UNIVERSIDADE FEDERAL DE VIÇOSA

The algebraic renormalization of Yang-Mills-Chern-Simons theory

Daniel Oliveira Rocha Azevedo
Doctor Scientiae

VIÇOSA - MINAS GERAIS
2025

DANIEL OLIVEIRA ROCHA AZEVEDO

The algebraic renormalization of Yang-Mills-Chern-Simons theory

Thesis submitted to the Physics Graduate Program of the Universidade Federal de Viçosa in partial fulfillment of the requirements for the degree of *Doctor Scientiae*.

Adviser: Oswaldo Monteiro Del Cima

Co-adviser: Daniel H. Theodoro Franco

VIÇOSA - MINAS GERAIS
2025

Ficha catalográfica elaborada pela Biblioteca Central da Universidade Federal de Viçosa - Campus Viçosa

T

Azevedo, Daniel Oliveira Rocha, 1996-

A994a The algebraic renormalization of Yang-Mills-Chern-Simons
2025 theory / Daniel Oliveira Rocha Azevedo. – Viçosa, MG, 2025.

1 tese eletrônica (117 f.): il.

Texto em inglês.

Orientador: Oswaldo Monteiro Del Cima.

Tese (doutorado) - Universidade Federal de Viçosa, Departamento de Física, 2025.

Referências bibliográficas: f. 95-117.

DOI: https://doi.org/10.47328/ufvbbt.2025.258

Modo de acesso: World Wide Web.

1. Yang-Mills, Teoria de. 2. Campos de calibre (Física). 3. Simetria (Física). 4. Renormalização (Física). I. Del Cima, Oswaldo Monteiro, 1965-. II. Universidade Federal de Viçosa. Departamento de Física. Programa de Pós-Graduação em Física. III. Título.

CDD 22. ed. 530.1435

Bibliotecário(a) responsável: Bruna Silva CRB-6/2552

DANIEL OLIVEIRA ROCHA AZEVEDO

The algebraic renormalization of Yang-Mills-Chern-Simons theory

Thesis submitted to the Physics Graduate Program of the Universidade Federal de Viçosa in partial fulfillment of the requirements for the degree of *Doctor Scientiae*.

APPROVED: March 26, 2025.

Assent:

Daniel Oliveira Rocha Azevedo
Author

Oswaldo Monteiro Del Cima
Adviser

ACKNOWLEDGMENTS

First and foremost, to my parents. They are the giants on whose shoulds I stand on. And my sister, who has always been a beacon helping me find my own ways.

To my advisor, Prof. Oswaldo Del Cima, who has helped me grow and become who I am today, as a researcher and as a person.

To professors Piguet, Sorella, Daniel and Antônio, whose lessons are engraved in my memories. I'll be always looking up to you wherever I go.

To my friends and family, with whom I can always share my laughs and my worries (even tough I'm not very good at doing the latter).

To my partner Rebeca, who is with me in the good and the bad days, always helping me get through.

This work has been sponsored by the following Brazilian research agencies: Coordination for the Improvement of Higher Education Personnel (CAPES; Financing code 001), Minas Gerais State Foundation for Research Aid (FAPEMIG) and National Council of Scientific and Technological Development (CNPq).

ABSTRACT

AZEVEDO, Daniel Oliveira Rocha, D.Sc., Universidade Federal de Viçosa, March, 2025. **The algebraic renormalization of Yang-Mills-Chern-Simons theory**. Adviser: Oswaldo Monteiro Del Cima. Co-adviser: Daniel Heber Theodoro Franco.

In this thesis we study the quantization of Yang-Mills-Chern-Simons theory through the lens of the algebraic renormalization framework, which allows us to prove the extensions of the symmetries of the classical action to the quantum theory to all orders in perturbation theory in a regularization-scheme independent way. This is a particularly useful feature when dealing with theories containing the Levi-Civita totally anti-simmetric tensor, since a regularization which preserves the symmetries is not always available in such cases. Moreover, it allows us to determine the most general local counterterms that can be added to the classical action in a purely algebraic way. Based on the algebraic framework, we establish a local version of the Callan-Symanzik equation which proves the non-renormalization of the Chern-Simons mass parameter and the finiteness of the Yang-Mills-Chern-Simons theory as a whole in the Landau gauge. We then analyse the theory in more general scenarios, including non-covariant gauges and taking into account the existence of Gribov copies which appear in the infrared regime of the theory. In both of these, we recover the finite character of the theory.

Keywords: Yang-Mills-Chern-Simons; Algebraic Renormalization; BRST symmetry

RESUMO

AZEVEDO, Daniel Oliveira Rocha, D.Sc., Universidade Federal de Viçosa, março de 2025. **A renormalização algébrica da teoria de Yang-Mills-Chern-Simons**. Orientador: Oswaldo Monteiro Del Cima. Coorientador: Daniel Heber Theodoro Franco.

Nesta tese, estudamos a quantização da teoria de Yang-Mills-Chern-Simons através do formalismo da renormalização algébrica, o que nos permite provar as extensões das simetrias da ação clássica à teoria quântica para todas as ordens na teoria de perturbação de uma forma independente do esquema de regularização. Este é um recurso particularmente útil ao lidar com teorias que contêm o tensor totalmente antissimétrico de Levi-Civita, uma vez que uma regularização que preserva as simetrias nem sempre está disponível em tais casos. Além disso, permite-nos determinar os contratermos locais mais gerais que podem ser adicionados à ação clássica de uma forma puramente algébrica. Com base no formalismo algébrico, estabelecemos uma versão local da equação de Callan-Symanzik que prova a não renormalização do parâmetro de massa de Chern-Simons e a finitude da teoria de Yang-Mills-Chern-Simons como um todo no calibre de Landau. Em seguida, analisamos a teoria em cenários mais gerais, incluindo calibres não covariantes e levando em consideração a existência de cópias de Gribov que aparecem no regime infravermelho da teoria. Em ambos, recuperamos o caráter finito da teoria.

Palavras-chave: Yang-Mills-Chern-Simons; Renormalização Algébrica; Simetria BRST

Contents

Chapter 1

Introduction

When non-Abelian gauge theory was first introduced in 1954 by C.N. Yang and R.L. Mills [1] (and independently by R. Shaw, who never published his results) and later generalized by R. Utyiama [2], it was treated more as a mathematical exercise than as physically significant. This impression came from a general disbelief that field theory would be applicable for anything other then quantum electrodynamics, which had its problematic ultraviolet behavior tamed in the late 40's by the works of H. Bethe [3], R. Feynman [4,5], J. Schwinger [6–8], S. Tomonaga [9–11] and F. Dyson [12]. The proliferation of new subatomic particles and the incapacity of theorists at the time to properly address them led many to search for more phenomenologically based approaches, such as the S-matrix theory (for example, [13]). The main issue was that the gauge invariance principle [14] required that all new particles, responsible for the interactions, needed to be massless, and no such particles were found so far.

This view, however, started to change in the mid 60's. The idea of spontaneous symmetry breaking, described by the Brout–Englert–Higgs mechanism [15–17], explained how gauge fields could acquire mass while preserving gauge invariance. This mechanism paved the way for a consistent description of the weak nuclear force, resulting in the Glashow-Salam-Weinberg theory or electroweak theory [18–20], which also presented a unified description of the weak and electromagnetic interactions. Around the same time, the new particles (hadrons) discovered in the preceding decades were organized by M. Gell-Mann [21] and Y. Ne'eman [22], in a scheme based in the $SU(3)$ symmetry group, called the Eightfold Way. These hadrons were later described as non-fundamental particles composed of quarks [23–25], whose dynamics is described by a theory which would later turn into our modern understanding of Quantum Chromodynamics (QCD). With these two pieces together, the Standard Model (SM) was formed.

The years following were marked by major advances in the physics of the SM. The renormalization of Yang-Mills theories was proven at the start of the decade by G. 't Hooft, both at the symmetric [26] and broken [27] phases. Moreover, the theoretical framework established allowed for the prediction of a fourth quark [28, 29], which was later confirmed to exist with the detection of the J/ψ (charmonium) [30,31], in a remarkable event known afterwards as the November revolution. The discovery of the weak neutral current [32], confirming a prediction of the electroweak theory, and the discovery of asymptotic freedom [33, 34], indicating a possible explanation for the confinement of quarks, were further evidence for the robustness of the SM. With the identification of the Higgs boson [35,36] in 2012, the last piece of the Standard Model was set. However, there are still phenomena it does not contemplate, like the neutrino masses, which require some extension of the theory.

Another important event coming from those years was the identification of the Becchi-Rouet-Stora-Tyutin (BRST) symmetry and the impact it had on theoretical physics. The quantization of Yang-Mills theories in the continuum (and of gauge theories in general) requires the introduction of a gauge-fixing term. In the framework of path integral quantization, the gauge-fixing is necessary to avoid the overcounting of gauge field configurations related by a gauge transformation. In non-Abelian gauge theories, this term is introduced through the Faddeev-Popov procedure [37]. This leads to a breakdown of gauge invariance for the total action, but also to the emergence of BRST symmetry [38–41]. This symmetry allowed for a sophisticated characterization of the renormalization of gauge theories as a cohomology problem, namely the cohomology of the Slavnov-Taylor operator, which expresses the invariance of the total action under BRST transformations, and for the consistent construction of the physical space of the theory [42], without unphysical and non-propagating modes. With the BRST symmetry, one can study the renormalization of gauge theories in a purely algebraic manner (see [43]), without relying on a specific renormalization-scheme.

This discussion on renormalization works really well for the perturbative regime, but fails when we try to asses strongly coupled theories. To this date, we still don't know exactly what mechanism drives confinement in QCD, that is, what makes it go from a theory of quarks and gluons at high energies to one of hadrons and mesons atlow energies. In the late 70's, this motivated some research on $2 + 1$ dimensional Yang-Mills theories, since they exhibit an explicitly confining phase [44–46]. Moreover, these theories allow for the introduction of a gauge-invariant mass term for the gauge fields [47–49], called the Chern-Simons term, endowing this lower dimensional theory with a vastly different structure than its four dimensional counterpart. Beyond serving as a theoretical laboratory, where we can study phenomena which are of more complicated description in four dimensions, they are also of interest for the description of some condensed matter systems, like high-temperature superconductors [50, 51] and the fractional quantum Hall effect [52–54].

In view of all this considerations, in the following chapters we will present a study of Yang-Mills theories in three spacetime dimensions, endowed with a Chern-Simons mass term, which defines the so called Yang-Mills-Chern-Simons (YMCS) theory. We consider this type of theory for a general $SU(N)$ group. The quantization of such theories will be made using the BRST framework through the algebraic renormalization program. In chapter 2, we present a general introduction to the Quantum Action Principle (QAP) [55–62] , which is key in establishing the validity of the algebraic method. In chapter 3, we introduce the YMCS theory in the Landau gauge and prove its renormalizability to all orders in perturbation theory. Finally, we prove the finite character of YMCS using a local formulation of the Callan-Symanzik equation. In chapter 4, we introduce a discussion of non-covariant gauges, which do not allow for a direct implementation of the QAP. We then construct a more general gauge fixing, which interpolates between covariant and non-covariant gauges, and show that the assumptions of the QAP are valid for it. Lastly, we prove the renormalizability of YMCS theory in the interpolating gauge. In chapter 5 we introduce a discussion on the Gribov problem and how it affects non-Abelian theories in the infrared regime. We then present a partial solution to this issue through the formulation of the Gribov-Zwanziger action and its subsequent refinement. At last, we implement the elimination of the so-called Gribov copies coming from the Gribov problem to the YMCS action in linear covariant gauges and prove that it maintains the finiteness of the usual YMCS theory. In chapter 6, we gather our final remarks and the perspectives for future work.

For our conventions, we use natural units $\hbar = c = 1$, with the exception of chapter 2, where we keep $\hbar$ for clarity in the exposition of some arguments, and use the Minkowski metric in (2+1)d $\eta_{\mu\nu} = (1, -1, -1)$, except for chapter 5, where we adopt the Euclidean signature $\eta_{\mu\nu} = \delta_{\mu\nu} = (1, 1, 1)$. Whenever the number of dimensions is not explicitly stated (for example, in the integral measure d^3x), it is assumed that we are working in a general dimension d.

Chapter 2

An introduction to the Quantum Action Principle

2.1 A Brief Remark on the Generating Functionals

2.1.1 The Green functional

Before the presentation of the algebraic renormalization of the Yang-Mills-Chern-Simons theory, we shall first introduce how the Quantum Action Principle emerges from the formulation of renormalized perturbation theory and the major role played by symmetries in the renormalization of a theory. We start by discussing the Green functions (or correlation functions) of a theory, given by the vacuum expectation value of time-ordered products of the field operators $\phi_i(x)$

$$G_{i_1 \ldots i_n}(x_1, \ldots, x_n) = \langle T\phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n)\rangle, \tag{2.1}$$

where the index i generically represents any tensorial or internal structure the field may posses.

