1}{\rho_1}\int_0^L v\overline{w}dx+\frac{\alpha_1}{\rho_1}\int_0^L\eta\overline{w}dx \quad (51)$$

Combining of (45), (49), (50) and (51) yields that

$$\|v\|^2_{L^2(0,L)}+\|\eta\|^2_{L^2(0,L)} \leqslant C\Big(\|\beta_1 v+\beta_2\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big), \quad (52)$$

for $|\lambda|>1$ and a positive constant C. It follows from the estimates (45), (49) and (52) that

$$\|(i\lambda I-A)^{-1}F\|_{\mathscr{H}} \leqslant C\|F\|_{\mathscr{H}}$$

when $|\lambda|>1$, for a positive constant C.

$(b)-(b.2)$. By (37) we have

$$\|\theta_x\|^2_{L^2(0,L)}+\|\beta_1v_x+\beta_2\eta_x\|^2_{L^2(0,L)} \leqslant C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}. \quad (53)$$

Hence by (32) and (33) we obtain

$$\|\beta_1u_x+\beta_2w_x\|^2_{L^2(0,L)} \leqslant C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \quad |\lambda|>1, \quad (54)$$

for a positive constant C. Taking the inner product of (41) with $(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12})u+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)w$, using that $i\lambda u=v+f, i\lambda w=\eta$ and

$$\frac{\left(\frac{\beta_1}{\rho_1}b_{11}+\frac{\beta_2}{\rho_2}b_{12}\right)}{\beta_1}=\frac{\left(\frac{\beta_1}{\rho_1}b_{12}+\frac{\beta_2}{\rho_2}b_{22}\right)}{\beta_2}=\gamma, \quad \beta_1\beta_2\neq 0,$$

we get

$$\int_0^L \left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)u_x+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)w_x\right|^2dx$$
$$\leqslant |\gamma|\int_0^L\left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)u_x+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)w_x\right||\beta_1 v_x$$
$$+\beta_2\eta_x|dx+\alpha\left|\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right|\int_0^L|u-w|\left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)u\right.$$
$$\left.+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)w\right|dx+\int_0^L|\beta_1 v+\beta_2\eta|\left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)v\right.$$
$$\left.+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)\eta\right|dx+\int_0^L|\beta_1 v+\beta_2\eta|\left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)f\right.$$
$$\left.+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)g\right|dx+\left(\frac{\beta_1^2}{\rho_1}+\frac{\beta_2^2}{\rho_2}\right)\int_0^L|\theta_x|\left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)u\right.$$
$$\left.+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)w\right|dx+\int_0^L|\beta_1 h+\beta_2 p|\left|\left(\frac{\beta_1}{\rho_1}a_{11}+\frac{\beta_2}{\rho_2}a_{12}\right)u\right.$$
$$\left.+\left(\frac{\beta_1}{\rho_1}a_{12}+\frac{\beta_2}{\rho_2}a_{22}\right)w\right|dx. \tag{55}$$

Substituting $u-w=\frac{(f-g)}{i\lambda}+\frac{(v-\eta)}{i\lambda}$ in (55), using the Cauchy-Schwartz inequality and the estimate (53) we get

$$\|(\rho_2\beta_1a_{11}+\rho_1\beta_2a_{12})u_x+(\rho_2\beta_1a_{12}+\rho_1\beta_2a_{22})w_x\|^2_{L^2(0,L)}$$
$$\leqslant C\left(\|\beta_1 v+\beta_2\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+\frac{1}{|\lambda|}\|U\|^2_{\mathscr{H}}\right), \tag{56}$$

for $|\lambda|>1$ and $C>0$. From (4), (54) and (56) we have

$$\|u_x\|^2_{L^2(0,L)}+\|w_x\|^2_{L^2(0,L)}\leqslant C\Big(\|\beta_1 v+\beta_2\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}$$
$$+\frac{1}{|\lambda|}\|U\|^2_{\mathscr{H}}\Big), \tag{57}$$

for $|\lambda|>1$ and $C>0$. Therefore, by (50) and (51), with $\alpha_1=0$, (53) and (57) we obtain

$$\|v\|^2_{L^2(0,L)}+\|\eta\|^2_{L^2(0,L)}\leqslant C\Big(\|\beta_1 v+\beta_2\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}$$
$$+\frac{1}{|\lambda|}\|U\|^2_{\mathscr{H}}). \tag{58}$$

Combining of (53), (57) and (58) yields

$$\|U\|^2_{\mathscr{H}}\leqslant C\left(\|\beta_1 v+\beta_2\eta\|_{L^2(0,L)}\|U\|_{\mathscr{H}}+\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+\frac{1}{|\lambda|}\|U\|^2_{\mathscr{H}}\right)$$

when $|\lambda|>1$, for a positive constant C. Finally, from (53) it follows

$$\left(1-\frac{C}{|\lambda|}\right)\|U\|_{\mathscr{H}}\leqslant C\|F\|_{\mathscr{H}}.$$

Thus

$$\|(i\lambda I-A)^{-1}\|_{\mathscr{H}}\leqslant C, |\lambda|>\max\{1,C\},$$

and the proof is complete. □

4. A lack of exponential stability

In this section to simplify the computation, without loss of generality, we assume $L=\pi$. Moreover, we will suppose

$$\beta_2(\beta_1\rho_2a_{11}+\beta_2\rho_1a_{12})=\beta_1(\beta_2\rho_1a_{22}+\beta_1\rho_2a_{12}). \tag{59}$$