From the Green functions, we can obtain all physical information of a theory. Indeed, one can show that a theory can be fully reconstructed from its vacuum expectation values (cf. [63] for a complete discussion). In particular, correlation functions are the building blocks for the LSZ (Lehmann, Symanzik, Zimmermann) [64] construction of the scattering matrix, also called S-matrix, which gives us the probability of obtaining a certain outgoing state of a system from the incoming state (cf. [65] for a pedagogical introduction).

Within the scope of perturbation theory, correlation functions are constructed as formal power series in the coupling constants, and are related to the diagrammatic expansion using Feynman graphs. We talk of formal series in the sense that the results from the perturbative expansion are expected to hold order by order and no assumptions about convergence of their sums are being made, all interest lying on the coefficients. With that in mind, we define the generating functional of the Green functions as

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!\,\hbar^n} \int dx_1 \dots dx_n G_{i_1 \dots i_n}(x_1, \dots, x_n) J^{i_1}(x_1) \dots J^{i_n}(x_n), \quad (2.2)$$

where the sources J^{i_n} are functions of the Schwartz space $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$, i.e. are rapidly decreasing C^∞ functions. This define the Green functions to be tempered distributions (cf. [63, 66] for a general discussion on the theory of distributions).

From the generating functional $Z[J]$, one can obtain the Green functions $G_{i_1 \dots i_n}(x_1, \dots, x_n)$ through functional derivatives with respect to the sources $J^{i_j}(x)$. Equivalently, we can write $Z[J]$ as

$$\begin{aligned} Z[J] = Z[0] &+ \frac{i}{\hbar} \int dx \left. \frac{\delta Z}{\delta J^i(x)} \right|_{J=0} J^i(x) \\ &+ \frac{i^2}{2\hbar^2} \int dx_1 dx_2 \left. \frac{\delta^2 Z}{\delta J^{i_1}(x_1) \delta J^{i_2}(x_2)} \right|_{J=0} J^{i_1}(x_1) J^{i_2}(x_2) + \dots \\ &+ \frac{i^n}{n!\,\hbar^n} \int dx_1 \dots dx_n \left. \frac{\delta^n Z}{\delta J^{i_1}(x_1) \dots \delta J^{i_n}(x_n)} \right|_{J=0} J^{i_1}(x_1) \dots J^{i_n}(x_n) + \dots . \end{aligned} \quad (2.3)$$

Therefore, we can identify

$$(-i\hbar)^n \left. \frac{\delta^n Z}{\delta J^{i_1}(x_1) \dots \delta J^{i_n}(x_n)} \right|_{J=0} = \langle T \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n) \rangle. \quad (2.4)$$

Using the previous result and taking into account that the transition amplitude for a given scattering process, i.e. a certain final state $|f\rangle$ coming from an initial state $|i\rangle$, is given by the sum over all possible field configurations of the exponential of the action [67, 68], we can formally define the generating functional as the path integral

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}(S[\phi] + \int dx \phi_i(x) J^i(x))} , \quad (2.5)$$

where $\mathcal{N}$ is a normalization factor, $\mathcal{D}\phi$ is the functional measure of integration and $S[\phi]$ is the action functional which determines the classical theory. With this expression for the generating functional, the correlation functions are given by

$$\langle T\phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)\rangle = \mathcal{N}\int \mathcal{D}\phi\ \phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)\ e^{\frac{i}{\hbar}S[\phi]}\ . \tag{2.6}$$

It is useful to split the action into its free and interacting parts

$$S[\phi] = S_0[\phi] + S_{int}[\phi], \tag{2.7}$$

the former being composed only of quadratic terms on the fields, while the latter exhibits terms which are the product of three or more fields. This can be written in the general form as

$$S[\phi] = \int dx\ \frac{1}{2}\phi_i(x)K^{ij}(\partial_x)\phi_j(x) + S_{int}[\phi], \tag{2.8}$$

where $K^{ij}(\partial)$ is some invertible differential operator, called the wave operator. One can perform a change of variables

$$\phi_i(x) \to \phi_i(x) - \int dy\ J^j(y)\Delta_{ij}(x,y), \tag{2.9}$$

where $\Delta_{ij}(x,y)$ is the inverse of the wave operator ($\Delta_{ik}(x,y)K^{kj}(\partial_x) = \delta_i^j\delta(x-y)$).The measure $\mathcal{D}\phi$ is left invariant, since the Jacobian of the transformation is the identity. After the change of variables, the free action plus source terms reads

$$S_0[\phi'] + \int dx\ \phi'_i(x)J^i(x) = S_0[\phi] - \frac{1}{2}\int dxdy\ J^i(x)\Delta_{ij}(x,y)J^j(y). \tag{2.10}$$

The generating functional of the free action can then be written as

$$Z_0[J] = \mathcal{N}\int \mathcal{D}\phi\ e^{\frac{i}{\hbar}S_0[\phi]}e^{-\frac{i}{2\hbar}\int dxdy\ J^i(x)\Delta_{ij}(x,y)J^j(y)}\ . \tag{2.11}$$

We see that, up to a normalization factor (which we can choose to be 1), the two-point correlation function of the free theory, i.e. the propagator, is proportional to the inverse of the wave operator

$$\langle T\phi_i(x)\phi_j(y)\rangle_0 = -i\hbar\Delta_{ij}(x,y). \tag{2.12}$$

Moreover, we see that higher point correlation functions obey the following rule

$$\langle T\phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)\rangle_0 = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (i\hbar)^n\Delta_{i_1i_2}\ldots\Delta_{i_{n-1}i_n} + \text{perm.}, & \text{if } n \text{ is even,} \end{cases} \tag{2.13}$$

where we need to take into account all possible permutations of the n indices. This result is known as Wick's theorem, which appears as a natural consequence in the functional formalism [67]. However, the free theory says nothing to us about how the fields interact. In fact, we see from (2.13) that higher point correlation functions are simply disconnected products of propagators.

As we shall see in the following, we can obtain the Green functions of the full interacting theory as a perturbation of the free one. To show this, let us assume a functional $R[\phi]$ of the free fields $\phi_i(x)$, which admits a power series expansion

$$R[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \left. \frac{\delta^n R[\phi]}{\delta\phi_{i_1}(x_1)\dots\delta\phi_{i_n}(x_n)} \right|_{\phi=0} \phi_{i_1}(x_1)\dots\phi_{i_n}(x_n). \tag{2.14}$$

We can readily see from (2.6) that the expectation value of such functional in the free theory is given by

$$\langle TR[\phi]\rangle_0 = \mathcal{N} \int \mathcal{D}\phi R[\phi] e^{\frac{i}{\hbar}S_0[\phi]} \ . \tag{2.15}$$

Choosing $R[\phi] = \phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)e^{\frac{i}{\hbar}S_{int}[\phi]}$ and the normalization factor to be $\mathcal{N} = \left(\int \mathcal{D}\phi \ e^{\frac{i}{\hbar}S_{int}[\phi]}e^{\frac{i}{\hbar}S_0[\phi]}\right)^{-1}$, we obtain the Gell-Mann-Low formula

$$\begin{aligned} \langle T\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)\rangle &= \frac{\int \mathcal{D}\phi \ \phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)e^{\frac{i}{\hbar}S_{int}[\phi]}e^{\frac{i}{\hbar}S_0[\phi]}}{\int \mathcal{D}\phi \ e^{\frac{i}{\hbar}S_{int}[\phi]}e^{\frac{i}{\hbar}S_0[\phi]}} \\ &= \frac{\langle T\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)e^{\frac{i}{\hbar}S_{int}[\phi]}\rangle_0}{\langle Te^{\frac{i}{\hbar}S_{int}[\phi]}\rangle_0}. \end{aligned} \tag{2.16}$$

Expressing the interaction term as a power series results in the perturbative expansion of the Green functions, diagrammatically represented by Feynman graphs. From the leading term we recover the tree-level interacting theory and the other terms in the series give the loop diagrams of n^{th} order. This perturbative expansions can be expressed in another way by noticing that [67, 69]

$$\frac{\delta}{\delta J^i(x_i)} e^{\frac{i}{\hbar}\int dx \phi_i(x)J^i(x)} = \frac{i}{\hbar}\phi_i(x_i)e^{\frac{i}{\hbar}\int dx\phi_i(x)J^i(x)} \ , \tag{2.17}$$

so that we can associate to the fields $\phi_i(x)$ a functional derivative operator

$$\phi_i(x_i) \rightarrow \frac{\hbar}{i}\frac{\delta}{\delta J^i(x_i)}. \tag{2.18}$$

Now, if we expand the interaction term into a power series, as stated before, we can rewrite its field content in terms of this functional operator. Therefore, the interaction part of the action can be extracted from the functional integral over the free fields, which modifies (2.5) into

$$Z[J] = \mathcal{N} e^{\frac{i}{\hbar} S_{int}\left(\frac{\hbar}{i}\frac{\delta}{\delta J^i(x_i)}\right)} Z_0[J]. \tag{2.19}$$

Notice that this expression is only possible when dealing with perturbative theories. This expression leads to the loop expansion of the correlation functions. However, it generates disconnected and vacuum bubble diagrams along with the connected ones. To obtain only connected graphs, we will need to look to the connected and vertex functionals, as we shall see in the next section.

2.1.2 The Connected and Vertex Functionals

To see how to get rid of disconnected diagrams, let us evaluate the two-point correlation function of

$$Z^c[J] = -i\hbar \ln Z[J], \tag{2.20}$$

which is defined as the generating functional of connected Green functions. Using the definition of equation (2.4), the two-point connected correlation function is given by

$$\begin{aligned} \frac{\delta^2 Z^c[J]}{\delta J^i(x)\delta J^j(y)} &= -i\hbar \frac{1}{Z}\frac{\delta^2 Z[J]}{\delta J^i(x)\delta J^j(y)} + i\hbar \frac{1}{Z^2}\frac{\delta Z[J]}{\delta J^i(x)}\frac{\delta Z[J]}{\delta J^j(y)} \\ &= \frac{i}{\hbar}\left(\langle T\phi_i(x)\phi_j(y)\rangle - \langle \phi_i(x)\rangle\langle\phi_j(y)\rangle\right), \end{aligned} \tag{2.21}$$

expressed in terms of the general Green functions. Here, the second term is responsible for eliminating the disconnected contributions coming from $Z[J]$, leaving only connected terms between the points x and y. This can be generalized for all n-point correlation functions [65]. As a formal power series, the connected functional can be expressed as

$$Z^c[J] = \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!\,\hbar^{n-1}} \int dx_1 \dots dx_n J^{i_1}(x_1)\dots J^{i_n}(x_n) \langle T\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)\rangle_c \ , \tag{2.22}$$

from which we can define

$$\langle T\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)\rangle_c = (-i\hbar)^{n-1} \left.\frac{\delta^n Z^c[J]}{\delta J^{i_1}(x_1)\dots\delta J^{i_n}(x_n)}\right|_{J=0} . \tag{2.23}$$

The vertex functional, also known as the one-particle irreducible (1PI) generating functional, provides connected Green functions with amputated external legs, that is, the connected function stripped of the full two point functions of all external legs. By defining the "classical fields" $\varphi_i(x)$, which are the vacuum expectation value of the quantum fields $\phi_i(x)$, as

$$\varphi_i = \frac{\delta Z^c}{\delta J^i}, \tag{2.24}$$

we obtain the vertex functional as a Legendre transformation of the connected one

$$\Gamma[\varphi] = Z^c[J] - \int dx\; J^i(x)\varphi_i(x)\Big|_{\varphi_i = \frac{\delta Z^c}{\delta J^i}}, \tag{2.25}$$

where $\varphi_i(x)$ are fast decreasing functions belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$, as the sources $J^i(x)$. As such, it can be expressed as a formal power series

$$\Gamma[\varphi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \varphi_{i_1}(x_1) \dots \varphi_{i_n}(x_n) \Gamma^{i_1 \dots i_n}(x_1 \dots x_n), \tag{2.26}$$

where $\Gamma^{i_1 \dots i_n}(x_1 \dots x_n) = \langle T\phi_{i_1}(x_1) \dots \phi_{i_n}(x_n)\rangle_{1PI}$ is the n-points 1PI Green function. From the definition (2.25), we get

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi_i(x)} = -J^i(x), \tag{2.27}$$

We assume that the fields vacuum expectation values are zero. From these functional identities, we see that

$$\frac{\delta^2 \Gamma[\varphi]}{\delta J^i(x)\delta\varphi_j(y)} = -\frac{\delta J^j(y)}{\delta J^i(x)}, \tag{2.28}$$

but since $\varphi = \frac{\delta Z^c}{\delta J}$, we can manipulate the expression to be

$$\int dz \frac{\delta^2 \Gamma[\varphi]}{\delta\varphi_k(z)\delta\varphi_j(y)} \frac{\delta^2 Z^c[J]}{\delta J^k(z)\delta J^i(x)}\Big|_{J,\varphi=0} = -\delta_i^j \delta(x-y), \tag{2.29}$$

which tells us that

$$\frac{\delta^2 \Gamma[\varphi]}{\delta\varphi_k(z)\delta\varphi_j(y)} = -\left(\frac{\delta^2 Z^c[J]}{\delta J^k(z)\delta J^j(y)}\right)^{-1}. \tag{2.30}$$

This represents the generalization of the tree-level identity involving the propagator and wave operator, in which now all perturbative corrections are taken into account. Moreover, from this result it is possible to show that the connected Green functions can be decomposed as products of 1PI functions, from the three-point function onward.

The vertex functional can also be written as a formal power series in $\hbar$

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}[\varphi], \tag{2.31}$$

where the index n indicates the number of loops of the diagrams contributing to the sum. We prove this statement in the following. The diagrams containing zero loops are those present in the tree-level action, so that we can identify

$$\Gamma^{(0)}[\varphi] = S[\varphi]. \tag{2.32}$$

Since in a given diagram each vertex V contributes a factor of $\hbar^{-1}$, each internal line I corresponds to a propagator carrying a factor of $\hbar$ and there is a global $\hbar$ factor coming from (2.20), the overall power of $\hbar$ is

$$N(\hbar) = I + 1 - V = L, \tag{2.33}$$

where L is the number of loops and the identity is due to Euler's formula. Therefore, the expansion in $\hbar$ is also a loop expansion of the vertex functional.

2.1.3 Composite operators

For the sake of completeness and for later use, we discuss how to introduce composite operators in a given theory, these being field operators $Q^p(x)$, where p represents any tensorial or color structure, corresponding to local field polynomials in the classical theory. They are particularly useful in theories which are invariant under non-linear symmetries and will be used to control the renormalization of such transformations. Consider the classical insertion of the composite operator $Q^p(x)$ in a theory with a classical interaction action $S_{int}[\phi]$, i.e.,

$$S_{int}[\phi, \rho] = S_{int}[\phi] + \int dx \; \rho_p(x) Q^p(x), \tag{2.34}$$

depending on the external sources $\rho_p(x)$. This allows us to write a new Green functional

$$\begin{aligned} Z[J, \rho] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{i^n}{n!\,\hbar^n} \frac{i^m}{m!\,\hbar^m} \int dx_1 \dots dx_n \int dy_1 \dots dy_m \times \\ \times J^{i_1}(x_1) \dots J^{i_n}(x_n) \rho_{p_1}(y_1) \dots \rho_{p_m}(y_m) \times \\ \times \langle T \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n) Q^{p_1}(y_1) \dots Q^{p_m}(y_m) \rangle, \end{aligned} \tag{2.35}$$

where now we can obtain all the Green functions with insertions of the composite operator $Q^p(x)$ [43]. The connected and vertex functional generalizations follow in the same manner. In particular, we get

$$(-i\hbar)\left.\frac{\delta Z[J,\rho]}{\delta\rho_p(x)}\right|_{\rho=0} := Q^p(x)\cdot Z[J] \tag{2.36}$$

which correspond to the Green functional with the insertion of the composite operator $Q^p(x)$, which generates n-point functions $\langle T\phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)Q^p(y)\rangle$ with a vertex coming from the composite operator. This Green function contains Feynman graphs with a new vertex associated with the insertion. As before, similar results hold for the connected and vertex functionals. In particular, in view of the loop expansion of the vertex functional, we have

$$\left.\frac{\delta\Gamma[\varphi,\rho]}{\delta\rho_p(x)}\right|_{\rho=0} = Q^p(x)\cdot\Gamma[\varphi] = Q^p(x) + O(\hbar), \tag{2.37}$$

where the zeroth order term is simply the classical insertion to the interaction term (2.34).

2.2 Symmetries and the Quantum Action Principle

A key aspect of renormalization is to prove that the classical symmetries of a theory can be extended to a quantum analogue. In other words, that the Green functions of the theory fulfill such symmetries, expressed by Ward identities, to all orders in perturbation theory. However, is not always possible to find an invariant regularization procedure, capable of taming the divergencies in correlation functions while maintaining its invariant character, which may lead to a breaking of the symmetries of the quantum theory. The Quantum Action Principle enters then as a tool to identify true symmetry breakings, i.e. symmetries of the classical action that are not carried on to the quantum theory, and breakings which appear solely due to the renormalization procedure. Those are called anomalies and noninvariant counterterms, respectively. We will discuss more about that later on.

Let us begin by explaining how to functionally characterize a symmetry through the so called Ward identities, which generalize the classical symmetries to a quantum analogue. Suppose there is a continuous symmetry of the action (2.7), that is, a transformation of the fields which leaves the action invariant, that has the infinitesimal form

$$\delta\phi_i(x) = i\epsilon^a R_{ai}(x), \tag{2.38}$$

where ϵ^a is an infinitesimal parameter of the transformation and $R_{ai}(x)$ are local functionals of the fields and their derivatives. The set of possible transformations of the fields (2.38) constitutes a continuous group G, also called a Lie group. Therefore, these transformations belong to a certain representation of G. The generators τ_a of G, belonging to the algebra $\mathfrak{g}$, obey the relations

$$[\tau_a, \tau_b] = i f_{ab}{}^c \tau_c \ , \tag{2.39}$$

with $f_{ab}{}^c$ being the totally antisymmetric structure constants of the group G, which fulfill the Jacobi identities

$$\underset{a,b,c}{\circlearrowleft} f_{ab}{}^d f_{cd}{}^e = 0, \tag{2.40}$$

where $\circlearrowleft$ is the cyclic sum symbol. We then impose that $R_{ai}(x)$ obey the functional relations

$$\int dz \left(R_{bj}(z) \frac{\delta R_{ai}(x)}{\delta \phi_j(z)} - R_{aj}(z) \frac{\delta R_{bi}(x)}{\delta \phi_j(z)} \right) = i f_{ab}{}^c R_{ci}(x). \tag{2.41}$$

The transformations can be implemented by a Ward operator $\mathcal{W}_a$ on a general functional $F[\phi]$

$$\delta F[\phi] = -i\epsilon^a \mathcal{W}_a F[\phi] = i\epsilon^a \int dx R_{ai}(x) \frac{\delta F[\phi]}{\delta \phi_i(x)} \tag{2.42}$$

where we define

$$\mathcal{W}_a = -\int dx R_{ai}(x) \frac{\delta}{\delta \phi_i(x)}. \tag{2.43}$$

The Ward operators obey the same commutation relations of the Lie algebra $\mathfrak{g}$. We can verify this explicitly

$$\begin{aligned}
[\mathcal{W}_a, \mathcal{W}_b]F &= \int dydz \left[R_{ai}(y) \frac{\delta}{\delta \phi_i(y)} \left(R_{bj}(z) \frac{\delta F}{\delta \phi_j(z)} \right) - R_{bi}(y) \frac{\delta}{\delta \phi_i(y)} \left(R_{aj}(z) \frac{\delta F}{\delta \phi_j(z)} \right) \right] \\
&= \int dydz \left[R_{ai}(y) \frac{\delta R_{bj}(z)}{\delta \phi_i(y)} - R_{bi}(y) \frac{\delta R_{aj}(z)}{\delta \phi_i(y)} \right] \frac{\delta F}{\delta \phi_j(z)} \\
&= \int dz (-i f_{ab}{}^c R_{cj}(z)) \frac{\delta F}{\delta \phi_j(z)} = i f_{ab}{}^c \mathcal{W}_c F,
\end{aligned} \tag{2.44}$$

where we used (2.41) from the second to the third equality.

In the case of linear symmetries, that is, transformations which depend linearly on the fields, we have

$$\delta \phi_i(x) = i\epsilon^a T_{ai}{}^j \phi_j(x), \tag{2.45}$$

with the matrix elements $T_{ai}{}^{j}$ obeying the commutation relations, defining a representation of the Lie group G. Under this representation, the identities which express the invariance of the action $S[\phi]$ are functionally implemented by

$$\mathcal{W}_a S[\phi] = -\int dx T_{ai}{}^{j}\phi_j(x)\frac{\delta}{\delta\phi_i(x)}S[\phi] = 0. \tag{2.46}$$

Bearing in mind that the classical action is the tree-level approximation of the vertex functional, one can obtain the connected and Green generating functionals tree approximation from (2.25) and (2.20). One can then establish the equivalent invariance of the Green generating functional

$$\mathcal{W}_a Z^{(0)}[J] = -\int dx J^i(x) T_{ai}{}^{j}\frac{\delta}{\delta J^j(x)}Z^{(0)}[J] = 0. \tag{2.47}$$

Therefore, one can establish relations between Green functions which are consequences of this symmetry. They read

$$\sum_{n=1}^{N}\langle T\phi_{i_1}(x_1)\ldots\phi_{i_{n-1}}(x_{n-1})T_{ai_n}{}^{j_n}\phi_{j_n}(x_n)\ldots\phi_{i_N}(x_N)\rangle = 0, \tag{2.48}$$

resulting in a sum of elementary Green functions. These are the so-called Ward Identities. Such symmetries will not be subjected to the process of renormalization.

2.2.1 Non-linear symmetries

In the general case, where R_{ai} is not a linear function in the fields, the Ward identity would not result in a sum of elementary Green functions, due to the insertion of local composite operators. Thus, one can express the total classical action as

$$\Gamma^{(0)} = S[\phi] + S_{ext}[\rho,\phi] = S[\phi] + i\int dx\epsilon^a\rho^i(x)R_{ai}(x), \tag{2.49}$$

where we couple the field transformation to an external source ρ^i. The total action $\Gamma^{(0)}$, however, is not invariant under the transformation of the fields (2.38), due to the introduction of the external sources, so we can't directly derive Ward identities for it as in the linear case. A way to do so is to treat the infinitesimal parameters ϵ^a as Grassmann numbers transforming in the adjoint representation of the gauge group G, that is

$$\{\epsilon^a, \epsilon^b\} = 0, \quad \delta\epsilon^a = -\frac{1}{2}f_{bc}{}^{a}\epsilon^b\epsilon^c. \tag{2.50}$$

With this redefinition, the transformation rules implemented by the operator δ are nilpotent

$$\delta^2 = 0. \tag{2.51}$$

From now on, the operator δ will be denoted as s. The total action now is invariant under the transformations implemeted by s

$$\begin{aligned} s\Gamma^{(0)} &= sS[\phi] + is \int dx \epsilon^a \rho^i(x) R_{ai}(x) \\ &= i \int dx s\epsilon^a \rho^i(x) R_{ai}(x) - i \int dx \epsilon^a s\rho^i(x) R_{ai}(x) - i \int dx \epsilon^a \rho^i(x) sR_{ai}(x) \\ &= -\frac{i}{2} \int dx f_{bc}{}^a \epsilon^b \epsilon^c \rho^i(x) R_{ai}(x) - i \int dx \epsilon^a \rho^i(x) \int dz \delta\phi_j(z) \frac{\delta R_{ai}(x)}{\delta\phi_j(z)} \\ &= -\frac{i}{2} \int dx f_{bc}{}^a \epsilon^b \epsilon^c \rho^i(x) R_{ai}(x) + \int dx dz \rho^i \epsilon^a \epsilon^b R_{bj}(z) \frac{\delta R_{ai}(x)}{\delta\phi_j(z)} \\ &= -\frac{i}{2} \int dx f_{bc}{}^a \epsilon^b \epsilon^c \rho^i(x) R_{ai}(x) \\ &\qquad\qquad + \frac{1}{2} \int dx dz \rho^i \epsilon^a \epsilon^b \left(R_{bj}(z) \frac{\delta R_{ai}(x)}{\delta\phi_j(z)} - R_{aj}(z) \frac{\delta R_{bi}(x)}{\delta\phi_j(z)} \right) \\ &= -\frac{i}{2} \int dx f_{bc}{}^a \epsilon^b \epsilon^c \rho^i(x) R_{ai}(x) + \frac{i}{2} \int dx \rho^i \epsilon^a \epsilon^b f_{ab}{}^c R_{ci}(x) = 0, \end{aligned} \tag{2.52}$$

where we used (2.38), (2.41) and (2.50) in the intermediate steps, as well as the cyclic permutation of the structure constants and the invariance of the external source ($s\rho^i = 0$). The invariance of the total tree-level action can be functionally expressed as

$$\mathcal{S}(\Gamma^{(0)}) = \int dx \frac{\delta\Gamma^{(0)}}{\delta\rho^i} \frac{\delta\Gamma^{(0)}}{\delta\phi_i} - \frac{1}{2} f_{bc}{}^a \epsilon^b \epsilon^c \frac{\delta\Gamma^{(0)}}{\delta\epsilon^a} = 0. \tag{2.53}$$

This equation is known as a Slavnov-Taylor identity. The nilpotent operator s is called the BRST operator [38, 40, 41] and is of great importance in the study of gauge theories, as it turns out that the renormalization of such theories is fully encompassed by the study of the cohomology of the BRST operator [70]. Beyond that, it is also useful at the classical level [71, 72]. The previous presentation was done considering global (rigid) transformations, where the parameters ϵ^a do not depend on the space-time position x, but it works similarly for local (gauge) transformations, where $\epsilon^a = \epsilon^a(x)$.