First of all, note that $\frac{\beta_1}{\rho_1}\left(\frac{a_{11}}{\beta_1}-\frac{a_{12}}{\beta_2}\right)$ is always positive. In fact, the hypothesis (59) implies that

$$\frac{\beta_1}{\rho_1}\left(\frac{a_{11}}{\beta_1}-\frac{a_{12}}{\beta_2}\right)=\frac{\beta_2}{\rho_2}\left(\frac{a_{22}}{\beta_2}-\frac{a_{12}}{\beta_1}\right),$$

and taking, for example, $\beta_1>0$ and $\beta_2<0$, we have that $\frac{a_{11}}{\beta_1}-\frac{a_{12}}{\beta_2}\leqslant 0$ implies $\frac{a_{22}}{\beta_2}-\frac{a_{12}}{\beta_1}\geqslant 0$. It results that $0<a_{11}\leqslant\frac{\beta_1}{\beta_2}a_{12}$ and $0<a_{22}\leqslant\frac{\beta_2}{\beta_1}a_{12}$ and therefore $a_{11}a_{22}\leqslant a_{12}^2$, but this is contradictory to our hypotheses over (a_{ij}).

We observe that in order to prove the next theorem, due to Theorem 3.1, it is sufficient to show that there exists a sequence of real number (λ_v) with $\lambda_v\to\infty$ and a bounded sequence (F_v) in $\mathscr{H}$ such that

$$\|(i\lambda_vI-\mathscr{A})^{-1}F_v\|_{\mathscr{H}}\to\infty,\quad v\to\infty.$$

Theorem 4.1. *Suppose that $\alpha_1=0,(b_{11},b_{12}),(b_{12},b_{22})$ and (β_1,β_2) are collinear and (59) holds. Then the semigroup $S_{\mathscr{A}}(t)$ is not exponentially stable.*

Proof. We will restrict our attention to the cases $\beta_1\beta_2\neq0$ and $\beta_1+\beta_2\neq0$. Consider a,b real numbers such that $a\beta_2-b\beta_1\neq0$. For each $n\in\mathbb{N}$, we take $F_n=(0,0,\rho_1^{-1}a\sin(nx),\rho_2^{-1}b\sin(nx),0)\in\mathscr{H}$, and denote by $U_n=(u_n,w_n,v_n,\eta_n,\theta_n)\in\mathscr{D}(\mathscr{A})$ the unique solution of the resolvent equation

$$(i\lambda I-\mathscr{A})U_n=F_n,\quad \lambda\in\mathbb{R}. \tag{60}$$

Due to the boundary conditions, the solutions are of the form $u_n=A_n\sin(nx),w_n=B_n\sin(nx)$ and $\theta_n=C_n\cos(nx)$ and we get the system

$$v_n=i\lambda u_n,\quad \eta_n=i\lambda w_n,$$
$$-\rho_1\lambda^2A_n+n^2(a_{11}+i\lambda b_{11})A_n+n^2(a_{12}+i\lambda b_{12})B_n$$
$$+\alpha(A_n-B_n)-\beta_1nC_n=a, \tag{61}$$
$$-\rho_2\lambda^2B_n+n^2(a_{12}+i\lambda b_{12})A_n+n^2(a_{22}+i\lambda b_{22})B_n$$
$$-\alpha(A_n-B_n)-\beta_2nC_n=b, \tag{62}$$
$$c\lambda C_n+\lambda\beta_1nA_n+\lambda\beta_2nB_n-i\kappa n^2C_n=0. \tag{63}$$

Multiplying (61) by $\frac{1}{\beta_1}$, (62) by $\frac{1}{\beta_2}$, subtracting the equations and using (59) we obtain

$$\left(-\lambda^2+\frac{(\beta_2a_{11}-\beta_1a_{12})n^2}{\rho_1\beta_2}\right)\left(\frac{\rho_1}{\beta_1}A_n-\frac{\rho_2}{\beta_2}B_n\right)+\alpha\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)(A_n-B_n)$$
$$=\frac{a}{\beta_1}-\frac{b}{\beta_2}.$$

Taking $\lambda=\lambda_n=\sqrt{\frac{\beta_2a_{11}-\beta_1a_{12}}{\rho_1\beta_2}}n=\sigma n$ in (60) we get

$$A_n-B_n=\frac{\beta_2a-\beta_1b}{\alpha(\beta_2+\beta_1)}=\tau.$$

Substituting $A_n=B_n+\tau$ into (61) and (63) we get

$$B_n=\frac{\beta_1\beta_2}{(\beta_1+\beta_2)a_{12}n+i\beta_2\sigma(b_{11}+b_{12})n^2}C_n$$
$$+\frac{a\beta_2-\alpha\beta_2\tau-\tau\beta_1a_{12}n^2-i\tau\sigma\beta_2b_{11}n^3}{(\beta_1+\beta_2)a_{12}n^2+i\beta_2\sigma(b_{11}+b_{12})n^3},$$

and

$$B_n=-\frac{\tau\beta_1}{\beta_1+\beta_2}-\frac{(c\sigma-i\kappa n)}{\sigma(\beta_1+\beta_2)n}C_n.$$

Thus,

$$C_n=-\frac{\sigma(\beta_1+\beta_2)(a\beta_2-\alpha\beta_2\tau)-i\tau\sigma^2\beta_2(\beta_2b_{11}-\beta_1b_{12})n^3}{(c\sigma-i\kappa n)[(\beta_1+\beta_2)a_{12}n+i\beta_2\sigma(b_{11}+b_{12})n^2]+\sigma\beta_1\beta_2(\beta_1+\beta_2)n}.$$

Since $b_{11}+b_{12}\neq0$, then

$$\lim_{n\to\infty}|C_n| = \frac{\tau\sigma|\beta_2 b_{11} - \beta_1 b_{12}|}{\kappa|b_{11}+b_{12}|} = 0.$$