2.2.2 Power-counting

As stated before, the perturbative quantization of a classical theory goes through the process of renormalization, that is, the removal of ultraviolet (UV) divergencies from the loop integrals in momentum space. These divergencies appear due to the local product of propagators, that is, in coinciding space-time points, which is not properly defined, since they are distribution valued objects. Renormalization is then the process of removing these divergencies in a physically meaningful way.

Since all Feynman graphs can be constructed using 1PI diagrams, we can focus on the divergencies of the latter. Considering a 1PI Feynman diagram γ with

- N_a amputated external legs corresponding to the fields $\Phi_a = \phi_i$ or the sources ρ_i with UV dimension d_a;
- N_V interaction vertices V_k of UV dimension d_k;
- I_{ij} internal lines corresponding to the propagators $\Delta_{ij} = \langle T\phi_i\phi_j\rangle$;
- L loops;

where UV dimension of the fields ϕ_i, sources ρ_p and propagators Δ_{ij} are defined as such

1. The quadratic part of the Lagrangian is bounded by the space-time dimension D, where the derivative ∂_μ has UV dimension 1;
2. The UV dimension of the propagators d_{ij} is given by the asymptotic behavior of its Fourier transform
$$\lim_{p\to\infty} \Delta_{ij}(p) \sim p^{d_{ij}}; \tag{2.54}$$
3. The UV dimensions of the fields are then restricted by the inequality
$$d_i + d_j \geq d_{ij} + D; \tag{2.55}$$
4. The UV dimension of the sources is defined as $d_p = D - d_{Q^p}$, where d_{Q^p} is the UV dimension of the composite operator coupled to it.

The UV dimension of the vertices is given by the sum of the dimensions of the fields and derivatives composing each vertex.

Therefore, to the diagram γ corresponds a momentum integral

$$J_\gamma(p) = \int dk_1 \dots dk_L I_\gamma(p, k), \tag{2.56}$$

where $p = (p_1, \ldots, p_{N_a})$, obeying momentum conservation $\sum_1^{N_a} p_i = 0$, are the external leg momenta and $k = (k_1, \ldots, k_L)$ are the independent loop momenta. One can find the superficial degree of divergence $d(\gamma)$ by rescaling the momenta as $p, k \to \lambda p, \lambda k$, so that the integrand behavior in the limit $\lambda \to \infty$ is given by

$$\int d(\lambda k_1) \ldots d(\lambda k_L) I_\gamma(\lambda p, \lambda k) \sim \lambda^{d(\gamma)}, \tag{2.57}$$

where $d(\gamma)$ is defined as

$$d(\gamma) = D + \sum_{N_V} (d_k - D) - \sum_a N_a d_a. \tag{2.58}$$

Theories in which the degree of divergence $d(\gamma)$ is bounded by a maximum value depending on the number of external legs

$$d_{max}(\gamma) = D - \sum_a N_a d_a, \tag{2.59}$$

that is, theories whose interaction vertices do not present dimension higher than D (in fact, the above expression is obtained assuming $d_k = D$ for all vertices) are called power-counting renormalizable theories. These are all theories where the number of divergent diagrams is finite, independently of the number of loops. A primary example is Quantum Electrodynamics. On the contrary, if the number of diagrams grows indefinitely with the number of loops, the theory is called power-counting non-renormalizable. That is the case of general relativity, for example. Finally, if the number of divergent diagrams decreases as we go to higher loop orders, the theory is called superrenormalizable. In the following, we'll be dealing with Yang-Mills-Chern-Simons theories, which belong to such class.

These results are formally expressed in the power-counting theorem, derived by Weinberg [73] and Zimmermann [74] for massive theories, and generalized to the massless case by Lowenstein and others [75–77]. They give a proper meaning to the subtraction of divergencies realized by the renormalization procedures. Moreover, they show how this subtraction leads to the introduction of local counterterms which are responsible for the renormalization of the parameters of the theory. These counterterms are fixed by normalization conditions and symmetry requirements. As a final comment, we mention that a sufficient condition for a theory to be physically meaningful is that the number of possible counterterms is finite, a feature that can be obtained by imposing that they obey symmetry restrictions, as we shall see in the following.

2.2.3 The Quantum Action Principle

The Ward identities derived so far are valid at the classical level, but we do not know if they remain valid for the quantized theory. The Quantum Action Principle [55–62] comes in as a way to constrain the possible breakings of the classical symmetries at the quantum level[1]. For the vertex functional, it reads

$$\int dx \frac{\delta\Gamma}{\delta\rho_a(x)}\frac{\delta\Gamma}{\delta\phi_i(x)} = \Delta^{ai}(x)\cdot\Gamma, \tag{2.60}$$

or equivalently, in a way which makes explicit use of the Ward operator,

$$\mathcal{W}^a\Gamma = \Delta^a\cdot\Gamma, \tag{2.61}$$

where the insertions Δ are local composite field operators, that is, integrated polynomials of the fields and their derivatives, coinciding with the classical insertion in the tree approximation. The insertions Δ are bounded by the space-time dimension D and fields dimensions d_i, and hold the same properties as the Ward operator (color indices, charge conjugation,*etc*). The expression above remain valid for linear symmetries, where no external source ρ needs to be introduced. In this case, the variation in relation to ρ is exchanged by the linear transformations of the fields.

The Quantum Action Principle is also valid for variations with respect to the parameters λ of the theory (masses, coupling constants, *etc*), where it reads

$$\frac{\partial}{\partial\lambda}\Gamma = \Delta\cdot\Gamma, \tag{2.62}$$

where the quantum insertions Δ hold the same properties as discussed previously.

In particular, an anomaly can be defined as the failure of the Ward identities to hold to a certain order in perturbation theory, that is, a symmetry of the classical action that cannot be extended to the full quantum theory. In terms of the vertex functional, for a non-anomalous symmetry, we want that

$$\mathcal{W}^a S = 0 \rightarrow \mathcal{W}^a\Gamma = 0. \tag{2.63}$$

But as we mentioned before, every regularization procedure may break the symmetry in the computation of the loop diagrams. If the symmetry breaking term cannot be absorbed by the introduction of suitable counterterms at some n^{th} order in the loop expansion, it represents a true anomaly and the symmetry is lost in the quantum theory. The argument follows as such:

[1]These results hold provided the theory is power-counting renormalizable.

assuming that the vertex functional violates the Ward identity at some order n

$$\mathcal{W}^a\Gamma = \hbar^n\Delta^a \cdot \Gamma = \hbar^n\Delta^a + O(\hbar^{n+1}), \tag{2.64}$$

where Δ^a are local insertions representing the breaking of the symmetry at that order. In a general manner, the breakings Δ^a can be written as

$$\Delta^a = \mathcal{A}^a + \mathcal{W}^a\hat{\Delta}, \tag{2.65}$$

where $\mathcal{A}^a$ represents a true anomaly and $\hat{\Delta}$ is a non-invariant counterterm which can be absorbed by a redefinition of the vertex functional $\Gamma \to \Gamma - \hbar^n\hat{\Delta}$ at order n. To define the presence or not of a true anomaly, we must study the cohomology problem posed by the Wess-Zumino condition [43, 70]

$$\mathcal{W}^a\Delta^b - \mathcal{W}^b\Delta^a = if_{ab}{}^c\Delta^c. \tag{2.66}$$

In this context, the anomalies represent non-trivial terms of the cohomology, that is, terms which cannot be expressed as $\mathcal{W}$-variation of some local insertion, while the non-invariant counterterms are represented by trivial ones.

In addition to anomalies and non-invariant counterterms, one can still have symmetric (invariant) counterterms added to the classical action through the renormalization process. These counterterms are responsible for the renormalization of the parameters and fields of the theory and need to be fixed by appropriate renormalization conditions. They are, in general, in a one-to-one correspondence to the terms in the classical action. This expresses the stability of the theory with respect to quantum fluctuations.

They can be understood as a perturbation of the total classical action $\Gamma^{(0)} \to \Gamma^{(0)} + \varepsilon\Gamma^{ct}$, so that it obeys the general Ward identity

$$\mathcal{W}(\Gamma^{(0)} + \varepsilon\Gamma^{ct}) = 0 + O(\varepsilon^2), \tag{2.67}$$

up to second order in the infinitesimal parameter ε. Given that we can generally write the Ward operator acting on the vertex functional as

$$\mathcal{W}\Gamma = \int dx \frac{\delta\Gamma}{\delta\rho_a(x)}\frac{\delta\Gamma}{\delta\phi_i(x)}, \tag{2.68}$$

the counterterm action Γ^{ct} must obey the linearized version of the Ward operator, which can easily be obtained from

$$\begin{aligned}\mathcal{W}(\Gamma^{(0)} + \varepsilon\Gamma^{ct}) = &\int dx \frac{\delta\Gamma^{(0)}}{\delta\rho_a(x)}\frac{\delta\Gamma^{(0)}}{\delta\phi_i(x)} \\ &+ \varepsilon\underbrace{\int dx\left(\frac{\delta\Gamma^{(0)}}{\delta\rho_a(x)}\frac{\delta\Gamma^{ct}}{\delta\phi_i(x)} + \frac{\delta\Gamma^{(0)}}{\delta\phi_i(x)}\frac{\delta\Gamma^{ct}}{\delta\rho_a(x)}\right)}_{\mathcal{W}_{\Gamma^{(0)}}\Gamma^{ct}} = 0 + O(\varepsilon^2),\end{aligned} \tag{2.69}$$

that is, the counterterm action must obey

$$\mathcal{W}_{\Gamma^{(0)}}\Gamma^{ct} = 0, \tag{2.70}$$

where $\mathcal{W}_\Gamma$ is the linearized version of the Ward operator. The classical action is then stable if the perturbation can be reabsorbed by a redefinition of the initial fields and parameters

$$\Gamma^{(0)}[\phi, \lambda, \rho] + \varepsilon\Gamma^{ct}[\phi, \lambda, \rho] = \Gamma^{(0)}[\phi_0, \lambda_0, \rho_0] + O(\varepsilon^2), \tag{2.71}$$

where the quantities $\phi_0, \lambda_0, \rho_0$ are related to the original ones by

$$\phi_0 = Z_\phi^{1/2}\phi, \quad \lambda_0 = Z_\lambda\lambda, \quad \rho_0 = Z_\rho\rho. \tag{2.72}$$

The quantities Z_i are defined order by order as a series in $\hbar$. If this condition is satisfied, we say that the theory is multiplicatively renormalizable.

What we call algebraic renormalization is an iterative program to construct the most general anomaly, non-invariant and symmetric counterterms possible at any perturbative order, based on the Quantum Action Principle. Therefore, at a given order n, we identify the possible quantum insertions which respect the symmetry and dimensional restrictions for a given Ward operator as a polynomial in the fields and their derivatives. In special, we look at this construction for the Slavnov-Taylor operator, which represents the BRST transformations at the functional level, in the cohomology sectors of ghost number 1, for possible anomalies and non-invariant counterterms, and ghost number 0, for the counterterms responsible for the renormalization of the parameters and fields of the tree-level action. This construction will be used throughout the rest of this thesis to study the renormalization of Yang-Mills-Chern-Simons theory, as we shall see in the following.

Chapter 3

Algebraic Renormalization of Yang-Mills-Chern-Simons in the Landau Gauge

3.1 Yang-Mills-Chern-Simons in the Landau Gauge

We now apply the results from the previous discussion to a concrete example, the Yang-Mills-Chern-Simons theory [47–49]. The tree-level Yang-Mills-Chern-Simons action for a general $SU(N)$ gauge group is given by

$$S_{YMCS} = \mathrm{Tr} \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m}{2} \epsilon^{\mu\rho\nu} \left(A_\mu \partial_\rho A_\nu + \frac{2g}{3} A_\mu A_\rho A_\nu \right) \right] , \quad (3.1)$$

where the gauge field is a Lie algebra valued field $A_\mu = A^a_\mu \tau^a$ in the adjoint representation of $SU(N)$, the field-strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, g is the coupling constant associated with the gauge group and m is the topological mass associated to the Chern-Simons term. We assume that the generators obey the normalization $\mathrm{Tr}(\tau^a \tau^b) = \delta^{ab}$, which renders the action in components

$$S_{YMCS} = \int d^3x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{m}{2} \epsilon^{\mu\rho\nu} \left(A^a_\mu \partial_\rho A^a_\nu + \frac{2g}{3} f^{abc} A^a_\mu A^b_\rho A^c_\nu \right) \right] . \quad (3.2)$$

The Yang-Mills term in the action is fully invariant under general gauge transformations

$$A_\mu \to A'_\mu = U^{-1} A_\mu U + \frac{i}{g} U^{-1} \partial_\mu U, \quad (3.3)$$

where $U = e^{ig\omega^a\tau^a}$ is an element of the gauge group $SU(N)$. In particular, an infinitesimal transformations takes the form

$$A_\mu \to A'_\mu = A_\mu - D_\mu\omega, \tag{3.4}$$

where $\omega = \omega^a(x)\tau^a$ is a Lie algebra valued infinitesimal local parameter of the gauge transformation and

$$D_\mu\omega = \partial_\mu\omega - g[A_\mu, \omega], \tag{3.5}$$

is the covariant derivative in the adjoint representation. However, unlike the Yang-Mills term, the Chern-Simons action is only invariant under infinitesimal gauge transformations, up to a total derivative. Under finite transformations, its variation is proportional to the second Chern character integral [78]

$$\mathcal{I} = \mathrm{Tr}\int d^3x\epsilon^{\mu\rho\nu}\left(U^{-1}\partial_\mu UU^{-1}\partial_\rho UU^{-1}\partial_\nu U\right). \tag{3.6}$$

Another feature of the Chern-Simons term is that it is odd under parity transformations. In three space-time dimensions, parity transformations can be defined as

$$\begin{aligned} x_0 &\to x_0, \\ x_1 &\to -x_1, \\ x_2 &\to x_2, \end{aligned} \tag{3.7}$$

and since the gauge field transforms covariantly, we get

$$\begin{aligned} A_0 &\to A_0, \\ A_1 &\to -A_1, \\ A_2 &\to A_2, \end{aligned} \tag{3.8}$$

so that under parity transformations

$$\epsilon^{\mu\rho\nu}\left(A^a_\mu\partial_\rho A^a_\nu + \frac{2g}{3}f^{abc}A^a_\mu A^b_\rho A^c_\nu\right) \xrightarrow{P} -\epsilon^{\mu\rho\nu}\left(A^a_\mu\partial_\rho A^a_\nu + \frac{2g}{3}f^{abc}A^a_\mu A^b_\rho A^c_\nu\right). \tag{3.9}$$

A final comment is due on the topological nature of the Chern-Simons term. Being the integral of a 3-form on a general manifold $\mathcal{M}$, it does not depend on any metric $g_{\mu\nu}$ defined over $\mathcal{M}$. Explicitly, this reads

$$\frac{\delta S_{CS}}{\delta g_{\mu\nu}} = 0. \tag{3.10}$$

This identity allows one to compute topological invariants using the general techniques of perturbative quantum field theory [79], making it a valuable tool from the mathematics perspective. It is also responsible for the ultraviolet finiteness of pure Chern-Simons theory [80, 81], meaning that its parameters and fields receive no radiative corrections coming from the loop diagrams expansion.

In the following, we present the complete gauge-fixed YMCS action. In order to make use of a local version of the Callan-Symanzik equation later, so as to prove its ultraviolet finiteness, we define the Yang-Mills-Chern-Simons theory on a general three-dimensional Riemannian manifold, asymptotically flat and topologically equivalent to $\mathbb{R}^3$ [82–84]. This is, however, a choice of presentation, since the vanishing of β functions for the Chern-Simons can be shown to hold purely from cohomological arguments [85].

3.1.1 The Complete Action and its Symmetries

We begin then with a theory defined over a Riemannian manifold $\mathcal{M}$, topologically equivalent to $\mathbb{R}^3$, which allows us to use the general results from renormalization theory in flat space-time [55–62, 86]. It is described by a dreiben $e^m_\mu(x)$, which connects the manifold metric tensor $g_{\mu\nu}(x)$ to the flat tangent space metric η_{mn} through the relation

$$g_{\mu\nu}(x) = e^m_\mu(x) e^n_\nu(x) \eta_{mn}. \tag{3.11}$$

Here μ denotes the global manifold index and m the local tangent space index. The inverse of the dreibein is denoted as $e^\mu_m(x)$ and its determinant is denoted as e.

Then, the gauge-fixed YMCS action in the Landau gauge, coupled to external sources to control the renormalization of the BRST transformations, is given by

$$\begin{aligned}\Gamma^{(0)} = \int d^3x \ \Big[& -\frac{e}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{m}{2} \epsilon^{\mu\rho\nu} \left(A^a_\mu \partial_\rho A^a_\nu + \frac{2g}{3} f^{abc} A^a_\mu A^b_\rho A^c_\nu \right) \\ & - e g^{\mu\nu} (\partial_\mu b^a A^a_\nu + \partial_\mu \bar{c}^a D_\nu c^a) + (A^{*a\mu} s A^a_\nu + c^{*a} s c^a) \Big] ,\end{aligned} \tag{3.12}$$

where c^a, $\bar{c}^a$ and b^a are the Faddeev-Popov ghost, antighost and the Nakanishi-Lautrup fields, respectively, and $A^{*a\mu}$ and c^{*a} are tensorial density sources ("antifields") coupled to the nonlinear transformations of A^a_μ and c^a. The BRST transformations of the fields are:

$$\begin{aligned} sA^a_\mu &= -D_\mu c^a = -(\partial_\mu c^a + f^{abc} A^b_\mu c^c), \\ sc^a &= \frac{g}{2} f^{abc} c^b c^c, \quad s\bar{c}^a = b^a, \quad sb^a = 0, \end{aligned} \tag{3.13}$$

and the antifields are BRST-invariant ($sA^{*a\mu} = 0$ and $sc^{*a} = 0$). These transformations ensure that the BRST operator s is nilpotent ($s^2 = 0$) and that the tree-level action is BRST-invariant ($s\Gamma^{(0)} = 0$). Since we are working in a general curved manifold, we also have invariance under diffeomorphisms

$$
\begin{aligned}
&\delta^{\varepsilon}_{diff} F_\mu = \varepsilon^\alpha \partial_\alpha F_\mu + (\partial_\mu \varepsilon^\alpha) F_\alpha, \quad \text{for } F_\mu = A^a_\mu, e^m_\mu \\
&\delta^{\varepsilon}_{diff} \Phi = \varepsilon^\alpha \partial_\alpha \Phi, \quad \text{for } \Phi = c^a, \bar{c}^a, b^a \\
&\delta^{\varepsilon}_{diff} A^{*a\mu} = \partial_\alpha(\varepsilon^\alpha A^{*a\mu}) - (\partial_\alpha \varepsilon^\mu) A^{*a\alpha}, \quad \delta^{\varepsilon}_{diff} c^{*a} = \partial_\alpha(\varepsilon^\alpha c^{*a}),
\end{aligned}
\tag{3.14}
$$

or in a condensed notation

$$
\delta^{\varepsilon}_{diff} \Phi = \mathcal{L}_\varepsilon \Phi, \quad \text{for } \Phi = A^a_\mu, e^m_\mu, c^a, \bar{c}^a, b^a, A^{*a\alpha}, c^{*a}, \tag{3.15}
$$

where $\mathcal{L}_\varepsilon$ is the appropriate Lie derivative of the field along the infinitesimal vector field ε^μ. Local Lorentz invariance is given by

$$
\delta^{\lambda}_{Lorentz} \Phi = \frac{1}{2} \lambda_{mn} \Omega^{[mn]} \Phi, \tag{3.16}
$$

where λ_{mn} is the infinitesimal parameter of the transformation and $\Omega^{[mn]}$ is the antisymmetric matrix generator of the Lorentz transformation, acting on the fields with the appropriate representation for each case.

To make use of the Quantum Action Principle, we first express the invariances of the action in terms of functional Ward identities. The BRST invariance is given by the Slavnov-Taylor identity

$$
\mathcal{S}(\Gamma^{(0)}) = \int d^3x \left(\frac{\delta \Gamma^{(0)}}{\delta A^{*a\mu}} \frac{\delta \Gamma^{(0)}}{\delta A^a_\mu} + \frac{\delta \Gamma^{(0)}}{\delta c^{*a}} \frac{\delta \Gamma^{(0)}}{\delta c^a} + b^a \frac{\delta \Gamma^{(0)}}{\delta \bar{c}^a} \right) = 0, \tag{3.17}
$$

with the linearized Slavnov-Taylor operator defined as

$$
\mathcal{S}_{\Gamma^{(0)}} = \int d^3x \left(\frac{\delta \Gamma^{(0)}}{\delta A^{*a\mu}} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Gamma^{(0)}}{\delta A^a_\mu} \frac{\delta}{\delta A^{*a\mu}} + \frac{\delta \Gamma^{(0)}}{\delta c^{*a}} \frac{\delta}{\delta c^a} + \frac{\delta \Gamma^{(0)}}{\delta c^a} \frac{\delta}{\delta c^{*a}} + b^a \frac{\delta}{\delta \bar{c}^a} \right). \tag{3.18}
$$

These operators obey the following algebraic relations, valid for any functional $\mathcal{F}$

$$
\begin{aligned}
&\mathcal{S}_{\mathcal{F}} \mathcal{S}(\mathcal{F}) = 0, \quad \forall \mathcal{F}, \\
&\mathcal{S}_{\mathcal{F}} \mathcal{S}_{\mathcal{F}} = 0, \quad \text{if } \mathcal{S}(\mathcal{F}) = 0.
\end{aligned}
\tag{3.19}
$$

These relations ensure the nilpotency of the Slavnov-Taylor operator, due to (3.17), so that $(\mathcal{S}_{\Gamma^{(0)}})^2 = 0$. The tree-level action is also subjected by a number of constraints and identities beyond the Slavnov-Taylor. They are the gauge-fixing condition, in this case the Landau gauge condition

$$\frac{\delta\Gamma^{(0)}}{\delta b^a} = \partial_\mu(eg^{\mu\nu}A^a_\nu). \tag{3.20}$$

The ghost equation

$$\mathcal{G}^a\Gamma^{(0)} = \frac{\delta\Gamma^{(0)}}{\delta\bar{c}^a} + \partial_\mu\left(eg^{\mu\nu}\frac{\delta\Gamma^{(0)}}{\delta A^{*a\mu}}\right) = 0, \tag{3.21}$$

which comes from the commutator of (3.17) with (3.20). It restrains the dependence of the theory on the anti-ghost to the combination $\tilde{A}^{*a\mu} = A^{*a\mu} + eg^{\mu\nu}\partial_\nu c^a$. As a special feature of the Landau gauge, we have also an anti-ghost equation

$$\begin{aligned} \bar{\mathcal{G}}^a\Gamma^{(0)} &= \int d^3x\left(\frac{\delta\Gamma^{(0)}}{\delta c^a} + gf^{abc}\bar{c}^b\frac{\delta\Gamma^{(0)}}{\delta b^c}\right) = \Delta^a_{cl}, \\ \text{with } \Delta^a_{cl} &= \int d^3x g f^{abc}(A^{*b\mu}A^c_\mu - c^{*b}c^c). \end{aligned} \tag{3.22}$$

The equations (3.20-3.22) above, being linearly broken in the quantum fields, therefore composed of linear sums of C^∞ distributions, will not be subjected to renormalization, that is, they hold for the quantum theory as well [43]. This result will simplify the search for possible counterterms, as we shall see later. In particular, the anti-ghost equation implies the nonrenormalization of the ghost fields and of the classical breaking Δ^a_{cl} coefficients, that is, the structure constants f^{abc} [87].

Beyond that, we have the Ward identities corresponding to the diffeomorphisms and Lorentz transformations

$$\mathcal{W}_X\Gamma^{(0)} = \int d^3x \sum_\Phi \delta_X\Phi\frac{\delta\Gamma^{(0)}}{\delta\Phi} = 0, \tag{3.23}$$

where $X = (diff, Lorentz)$ and Φ runs over all fields and anti-fields of the theory. Finally, there is a rigid gauge invariance, coming from the commutation of (3.17) and (3.22)

$$\mathcal{W}^a_{rigid}\Gamma^{(0)} = \int d^3x \sum_\phi f^{abc}\phi^b\frac{\delta\Gamma^{(0)}}{\delta\phi^c} = 0, \tag{3.24}$$

where $\phi = A, c, \bar{c}, b, A^*, c^*$.