Therefore

$$\lim_{n\to\infty}|B_n| = \frac{|\tau\beta_1|}{|\beta_1+\beta_2|} \neq 0,$$

and since

$$\lim_{n\to\infty}\|w_{nx}\|^2_{L^2(0,\pi)} = \lim_{n\to\infty}\frac{n^2\pi}{2}|B_n|^2 = \infty,$$

we conclude that

$$\lim_{n\to\infty}\|U_n\|_{\mathscr{H}} = \infty. \tag{64}$$

Thus, the proof is complete. □

Remark 4.2. In case that β_1 or $\beta_2 = 0$, it follows from (59) that $a_{12} = 0$. Suppose, for example, that $\beta_2 = 0$. It results by the hypothesis of the Theorem 4.1 that $b_{12} = b_{22} = 0$. We can take in the proof of this theorem $a = 0, b = -1$ and $\lambda = \lambda_n = \sqrt{\frac{a_{22}n^2+\alpha}{\rho_2}}$. Then it follows by (62) that $A_n = \frac{1}{\alpha}$. Thus

$$\lim_{n\to\infty}\|u_{nx}\|_{L^2(0,\pi)} = \infty$$

and we have (64). When $\beta_1 + \beta_2 = 0$, condition (59) implies that $\rho_2(a_{11}+a_{12}) = \rho_1(a_{12}+a_{22})$. In this case, adding (61) and (62) we get

$$\left(-\lambda^2 + \frac{a_{11}+a_{12}}{\rho_1}n^2\right)(\rho_1 A_n + \rho_2 B_n) = a + b.$$

We can take $\lambda = \lambda_n = \sqrt{\frac{a_{11}+a_{12}}{\rho_1}n^2 + a + b}$ and obtain $\rho_1 A_n + \rho_2 B_n = -1$. We can use the same procedure as in the proof of the Theorem 4.1 to get (64).

Theorem 4.3. *Suppose that* $\alpha_1 > 0$ *and* $b_{11} = b_{22} = -b_{12}$ *and* $\beta_1 = -\beta_2$ *and* $\rho_2(a_{11}+a_{12}) = \rho_1(a_{22}+a_{12})$. *Then the semigroup* $S_{\mathscr{A}}(t)$ *is not exponentially stable.*

Proof:. For each $n \in \mathbb{N}$, we take $F_n = (0, 0, \rho_1^{-1}\sin(nx), \rho_2^{-1}\sin(nx), 0) \in \mathscr{H}$, and denote by $U_n \in \mathscr{D}(\mathscr{A})$ the unique solution of the resolvent Eq. (60). The solutions are of the form $u_n = A_n\sin(nx), w_n = B_n\sin(nx)$ and $\theta_n = C_n\cos(nx)$ and we get $v_n = i\lambda u_n, \eta_n = i\lambda w_n$,

$$-\rho_1\lambda^2 A_n + n^2(a_{11} + i\lambda b_{11})A_n + n^2(a_{12} - i\lambda b_{11})B_n + \alpha(A_n - B_n) + i\alpha_1\lambda(A_n - B_n) - \beta_1 n C_n = 1, \tag{65}$$

$$-\rho_2\lambda^2 B_n + n^2(a_{12} - i\lambda b_{11})A_n + n^2(a_{22} + i\lambda b_{11})B_n - \alpha(A_n - B_n) - i\alpha_1\lambda(A_n - B_n) + \beta_1 n C_n = 1, \tag{66}$$

$$c\lambda C_n + \lambda\beta_1 n A_n - \lambda\beta_1 n B_n - i\kappa n^2 C_n = 0. \tag{67}$$

Adding (65) and (66) we get

$$\left(-\lambda^2 + \frac{(a_{11}+a_{12})n^2}{\rho_1}\right)(\rho_1 A_n + \rho_2 B_n) = 2. \tag{68}$$

We take $\lambda = \lambda_n = \sqrt{\frac{(a_{11}+a_{12})n^2}{\rho_1} + 2}$. Hence in (68) we obtain

$$\rho_1 A_n + \rho_2 B_n = -1 \quad \text{or} \quad A_n = \frac{-(1+\rho_2 B_n)}{\rho_1} \tag{69}$$

Substituting (69) into (65) and (67) and performing straightforward calculations we obtain

$$n^2[(1+\rho_2 B_n)(a_{11}+a_{12}) - (a_{11}+i\lambda b_{11}) - \rho_2(a_{11}+i\lambda b_{11})B_n + \rho_1(a_{12} - i\lambda b_{11})B_n] + 2\rho_1\rho_2 B_n - (\alpha + i\lambda\alpha_1)(1 + (\rho_1+\rho_2)B_n) + \frac{\lambda\beta_1^2 n^2(1+(\rho_1+\rho_2)B_n)}{i\kappa n^2 - c\lambda} = -\rho_1.$$

Therefore

$$\lim_{n\to\infty}|B_n| = \frac{1}{\rho_1+\rho_2} \neq 0,$$

and we have (64). □

5. Numerical examples

The following numerical examples show the asymptotic behavior of the solution of (10) due to the exponential stability when the conditions (4) and (5)–(8) are verified, or the conditions (6) and (7) are verified, and the lack of exponential stability, when they are not verified.