Field	A	c	$\bar{c}$	b	A^*	c^*	g
d	1/2	-1/2	3/2	3/2	5/2	7/2	1/2
$\Phi\Pi$	0	1	-1	0	-1	-2	0
d_W	-1/2	-1/2	3/2	3/2	1/2	1/2	1/2

Table 3.1: Dimension, ghost number and Weyl dimension of the field content and coupling constant of the YMCS action.

3.1.2 Renormalizability

To prove the renormalizability of the theory is to prove that we can construct a renormalized vertex functional $\Gamma = \Gamma^{(0)} + O(\hbar)$, which obeys the same symmetry constraints as the tree-level action $\Gamma^{(0)}$ and corresponds to it at zero order in the $\hbar$ perturbative expansion (2.31). To do so, we make use of the Quantum Action Principle, which allows us to establish a recursive procedure valid at all orders in perturbation theory. This is the core of the algebraic renormalization procedure [43].

The first step is then to check the power-counting renormalizability of the Yang-Mills-Chern-Simons theory defined by (3.12). A usual feature of lower-dimensional field theories is that they are superrenormalizable, meaning that the number of divergent graphs decreases as we go to higher loop orders and are, therefore, finite. This can be interpreted as a result of the coupling constant possessing non-zero mass dimension. In the case of YMCS, the fields UV dimension is equal to their canonical dimension, and therefore constrained by the dimension of the action being bounded by three, and the coupling constant has mass dimension 1/2. Beyond that, we choose the dimensions so as to render the BRST operator massless. We collect the dimensions in Table 3.1, as well as their ghost numbers $\Phi\Pi$ (associated with the Grassmann parity of the field) and Weyl dimension d_W, which is defined by how the field transforms under a rescaling of the metric tensor

$$g_{\mu\nu} \to e^{-2\omega(x)} g_{\mu\nu} \implies \Phi \to \Phi e^{d_W \omega}. \tag{3.25}$$

The superrenormalizability can be seen directly from the superficial degree of divergence. For a given 1PI Feynman diagram γ, we have

$$d(\gamma) = 3 - \sum_{\phi} d_\phi N_\phi - \frac{1}{2} N_g, \tag{3.26}$$

where N_g is the power of the coupling constant appearing in the integral representation of the diagram. This dependence indicates its superrenormalizability, since higher loops implies higher powers in the coupling constant, and therefore lower degree of divergence. In fact, from (3.26), we get that the YMCS theory is only divergent up to two loops, which allows the explicit verification of its ultraviolet finiteness [48, 88–90].

Following [91], we consider g as an external field of mass dimension 1/2, so that we can use the results from the Quantum Action Principle to restrict the dimension of the counterterms. In this manner, the degree of divergence of a diagram γ is given by

$$d(\gamma) = 3 - \sum_{\Phi=\phi,g} d_\Phi N_\Phi. \tag{3.27}$$

The dimension of the counterterms is then restricted by the dimension of the action, being bounded by 3. But since they come from radiative corrections, which means at least power g^2 in the coupling constant, the field polynomials have dimensions bounded by 2, not taking g into account.

We are now able to discuss the possible anomalies and counterterms of the theory. As stated before, the gauge-fixing condition (3.20), ghost (3.21) and antighost (3.22) equations, being linearly broken in the quantum fields, do not need to go through renormalization. The rigid invariance can also be shown to hold for the quantum theory [43]. As for the diffeomorphisms and Lorentz Ward identities (3.23), they have been shown to be free of anomalies for the class of manifolds we consider here [92, 93] and, therefore, can be assumed as valid to all orders.

The question about renormalizability therefore rests on the validity of the Slavnov-Taylor identity (3.17) for the vertex functional Γ. As it is well known, this corresponds to the study of the cohomology of the linearized Slavnov-Taylor operator $\mathcal{S}_{\Gamma^{(0)}}$, with its nilpotency defined by the algebraic relations (3.19). It follow as such: assuming that the Slavnov-Taylor identity holds to order $n-1$ in $\hbar$, that is

$$\mathcal{S}(\Gamma) = O(\hbar^n), \tag{3.28}$$

we want to prove that it can be extended to order n, so that

$$\mathcal{S}(\Gamma) = O(\hbar^{n+1}). \tag{3.29}$$

Using the results from the Quantum Action Principle, we know that a possible breaking is a quantum insertion $\Delta\cdot\Gamma$ in the space of integrated local polynomials in the fields, bounded by dimension 2 (not counting the coupling constant) and with ghost number 1, at order n in the $\hbar$ expansion

$$\mathcal{S}(\Gamma) = \hbar^n \Delta\cdot\Gamma = \hbar^n\Delta + O(\hbar^{n+1}). \tag{3.30}$$

Since we can expand the linearized operator for the vertex functional as

$$\mathcal{S}_\Gamma = \mathcal{S}_{\Gamma^{(0)}} + O(\hbar), \tag{3.31}$$

the nilpotency of the Slavnov-Taylor operator implies that the breaking term obeys the following identity

$$\mathcal{S}_{\Gamma^{(0)}} \Delta = 0. \tag{3.32}$$

This consistency condition describes the cohomology of the operator in the sector of ghost number 1. The solution of (3.32) has the general form

$$\Delta = \mathcal{A} + \mathcal{S}_{\Gamma^{(0)}} \hat{\Delta}, \tag{3.33}$$

where $\mathcal{A}$ represents an anomaly, being a non-trivial cocycle, while $\mathcal{S}_{\Gamma^{(0)}} \hat{\Delta}$ belongs to the space of coboundaries, that is, solutions that are themselves variations of another term, and therefore are trivial solution due to the nilpotency of the operator $\mathcal{S}_{\Gamma^{(0)}}$. We see that the Slavnov-Taylor operator defines equivalence classes of elements which differ by a coboundary, that is

$$\Delta' \sim \Delta'' \implies \Delta' - \Delta'' = \mathcal{S}_{\Gamma^{(0)}} \tilde{\Delta}. \tag{3.34}$$

The breaking Δ, which is a solution of (3.32), is an integrated local functional of the fields, with ghost number 1 and dimension 2. Since all identities other than the Slavnov-Taylor were argued to hold for the quantum theory, it obeys the restrictions

$$\frac{\delta}{\delta b^a} \Delta = \mathcal{G}^a \Delta = \bar{\mathcal{G}}^a \Delta = \mathcal{W}_X \Delta = 0, \tag{3.35}$$

where $\mathcal{W}_X$ stands for the diffeomorphisms, Lorentz and rigid Ward identities. Moreover, the cohomology sector of ghost number 1 does not depend on the antifields [93, 94], which further restricts the fields dependence of Δ to only the gauge fields A^a_μ and the ghosts c^a, and the latter only through its derivatives since the anti-ghost equation (3.22) is an integrated one

$$\int d^3x \frac{\delta}{\delta c^a} \Delta = 0. \tag{3.36}$$

It is known that in three dimensions there are no anomalies, that is, the cohomology sector of ghost number 1 is empty [92, 94], up to some contribution from Abelian ghosts. These, however, do not contribute to the anomaly, due to their soft coupling [95, 96]. Moreover, we assume a simple compact $SU(N)$ Lie group, so no Abelian factor appears in the analysis. Therefore, the Slavnov-Taylor identity can be consistently implemented for the vertex functional Γ at all orders in perturbation theory.

As for the symmetric counterterms, that is, the sector of ghost number 0, responsible for the renormalization of the fields and parameters, we see that the bound to mass dimension 2 terms (since we must take into account the g^2 factor) to be added at each order to the tree-level action is rather restrictive. In fact, terms like the Yang-Mills kinetic term $F^{a\mu\nu}F^a_{\mu\nu}$, being of mass dimension 3, do not belong to the cohomology. The only term allowed by this constraint is the Chern-Simons term, so that

$$\begin{aligned}\Gamma^{ct} &= z_m \int d^3x\, \frac{m}{2}\epsilon^{\mu\rho\nu}\left(A^a_\mu \partial_\rho A^a_\nu + \frac{2g}{3} f^{abc} A^a_\mu A^b_\rho A^c_\nu\right)\\ &= z_m m \frac{\partial}{\partial m}\Gamma^{(0)},\end{aligned} \tag{3.37}$$

where z_m is an arbitrary parameter.

It follows from (3.37) that the Chern-Simons topological mass m is the only parameter that may suffer radiative corrections, since the only possible counterterm is related to it. Therefore, the coupling constant g and the fields exhibit vanishing β-function and anomalous dimensions, respectively. This shows that the theory is renormalizable at all orders, i.e. its symmetries remain valid for the quantum theory, represented in terms of the vertex functional Γ, and radiative corrections only affect the Chern-Simons mass parameter, being possibly reabsorbed by an appropriate redefinition of the tree-level action parameters and fields. However, we shall see that, due to (3.37) not being locally BRST-invariant, there are no radiative corrections for the mass m.

3.2 Scaling Properties of Yang-Mills-Chern-Simons

The beta functions and anomalous dimensions of a quantum field theory are related to the scaling properties of such theory. Since invariant counterterms must be fixed by renormalization conditions at a certain energy scale μ, these quantities express the dependence of parameters and fields to a change of scale, that is, how the theory varies along the energy scale μ. This behavior is described by the Callan-Symanzik equation, which, in view of the Quantum Action Principle, can be written as [43]

$$(m\partial_m + \mu\partial_\mu)\Gamma = \Delta \cdot \Gamma, \tag{3.38}$$

where $\Delta \cdot \Gamma$ is a symmetric insertion bounded by the space-time dimension D. In our case, the insertion given by the counterterm (3.37) is written in a suitable basis to enter (3.38).

However, as we mentioned before, the counterterm, which is proportional to the Chern-Simons action, is not locally invariant under BRST transformations, meaning that the Lagrangian density $\mathcal{L}_{CS}$ is not invariant, but it's integral is, since this change is proportional to a total derivative ($s\mathcal{L}_{CS} = \partial\Lambda$). Therefore, a more appropriate set up to study the scale dependence of the theory is through a local version of the Callan-Symanzik equation, where the differences between the Chern-Simons Lagrangian and action can be better exploited. This local formulation can be achieved from the trace identity of the energy-momentum tensor, defined as the quantum insertion

$$\Theta_\mu{}^\nu \cdot \Gamma = e^{-1} e^m_\mu \frac{\delta\Gamma}{\delta e^m_\nu}. \tag{3.39}$$

Let us see how one can obtain such equation. From the conservation of the classical energy momentum tensor, due to the invariance of the classical action under diffeomorphisms, we get the following local equation

$$e\nabla_\mu \Theta_\nu{}^\mu = w_\nu(x)\Gamma^{(0)} + \nabla_\mu w_\nu{}^\mu(x)\Gamma^{(0)}, \tag{3.40}$$

where ∇_μ is the curved space covariant derivative[1], that is, it takes into account the diffeomorphism invariance, and the operators $w_\nu(x)$ and $w_\nu{}^\mu(x)$ are

$$w_\mu(x) = \sum_\Phi \nabla_\mu \Phi \frac{\delta}{\delta\Phi}, \tag{3.42}$$

and

$$w_\nu{}^\mu(x) = A^{*\mu}\frac{\delta}{\delta A^{*\nu}} - A^\mu \frac{\delta}{\delta A^\nu} - \delta^\mu_\nu \left(c^* \frac{\delta}{\delta c^*} + A^{*\lambda}\frac{\delta}{\delta A^{*\lambda}} \right), \tag{3.43}$$

which represent contact terms. We see that in the flat space limit the operator $w_\mu(x)$ is nothing more than the translation Ward operator.

The integrated trace of the energy-momentum is

$$\int d^3x e\Theta_\mu{}^\mu = \int d^3x e^m_\mu \frac{\delta\Gamma^{(0)}}{\delta e^m_\mu} \equiv N_e \Gamma^{(0)}, \tag{3.44}$$

where N_e is the counting operator for the dreibein field e^m_μ. The integral (3.44) is nothing but the rigid Weyl symmetry Ward operator [97] and it

[1]For a general tensor T, the action of the covariant derivative is given by

$$\nabla_\mu T^{\alpha\cdots}_{\beta\cdots} \equiv \partial_\mu T^{\alpha\cdots}_{\beta\cdots} + \Gamma^\alpha_{\mu\rho} T^{\rho\cdots}_{\beta\cdots} + \cdots - \Gamma^\rho_{\mu\alpha} T^{\alpha\cdots}_{\rho\cdots} - \cdots, \tag{3.41}$$

where $\Gamma^\rho_{\mu\nu}$ are the corresponding Christoffel symbols associated to the spin connection ω^{ab}_μ.

turns out to be an equation of motion, signaling that $\Theta_\mu{}^\nu$ is an improved energy-momentum tensor [98]. This follows from the identity

$$N_e\Gamma^{(0)} = \left(\sum_\Phi d_W(\Phi)N_\Phi + m\partial_m + \frac{1}{2}g\partial_g\right)\Gamma^{(0)}, \tag{3.45}$$

with $d_W(\Phi)$ the Weyl dimension and $N_\Phi = \int d^3x\Phi\frac{\delta}{\delta\Phi}$ the counting operator for the each field Φ in the action. Considering the integrand of equation (3.45) and taking (3.44) into account, we find the local trace identity

$$w(x)\Gamma^{(0)} \equiv \left(e^m_\mu(x)\frac{\delta}{\delta e^m_\mu(x)} - \sum_\Phi d_W(\Phi)N_\Phi\right)\Gamma^{(0)} = \Lambda(x), \tag{3.46}$$

which is a local version of equation (3.44), more easily seen as such in the form

$$e\Theta_\mu{}^\mu = \sum_\Phi d_W(\Phi)\Phi(x)\frac{\delta}{\delta\Phi(x)}\Gamma^{(0)} + \Lambda(x), \tag{3.47}$$

where the breaking of the scale invariance Λ, coming from the mass and coupling constant derivatives of the tree-level Lagrangian, that is, from the dimensionful couplings of the theory, is invariant under $\mathcal{S}_{\Gamma^{(0)}}$ and, being a soft breaking, of mass dimension lower than 3.