5.1. Example I. Amplitudes for sample sinusoidal initial condition

We consider here, a similar example of the previous section. That is, we choose $L = \pi$, and we suppose that the solutions are of the form$u_n = A_n(t)\sin(nx), w_n = B_n(t)\sin(nx), v_n = A_n\prime(t)\sin(nx)$, $w_n = B_n\prime(t)\sin(nx)$, and $\theta_n = C_n(t)\cos(nx)$. In this case, the amplitudes (A_n, B_n, C_n) verify the following system of ODEs:

$$\begin{aligned}
\rho_1 A_n'' &= -n^2(a_{11}A_n + a_{12}B_n + b_{11}A_n' + b_{12}B_n') \\
&\quad - \alpha(A_n - B_n) - \alpha_1(A_n' - B_n') + n\beta_1 C_n, \\
\rho_2 B_n'' &= -n^2(a_{12}A_n + a_{22}B_n + b_{12}A_n' + b_{22}B_n') \\
&\quad + \alpha(A_n - B_n) + \alpha_1(A_n' - B_n') + n\beta_2 C_n, \\
cC_n' &= -n\beta_1 A_n' - n\beta_2 B_n' - n^2\kappa C_n.
\end{aligned} \tag{70}$$

Thus, we consider the system (70) with the parameter values $a_{11} = a_{22} = 1.0, a_{12} = 0.0$, and $\rho_1 = \rho_2 = \alpha = c = \kappa = 1.0$.

Figs. 1 and 2 represent the evolution in time of the three amplitudes A_n, B_n, and C_n, and the derivatives A_n' and B_n' (which are the amplitudes of v and η, respectively), for $n = 100$. For the numerical simulation, we use the Runge-Kutta-Fehlberg method RKF45, with the standard solver `ode45()` of MATLAB. Fig. 1 corresponds to three different simulations with $\alpha_1 = 0.0$. The case (a) and (b) are simulations for $0 \leqslant t \leqslant 5.0$ and the case (c) is a simulation for $0 \leqslant t \leqslant 0.001$.

The first picture (a), represents a lack of exponential stability example when the hypothesis of Theorem 4.1 is verified:

$$\begin{aligned}
(b_{11}, b_{12}) = (1, -1), \quad (b_{12}, b_{22}) = (-1, 1) \quad \text{and} \quad (\beta_1, \beta_2) \\
= (1, -1), \quad \text{are collinear;} \quad \text{and} \quad \beta_2(\beta_1\rho_2 a_{11} + \beta_2\rho_1 a_{12}) \\
= \beta_1(\beta_2\rho_1 a_{22} + \beta_1\rho_2 a_{12}) = -1.
\end{aligned}$$

The second picture (b), represents an exponential stability example when the hypothesis of Theorem 3.3 is verified:

$$\begin{aligned}
\{(b_{11}, b_{12})(\beta_1, \beta_2)\} = \{(1, -1), (1, 1)\} \quad \text{and} \quad \{(b_{12}, b_{22})(\beta_1, \beta_2)\} \\
= \{(-1, 1), (1, 1)\}, \text{are linearly independent.}
\end{aligned}$$

The third picture (c), represents again an exponential stable case with

$$\begin{aligned}
\{(b_{11}, b_{12})(\beta_1, \beta_2)\} = \{(1, 0), (1, 1)\} \quad \text{and} \quad \{(b_{12}, b_{22})(\beta_1, \beta_2)\} \\
= \{(0, 1), (1, 1)\}, \text{linearly independent.}
\end{aligned}$$

We observe clearly in this picture (c), that the five amplitudes $A_n(t), B_n(t), A_n'(t), B_n'(t)$ and $C_n(t)$, tend to zero faster than the case (b), as $t \to \infty$. The numerical reason of this behavior is because in this case, which differs from case (b), the matrix $B = (b_{ij})$ is symmetric and positive definite. However both (b) and (c), are exponentially stable.

In the case $\alpha_1 > 0$, if we consider the same values of the parameter b_{ij} and β_i used in the before example, although not shown here,