The goal now is to promote this local trace identity (3.46) to the quantum theory, that is, make sure it holds for the quantum insertion $\Theta_\nu{}^\mu \cdot \Gamma$. In terms of the w operator, we want to show that

$$w\Gamma = \Lambda \cdot \Gamma, \tag{3.48}$$

where $\Lambda \cdot \Gamma$ is some quantum version of the classical soft scale breaking Λ. First, we notice that the following conditions hold for $w\Gamma$

$$\begin{aligned}
&\mathcal{S}_\Gamma w(x)\Gamma = 0, \quad \bar{\mathcal{G}}^a w(x)\Gamma = \frac{1}{2}\frac{\delta\Gamma}{\delta c^a(x)}, \\
&\frac{\delta\Gamma}{\delta b^a(y)}w(x)\Gamma = -\frac{3}{2}\frac{\partial}{\partial x^\mu}\delta(x-y)(eg^{\mu\nu}A^a_\nu)(y), \\
&\mathcal{G}^a(y)w(x)\Gamma = \frac{3}{2}\frac{\partial}{\partial x^\mu}\delta(x-y)\left(eg^{\mu\nu}\frac{\delta\Gamma}{\delta A^{*a\nu}}\right)(y),
\end{aligned} \tag{3.49}$$

coming from the commutation of the local operator $w(x)$ (3.46) and the ghost, anti-ghost, gauge condition and Slavnov-Taylor identities, which hold for the quantum theory. From the Quantum Action Principle, the trace identity can get corrected by some insertion $\Delta \cdot \Gamma$, so that

$$w(x)\Gamma = \Lambda \cdot \Gamma + \Delta \cdot \Gamma. \tag{3.50}$$

Since $\Lambda \cdot \Gamma$ obeys the full constraints (3.49) [83], this insertion must obey the homogeneous version of the constraints (3.49) imposed on $w(x)\Gamma$,

$$\mathcal{S}_\Gamma[\Delta \cdot \Gamma] = \bar{\mathcal{G}}^a[\Delta \cdot \Gamma] = \frac{\delta}{\delta b^a}[\Delta \cdot \Gamma] = \mathcal{G}^a[\Delta \cdot \Gamma] = 0, \tag{3.51}$$

as well as invariance or covariance under (3.23) and (3.24).

As in the previous cases, the insertion $\Delta \cdot \Gamma$ has an effective dimension bounded by 2, since a factor of g^2 must be included coming from radiative corrections. This restriction, added to the other constraints, makes it that there is no possible quantum breaking term for the local trace identity (3.46), since we now must have a local insertion which is invariant under the Slavnov-Taylor operator $\mathcal{S}_\Gamma$, ruling out the Chern-Simons Lagrangian. Therefore, $\Delta \cdot \Gamma = 0$ and the local trace identity (3.46) is valid for the vertex functional, which means

$$e\Theta_\mu{}^\mu \cdot \Gamma = \sum_\Phi d_W(\Phi)\Phi\frac{\delta}{\delta\Phi}\Gamma + \Lambda \cdot \Gamma. \tag{3.52}$$

Therefore, the Callan-Symanzik equation is given by

$$\left(m\partial_m + \frac{1}{2}g\partial_g\right)\Gamma = \int d^3x \Lambda \cdot \Gamma, \tag{3.53}$$

which is only affected by the soft breaking term, but no radiative corrections. This results implies the vanishing of the mass m and coupling constant g β-functions and the anomalous dimensions of the fields, as stated in the previous section. This shows that the Yang-Mills-Chern-Simons theory is UV finite, that is, it suffers no renormalization of its fields and parameters to all orders in perturbation theory.

The result obtained in this section will be of extreme importance in the following chapters, where we study Yang-Mills-Chern-Simons theory in more general scenarios.

Chapter 4

Yang-Mills-Chern-Simons in Non-covariant Gauges

4.1 Non-covariant Gauges

The study of theories possessing gauge freedom started with Maxwell's electrodynamics in non-covariant gauges, such as the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$), also known as radiation gauge, and the temporal gauge ($A_0 = 0$). In fact, the quantization of electrodynamics (QED) was first performed in these two gauge, and only many years after in a manifest Lorentz invariant way. However, there were some problems arising from the canonical quantization of electrodynamics, for example the non-local character of the canonical commutation relations [99]

$$[A^i(\mathbf{x}), E^j(\mathbf{x}')] = -i\left(\delta^{ij} - \frac{\nabla^i \nabla^j}{\nabla^2}\right)\delta(\mathbf{x} - \mathbf{x}'), \tag{4.1}$$

where ∇^i are the components of the Euclidean spatial differential operator, appearing as a result of the Coulomb gauge choice. The covariant approach, however, introduces negative norm states to the Hilbert space $\mathcal{H}$ of the theory, making it ill-defined. It was shown by Gupta [100] and Bleuler [101] that the true space in which the theory is defined only contains transverse photons, that is, it is a subspace obeying the supplementary condition

$$\mathcal{H}_{phys} = \{|\Psi\rangle, \langle\Psi'|\partial^\mu A_\mu|\Psi\rangle = 0\}. \tag{4.2}$$

This allowed for more practical calculation of the renormalization of QED, since Lorentz covariance and locality were restored. Moreover, one can see that this is nothing more than the BRST cohomology space in the case of Abelian theories.

The advantage of working with covariant gauges is that renormalization theory was well-established in this setting, which allowed for the perturbative treatment of gauge theories. In fact, all the main theorems of renormalization theory are valid for manifest Lorentz-invariant scenarios, such as the Power Counting Theorem [73–75] and the Quantum Action Principle [55–62]. Beyond that, they present a set of technical features [102] which facilitate the evaluation of loop integrals, i.e. the existence of a uniform prescription to dislocate the poles in the propagators (Feynman's $i\epsilon$ prescription), which allows for the Wick rotation of the integrals into Euclidean space, and the possibility to use the tensor method, also known as the Passarino-Veltman reduction [103], to reduce 1-loop and multiloop tensor integrals into known scalar integrals.

But non-covariant gauges also come with their own advantages [102,104]. First of all is the decoupling of the Faddeev-Popov ghost fields from the gauge fields for a wide range of such conditions. This is related to the elimination of spurious degrees of freedom from the theory, defining it in terms of only the propagating physical ones. For that reason, they are sometimes called physical gauges. Let us see how this happens, following a derivation due to Frenkel [105]. Take a Yang-Mills theory with the gauge condition

$$n^\mu A^a_\mu(x) = 0, \tag{4.3}$$

where n^μ is some arbitrary dimension one constant four-vector. The gauge-fixing Lagrangian is given by

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(n^\mu A^a_\mu(x))^2, \tag{4.4}$$

and it yields, together with the standard Yang-Mills Lagrangian, a gluon propagator of the form

$$\langle TA^a_\mu A^b_\nu\rangle(k) = -i\frac{\delta^{ab}}{k^2}\left[\eta_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n} + k_\mu k_\nu \frac{n^2}{(k\cdot n)^2}\right], \tag{4.5}$$

where we have taken $\alpha = 0$. This propagator is orthogonal to the vector n^μ, as can be seen by the contraction

$$n^\mu\langle TA^a_\mu A^b_\nu\rangle = -i\frac{\delta^{ab}}{k^2}\left[n_\nu - \frac{(k\cdot n)n_\nu + k_\nu n^2}{k\cdot n} + (k\cdot n)k_\nu \frac{n^2}{(k\cdot n)^2}\right] = 0. \tag{4.6}$$

The decoupling of the ghost comes from the gluon-ghost vertex, which is proportional to n^μ. From the gauge condition, we compute the ghost Lagrangian through the Faddeev-Popov operator. Given the gauge condition (4.3)

$$\begin{aligned} M^{ab}(x,y) &= \frac{\delta(n^\mu A'^a_\mu(x))}{\delta\omega^b(y)} = \int dz \frac{\delta(n^\mu A'^a_\mu(x))}{\delta A'^c_\nu(z)}\frac{\delta A'^c_\nu(z)}{\delta\omega^b(y)} \\ &= \delta^{ac} n^\mu D^{cb}_\mu \delta(x-y) \\ &= n^\mu(\delta^{ab}\partial_\mu + igf^{abc}A^c_\mu)\delta(x-y), \end{aligned} \tag{4.7}$$

Therefore, the ghost Lagrangian is given by

$$\mathcal{L}_{gh} = \bar{c}^a M^{ab} c^b. \tag{4.8}$$

The gluon-ghost vertex arises from the second therm in the Lagrangian, which is proportional to n^μ. This implies that the ghost fields decouple from the theory, since any diagram involving ghost fields automatically vanishes. Moreover, all S-matrix elements are independent of n^μ. This result also holds for the inhomogeneous condition $n\cdot A^a = b^a$, with b^a being arbitrary functions.

Another way to see this decoupling, although rather naive, is to notice that the use of the gauge condition $n \cdot A^a = 0$ turns the Faddeev-Popov operator into

$$M^{ab}(x,y) = \delta^{ab} n \cdot \partial \delta(x-y), \tag{4.9}$$

which no longer depends on the gauge fields and can be directly absorbed by the normalization factor of the generating functional, without the need to introduce ghost fields.

Beyond that, non-covariant gauges find extensive applications in supersymmetric theories. It was used to prove the ultraviolet finiteness of $N = 4$ super Yang-Mills [106–108] and the cancellation of anomalies in superstring theories [109]. The special class of algebraic non-covariant gauges are also free from the Gribov problem [110], that is, the choice of gauge-fixing successfully selects one representative element for each gauge orbit. More on the Gribov problem with be discussed in the next chapter.

We will focus on the class of algebraic non-covariant gauges [110]: they are defined by the introduction of a privileged direction in space-time, given by a constant vector $n^\mu = (n_0, -\mathbf{n})$, such as the one we already encountered in eq. (4.3). They can be divided into two different classes: axial gauges and planar gauges. In the axial case, we have the homogeneous conditions $n \cdot A^a = 0$, while the planar gauges are defined by the inhomogeneous conditions $n \cdot A^a = b^a$, where the b^a are arbitrary scalar functions. For this reason the planar gauge is also called the inhomogeneous axial gauge. There is a second division that can be made among them, regarding the type of constant vector n^μ, that is, the sign of its square norm: they are called temporal gauges for time-like vectors ($n^2 > 0$), light-cone gauges for light-like vectors ($n^2 = 0$) and spatial gauges for space-like vectors ($n^2 < 0$).

In all algebraic non-covariant gauges, we encounter a gauge propagator with the presence of spurious poles $(n \cdot k)^{-\alpha}, (\alpha = 1, 2, \dots)$, as we have seen in equation (4.5) of the previous example. The proper way to handle such poles is the first difficulty someone faces when dealing with such gauges. As we mentioned, there is no all encompassing prescription for the poles like Feynman's $i\epsilon$ prescription in covariant gauges. The main used ones are the Cauchy principle-value (PV) and the Mandelstam-Leibbrandt prescriptions.

The PV prescription is related to the Sokhotski–Plemelj theorem, for integrals defined on the real line [111]

$$\frac{1}{x \pm i\epsilon} = PV\frac{1}{x} \mp i\pi\delta(x), \quad \epsilon > 0 \tag{4.10}$$

which implies that the principle value of the pole $(n \cdot k)^{-\alpha}$ can be written as

$$PV\frac{1}{(n \cdot k)^{\alpha}} = \frac{1}{2}\lim_{\epsilon \to 0}\left(\frac{1}{(n \cdot k + i\epsilon)^{\alpha}} + \frac{1}{(n \cdot k - i\epsilon)^{\alpha}}\right), \tag{4.11}$$

for $\epsilon > 0$ and $\alpha = 1, \ldots, n$. This places the poles in the first and fourth quadrants in the complex q_0 plane (for $\mathbf{n} \cdot \mathbf{k} > 0$), which prevents the Wick rotation from Minkowski to Euclidean space, difficulting the evaluation of loop integrals. A similar relation holds for $\mathbf{n} \cdot \mathbf{k} < 0$, the poles now shifted to the second and third quadrants.