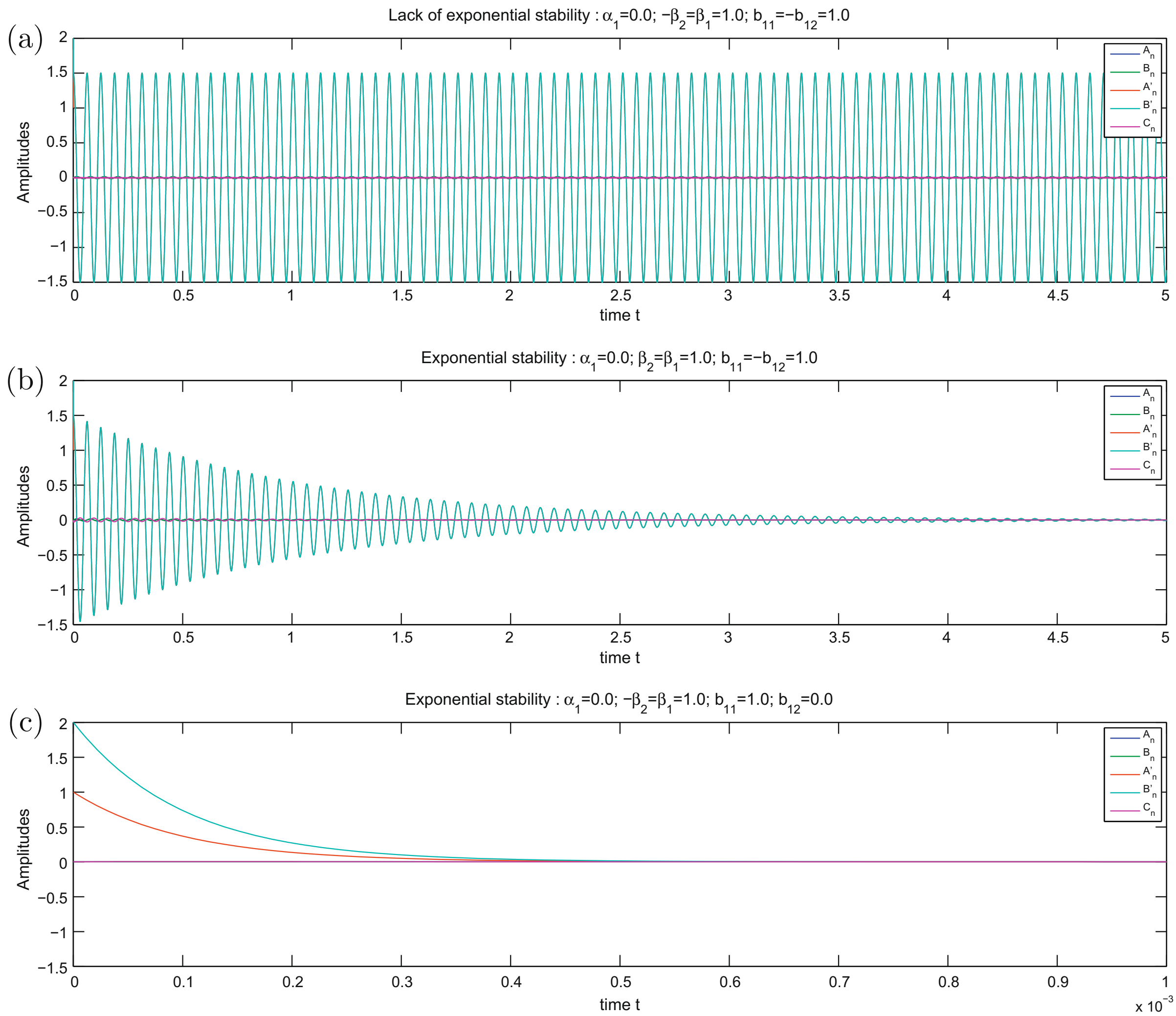

Fig. 1. Example I. Different evolution of the amplitudes for case $\alpha_1 = 0.0$: (a) lack of exponential stability ($\beta_1 = b_{11} = b_{22} = 1.0, \beta_2 = b_{12} = -1.0$); (b) exponential stability ($\beta_1 = \beta_2 = b_{11} = b_{22} = 1.0, b_{12} = -1.0$); (c) exponential stability ($\beta_1 = b_{11} = b_{22} = 1.0, \beta_2 = -1.0, b_{12} = 0.0$).

it is obtained almost identical behavior to those of Fig. 1. The reason is because the choice of this parameters values gives the exponential stability, or the lack thereof, independent of the value of α_1. However, in Fig. 2, we see a case where the value of α_1 affects the exponential stability of the solution.

In pictures (d), (e) and (f) of Fig. 2, we choose $b_{11} = b_{22} = b_{12} = 1.0$ and $\beta_1 = \beta_2 = 1.0$, that is sufficiently conditions for the lack of exponential stability when $\alpha_1 = 0.0$ (see picture (d)). On the other hand, the same values $b_{11} = b_{22} = b_{12} = \beta_1 = \beta_2 = 1.0$, ensure the exponential stability when $\alpha_1 > 0.0$ (see pictures (e) and (f)). Also we note for a numerical point of view, that the exponential decay to zero is faster when $\alpha_1 > 0$ is larger (compare pictures (e) and (f)).

5.2. Example II. Asymptotic behavior for a small initial condition

Here, we numerically compute the solution of the system (10), with $L = 1.0, T = 2.0$, and the initial condition:

$$v(x,0) = \begin{cases} 0 & \text{if} \quad 0.0 \leqslant x \leqslant 0.4, \\ 10(x-0.4) & \text{if} \quad 0.4 \leqslant x \leqslant 0.5, \\ 10(0.6-x) & \text{if} \quad 0.5 \leqslant x \leqslant 0.6, \\ 0 & \text{if} \quad 0.6 \leqslant x \leqslant 1.0, \end{cases}$$

$$\eta(x,0) = \begin{cases} 0 & \text{if} \quad 0.0 \leqslant x \leqslant 0.4, \\ 20(x-0.4) & \text{if} \quad 0.4 \leqslant x \leqslant 0.5, \\ 20(0.6-x) & \text{if} \quad 0.5 \leqslant x \leqslant 0.6, \\ 0 & \text{if} \quad 0.6 \leqslant x \leqslant 1.0, \end{cases} \tag{71}$$

and $u(x,0) = w(x,0) = \theta(x,0) = 0.0$. We remark that the initial condition defined in (71) are two peaks of height 1 and 2, respectively, and support in $(0.4;0.6)$. Additionally, we consider the same parameter values of the Example I, $a_{11} = a_{22} = 1.0, a_{12} = 0.0$, and $\rho_1 = \rho_2 = \alpha = c = \kappa = 1.0$.

In order to compare these numerical results with those of Example I, and the previous Section 4, we assume that

$$u(x,t) = \sum_{k=1}^{\infty} A_k(t)\sin(k-1)\pi x, \quad w(x,t) = \sum_{k=1}^{\infty} B_k(t)\sin(k-1)\pi x,$$
$$\theta(x,t) = \sum_{k=1}^{\infty} C_k(t)\cos(k-1)\pi x, \tag{72}$$

and therefore, we extend the initial conditions (71) by odd functions, in the interval $(-L,0)$. On the other hand, if we discretize the space dimension $(-L,L) = (-1,1)$ in $2N-1$ subintervals $I_j = (j\delta x, (j+1)\delta x)$, with $\delta x = 1/(2N)$, and $j = -N, \ldots, N-1$, and we approximate the solution $U(x,t)$ of the system (10) by piecewise functions equal to $U_j(t)$ in each subinterval, then we can take the Discrete Fourier Transform of the solution:

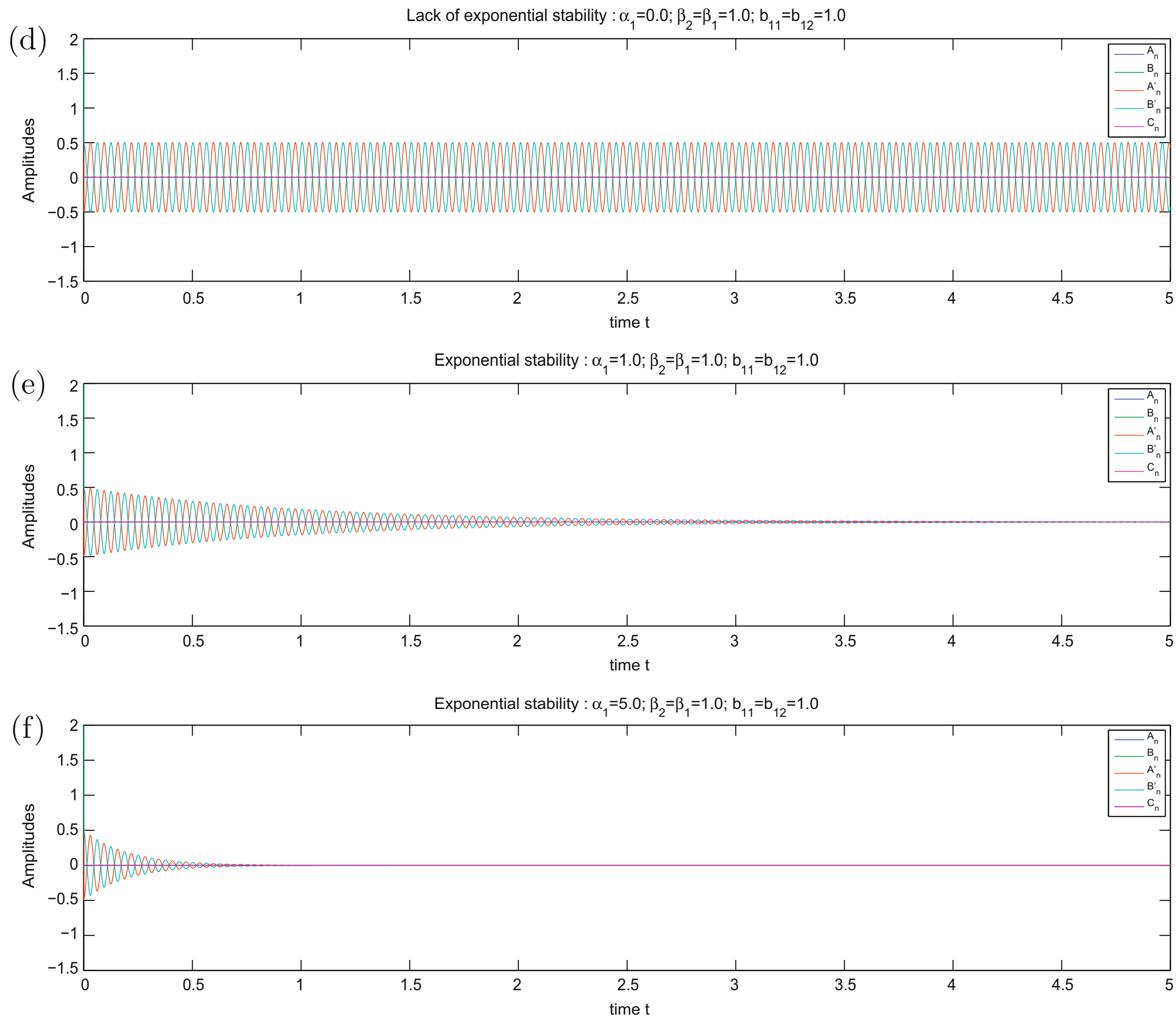

Fig. 2. Example I. Different evolution of the amplitudes for case $\beta_1 = \beta_2 = b_{11} = b_{22} = b_{12} = 1.0$: (d) lack of exponential stability (case $\alpha_1 = 0.0$); (e) exponential stability (case $\alpha_1 = 1.0$); (f) exponential stability (case $\alpha_1 = 5.0$).

$$\widetilde{U}_k(t) = \sum_{j=1}^{2N} U_j(t) e^{-\pi i(k-1)(j-1)/N}, \tag{73}$$

and we reconstruct the solution by the Inverse discrete Fourier transform:

$$U_j(t) = \frac{1}{2N} \sum_{k=1}^{2N} \widetilde{U}_k(t) e^{\pi i(k-1)(j-1)/N}. \tag{74}$$

We note that if we define $\widetilde{U}_k(t) = (\widetilde{u}_j(t), \widetilde{w}_j(t), \widetilde{v}_j(t), \widetilde{\eta}_j(t), \widetilde{\theta}_j(t))^T$, then $A_k(t) = -\Im(\widetilde{u}_j(t))$, $B_k(t) = -\Im(\widetilde{w}_j(t))$, $C_k(t) = \Re(\widetilde{\eta}_j(t))$ and the following system of ODEs is verified:

$$\begin{aligned}
&\widetilde{u}'_k = \widetilde{v}_k, \quad \widetilde{w}'_k = \widetilde{\eta}_k,\\
&\rho_1 \widetilde{v}'_k = -\pi^2(k-1)^2(a_{11}\widetilde{u}_k + a_{12}\widetilde{w}_k + b_{11}\widetilde{v}_k + b_{12}\widetilde{\eta}_k)\\
&\qquad - \alpha(\widetilde{u}_k - \widetilde{w}_k) - \alpha_1(\widetilde{v}_k - \widetilde{\eta}_k) - \pi i(k-1)\beta_1\widetilde{\theta}_k,\\
&\rho_2 \widetilde{\eta}'_k = -\pi^2(k-1)^2(a_{12}\widetilde{u}_k + a_{22}\widetilde{w}_k + b_{12}\widetilde{v}_k + b_{22}\widetilde{\eta}_k) + \alpha(\widetilde{u}_k - \widetilde{w}_k)\\
&\qquad + \alpha_1(\widetilde{v}_k - \widetilde{\eta}_k) - \pi i(k-1)\beta_2\widetilde{\theta}_k,\\
&c\widetilde{\theta}'_k = -\pi i(k-1)\beta_1\widetilde{v}_k - \pi i(k-1)\beta_2\widetilde{\eta}_k - \pi^2(k-1)^2\kappa\widetilde{\theta}_k.
\end{aligned} \tag{75}$$

We make simulations for $N = 1024$ using in this case the Stiff solver `ode15s()` of MATLAB to compute each one of the 1024 system of Eq. (75), and we reconstruct the solution by the Inverse discrete Fourier transform (74).

Figs. 3 and 4 represent the evolution of the solutions (u, w, v, η, θ), with the same parameter b_{ij} and β_i of Example I cases (d), (e) and (f): $b_{11} = b_{22} = b_{12} = 1.0$ and $\beta_1 = \beta_2 = 1.0$. Fig. 3 shows the lack of exponential stability with $\alpha_1 = 0.0$, and Fig. 4 shows the exponential stability with $\alpha_1 = 1.0$. In both figures, $u(x,t)$ is graph at top left, $w(x,t)$ at top right, $v(x,t)$ at bottom left, $\eta(x,t)$ at bottom right, and $\theta(x,t)$ in the center.

Finally, in Fig. 5 it is represented the norm $\mathscr{H}$ of the numerical solution of (10) for the 5 first cases of Example I ((a), (b), (c), (d) and (e)). More precisely, we plot the function:

$$t \mapsto \sqrt{\sum_{j=1}^{N} h\left(\left(\frac{u_j(t)-u_{j-1}(t)}{h}\right)^2 + \left(\frac{w_j(t)-u_{j-1}(t)}{h}\right)^2 + v_j^2(t) + w_j^2(t) + \left(\theta_j(t) - h\sum_{\ell}\theta_\ell(t)\right)^2\right)}.$$

We observe that in general, the cases of lack of exponential stability the curves diverge when $t \to \infty$ ((a) and (d)), and the curves tend to zero in the exponential stable cases ((b), (c) and (e)).

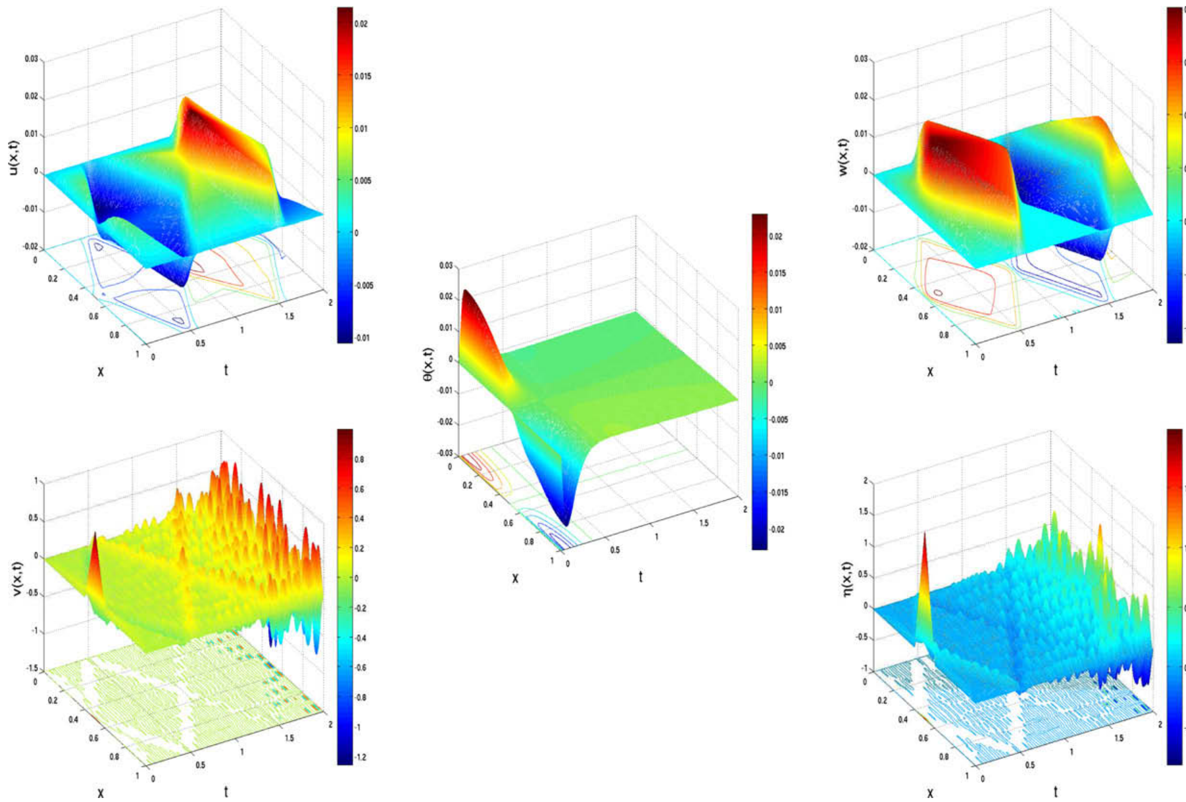

Fig. 3. Example II. Lack of exponential stability. Numerical solutions u, w, v, η, θ. Case $\alpha_1 = 0.0, \beta_1 = \beta_2 = b_{11} = b_{22} = b_{12} = 1.0$.

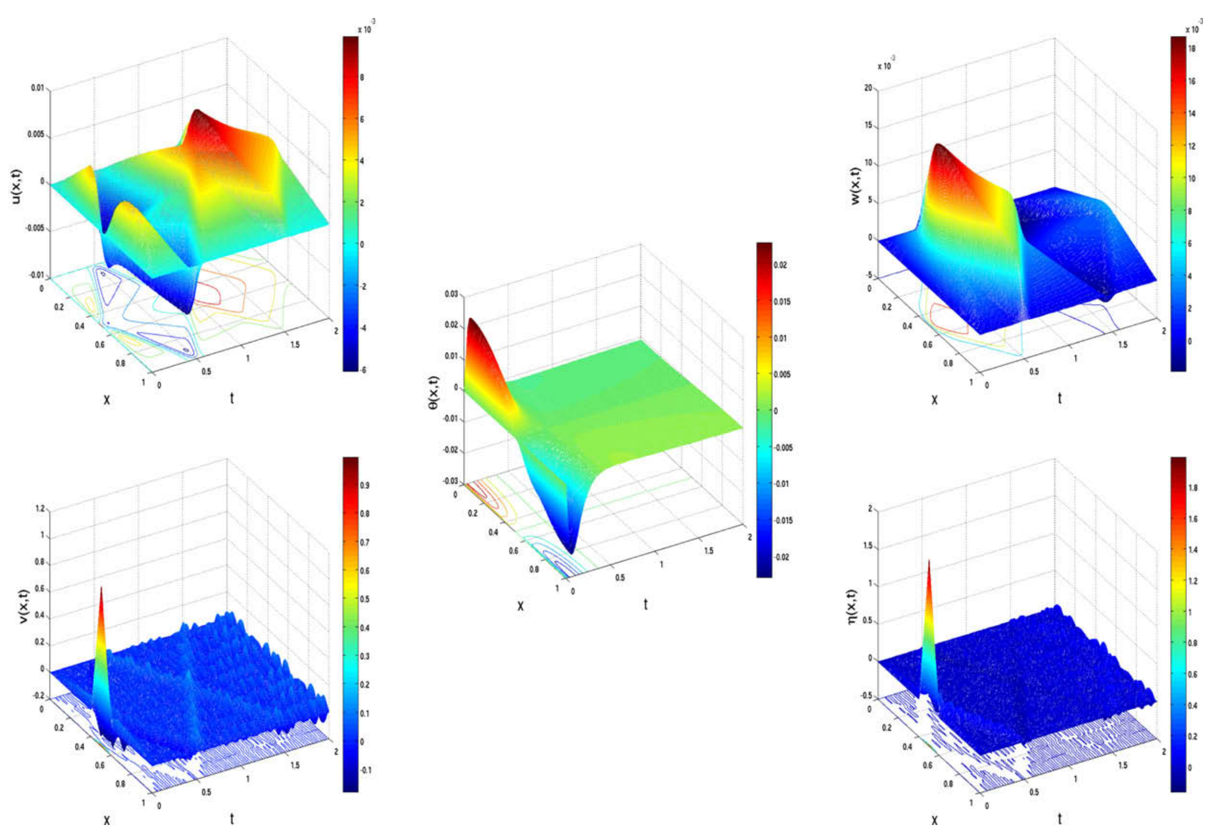

Fig. 4. Example II. Exponential stability. Numerical solutions u, w, v, η, θ. Case $\alpha_1 = 1.0, \beta_1 = \beta_2 = b_{11} = b_{22} = b_{12} = 1.0$.

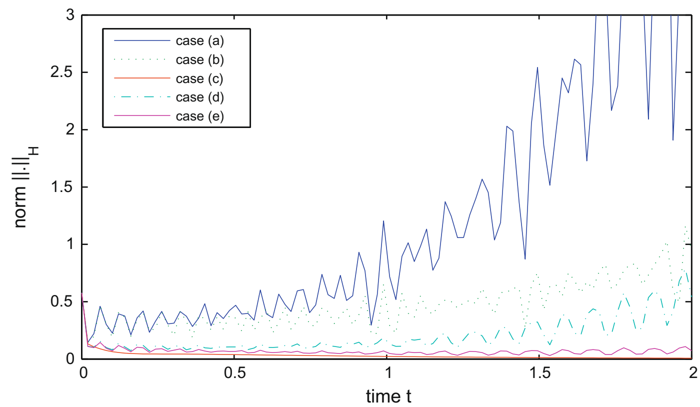

Fig. 5. Example II. Evolution in time of $t\mapsto\|U(\cdot,t)\|_{\mathscr{H}}$ for: (a) Lack of exponential stability $\alpha_1=0.0,\beta_1=b_{11}=b_{22}=1.0,\beta_2=b_{12}=-1.0$; (b) Exponential stability $\alpha_1=0.0,\beta_1=\beta_2=b_{11}=b_{22}=1.0,b_{12}=-1.0$; (c) Exponential stability $\alpha_1=0.0,\beta_1=b_{11}=b_{22}=1.0,\beta_2=-1.0,b_{12}=0.0$; (d) Lack of exponential stability $\alpha_1=0.0,\beta_1=\beta_2=b_{11}=b_{22}=b_{12}=1.0$; (e) Exponential stability $\alpha_1=1.0,\beta_1=\beta_2=b_{11}=b_{22}=b_{12}=1.0$.

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