The Mandelstam-Leibbrandt prescription [102], also called the $n^{*\mu}$-prescription, was initially proposed for the light-cone gauge and later generalized to the other classes [104] . It introduces the dual vector $n^{*\mu} = (n_0, \mathbf{n})$, so that

$$\frac{1}{(n \cdot k)} = \begin{cases} \lim_{\epsilon \to 0} \frac{1}{n \cdot k + i\epsilon sign(n^* \cdot k)}, \\ \lim_{\epsilon \to 0} \frac{n^* \cdot k}{(n^* \cdot k)(n \cdot k) + i\epsilon}, \end{cases} \tag{4.12}$$

where we assume $\epsilon > 0$. It has the main advantage to be causal and of placing the poles in the second and fourth quadrants, like Feynman's prescription, so that the integrals can be Wick rotated, overcoming a major computational trouble for non-covariant gauges.

However, even though the Mandelstam-Leibbrandt prescription was able to handle a few of the more serious problems of the PV one [110], it does fall short on some issues. The obvious one is lack of manifest Lorentz covariance. But it also the introduces another privileged direction, given by $n^{*\mu}$, bringing on another source of Lorentz violation. This new vector also generates new integrals, which complicate the handling of divergencies in the perturbative expansion. Another serious question comes from the non-polynomial character of the divergent parts of the loop integrals [104, 112]. According to the quantum action principle, the divergencies coming from loop integrals correspond to counterterms that are added to the classical action, them being local polynomials in the fields. The appearance of non-local divergencies coming from this non-polynomial dependence on the external momenta prevents us to naively apply some general results of renormalization theory, like the power counting theorem. But there is a way to circumvent this difficulties, which we will see in the following.

4.2 The Interpolating Gauge

Even though the use of non-covariant gauges led to non-local divergencies in the perturbative expansion of 1PI diagrams, they did not affect any physical quantities of the theory, such as S-matrix elements, and only local counterterms were necessary to render Yang-Mills theories finite at any order [113, 114]. These arguments, however, relied on the existent of an invariant regularization scheme, namely dimensional regularization. But there is another way to deal with such divergencies. One can introduce an interpolation parameter ζ, such that the gauge condition contain specific covariant and noncovariant conditions for certain values of ζ [115, 116]. The non-local divergencies can then be regularized with this new gauge parameter, ensuring that only local counterterms are needed for all ζ, except limiting values which define a non-covariant gauge condition. In this way, power counting now applies to the theory and the results from the quantum action principle can be employed. Moreover, this means that the restriction to a special regularization scheme falls over, since we can now work in a regularization independent way, and avoid the troubles of computing Feynman diagrams in non-covariant gauges [117]. The restriction of ζ to be a gauge parameter also guarantees that physical quantities, like correlation function of gauge invariant operators, do not depend on it.

We follow [116] to show how this works. Consider a gauge condition which interpolates between the light-cone (LC) gauge

$$n^\mu A^a_\mu = 0, \quad \text{with } n^2 = 0, \tag{4.13}$$

and the covariant gauge

$$\partial^\mu A^a_\mu = 0. \tag{4.14}$$

We can introduce this through the gauge-fixing action

$$S_{GF} = \int dx (b^a N^\mu A^a_\mu - \bar{c}^a N^\mu D^{ab}_\mu c^b), \tag{4.15}$$

where N^μ defines the gauge condition we want to impose. Since we want to interpolate between the light-cone gauge and the covariant gauge, it can be constructed by demanding that it goes to either one of these gauge conditions for limiting values of the interpolation parameter ζ, so that we can take

$$N^\mu = \frac{\partial^\mu + \zeta(n^* \cdot \partial)n^\mu}{\zeta(n^* \cdot \partial)}. \tag{4.16}$$

We can see the limiting cases for $\zeta \to 0$ and for $\zeta \to \infty$, where we get the covariant and light-cone conditions, respectively. Here we introduce the dual vector $n^{*\mu}$, in order to use the Mandelstam-Leibbrandt prescription for the poles. The interpolation between the two gauge conditions gets clearer by analyzing the gluon propagator of pure Yang-Mills theory with the gauge-fixing term 4.15

$$\begin{aligned} \langle T A^a_\mu A^b_\nu \rangle(k) = \frac{-i\delta^{ab}}{k^2 + i\epsilon} \Bigg[\eta_{\mu\nu} &- \frac{(k_\mu n_\nu + k_\nu n_\mu)\zeta(n^* \cdot k)}{k^2 + \zeta(n^* \cdot k)(n \cdot k) + i\epsilon(1+\zeta)} \\ &- \frac{k^2 k_\mu k_\nu}{(k^2 + \zeta(n^* \cdot k)(n \cdot k) + i\epsilon(1+\zeta))^2} \Bigg], \end{aligned} \tag{4.17}$$

where the LC propagator

$$\langle T A^a_\mu A^b_\nu \rangle(k)\big|_{\zeta \to \infty} = \frac{-i\delta^{ab}}{k^2 + i\epsilon} \left[\eta_{\mu\nu} - \frac{(k_\mu n_\nu + k_\nu n_\mu)(n^* \cdot k)}{(n^* \cdot k)(n \cdot k) + i\epsilon} \right] \tag{4.18}$$

arises naturally with the Mandelstam-Leibbrandt prescription. In the limit $\zeta \to 0$, we see that it leads to the Landau gauge gluon propagator, which is expected from the gauge-fixing action (4.15). This interpolation, however, is not exclusive to the Landau gauge and can be extended to other linear covariant gauges.

This interpolating gauge condition is non-local, due to the $(n^* \cdot \partial)$ term in the denominator of N^μ, a feature which could ruin the potential application of the quantum action principle. However, it can be localized with the introduction of auxiliary fields. It will also allow us to write the gauge condition in a manifest BRST-invariant way. Generalizing equation (4.15) to a linear covariant gauge, through the introduction of a gauge parameter α, and recalling that the gauge-fixing action does not affect physical quantities of the theory, which implies it must be a BRST-exact term, we cast (4.15) into the form

$$S_{GF} = s \int dx \left[\bar{c}^a \left(\frac{\partial^\mu}{\zeta(n^* \cdot \partial)} + n^\mu \right) A^a_\mu + \frac{1}{2} \alpha \bar{c}^a \frac{1}{\zeta^2 (n^* \cdot \partial)} b^a \right]. \tag{4.19}$$

To localize the gauge condition, we introduce two sets of auxiliary fields K^a and D^a, which can be removed by use of their respective equations of motion, so that the bosonic sector of the gauge-fixing is

$$S^{boson}_{GF} = \int dx \left[K^a(\zeta(n^* \cdot \partial) - \partial^\mu A^a_\mu) D + b^a D^a + b^a n^\mu A^a_\mu + \frac{1}{2} \alpha K^a K^a \right]. \tag{4.20}$$

To rewrite the fermionic sector, we need to introduce two ghost fields K^a_- and D^a_+, and require them to compose BRST-doublets with the auxiliary fields, so that the BRST transformations are

$$\begin{aligned} &sA^a_\mu = -D^{ab}_\mu c^b, \quad sc^a = \frac{1}{2} g f^{abc} c^b c^c, \quad s\bar{c}^a = b^a, \quad sb^a = 0, \\ &sK^a_- = K^a, \quad sK^a = 0, \quad sD^a = D^a_+, \quad sD^a_+ = 0, \end{aligned} \tag{4.21}$$

where the indices $-, +$ on the auxiliary ghost fields indicate their ghost number. The full local gauge-fixing action is then

$$S_{GF} = s \int dx \left[K^a_-(\zeta(n^{*\mu} \partial_\mu) - \partial^\mu A^a_\mu) D + \bar{c}^a (D^a + n^\mu A^a_\mu) + \frac{1}{2} \alpha K^a_- K^a \right]. \tag{4.22}$$

From this construction of the gauge-fixing action, one can immediately get that physical quantities do not depend on the gauge parameters ζ, α, n^μ and $n^{*\mu}$. That is because the total tree-level action, which is composed of

(4.22) together with an invariant action, giving the dynamics and interaction of the gauge fields (and possibly other fields), and an external source action, which must be introduced to control the renormalization of the BRST-transformations of A and c, only depend on these gauge parameters through some BRST-variation

$$\partial_\zeta \Gamma^{(0)} = s \int dx (K_-^a (n^* \cdot \partial) D), \quad \partial_\alpha \Gamma^{(0)} = s \int dx \left(\frac{1}{2} K_-^a K^a \right), \tag{4.23}$$
$$\partial_{n^\mu} \Gamma^{(0)} = s \int dx (\bar{c}^a A_\mu^a), \quad \partial_{n^{*\mu}} \Gamma^{(0)} = s \int dx (\zeta K_-^a \partial_\mu D),$$

meaning that these identities can be transformed in exact symmetries by extending the BRST symmetry, transforming the parameters into BRST-doublets and, therefore, ensuring that they can only appear in the trivial sector of the Slavnov-Taylor cohomology of ghost number 0. This proves that no physical quantity, whose renormalization is given by counterterms belonging to the non-trivial sector, depend on any of the gauge parameters nor on any of the auxiliary fields, which are also BRST-doublets.

We now proceed to briefly argument why only local counterterms appear and the power counting theorem is valid in this set up, following a reasoning which proves the quantum action principle for the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein (BPHZL) scheme [74, 76, 77]. The convergence of the BPHZL renormalized integrals in Minkowski space is guaranteed by the existence of convergent lower and upper bound integrals in Euclidean space [118], coming from a modification of Feynman's $i\epsilon$-prescription for the propagator poles

$$\frac{1}{k^2 + i\epsilon} \to \frac{1}{k^2 - m_s^2 + i\epsilon(\mathbf{k}^2 + m_s^2)}, \tag{4.24}$$

where $m_s^2 = m^2(s-1)$ is a mass term introduced for massless theories in order to avoid the appearance of IR divergencies coming from the renormalization process. By taking the limit $s = 1$ at the end of the subtraction procedure, we recover the original massless theory. For the non-covariant gluon propagator

(4.17), this prescription implies the modification

$$\frac{1}{k^2 + \zeta(n^* \cdot k)(n \cdot k) + i\epsilon(1+\zeta)} \to \frac{1}{k^2 + \zeta(n^* \cdot k)(n \cdot k) - m_s^2 + i\epsilon(\mathbf{k}^2 + m_s^2 + \zeta(\mathbf{k}\cdot\mathbf{n})^2)} \tag{4.25}$$

We can see that the non-covariant denominator of the propagator (4.17) can be obtained from the covariant one by the changing the Minkowski metric like

$$\eta_{00} = 1 + \zeta n_0^2, \quad \eta_{ij} = -(\delta_{ij} + \zeta n_i n_j), \tag{4.26}$$

so that the pole structure changes like

$$\begin{aligned} k^2 &\to \eta_{\mu\nu} k^\mu k^\nu \\ &= (1+\zeta n_0^2)k_0^2 - (\delta_{ij} + \zeta n_i n_j)k_i k_j \\ &= k_0^2 - k_i^2 + \zeta(n_0^2 k_0^2 - (n_i k_i)^2) \\ &= k^2 + \zeta(n^* \cdot k)(n \cdot k). \end{aligned} \tag{4.27}$$

Therefore, by using the modifications in the Minkowski metric presented above, the same prescription used with a covariant gauge-fixing can be obtained in the non-covariant one we are discussing. For non-negative values of ζ, the corresponding Euclidean metric is positive definite, and there are Euclidean majorant and minorant values for the integrals, which allows us to use the usual BPHZL subtraction procedure [74, 76, 77]. The theory is then well-defined for the interval $\zeta \in [0, \infty)$. In the limit $\zeta \to \infty$, this assumptions are no longer valid, and we may encounter non-local divergences proportional to $(n \cdot k)^{-r}$, but since this value characterizes the LC gauge, this is to be expected. For all other finite values, the integrals can always be made finite, so the usual power counting arguments remain valid, as well as the general results of the quantum action principle. The question of Lorentz invariance can be dealt with by treating the LC vectors n^μ and $n^{*\mu}$ as Lorentz vectors [119]. In this manner, the algebraic renormalization program can be extended to this class of non-covariant gauges.

4.3 Algebraic Renormalization of Yang-Mills-Chern-Simons in the Interpolating Gauge

The interpolating gauge-fixing presented in the previous section was first introduced to study the renormalization properties of Yang-Mills theories in four space-time dimensions [116], and later used to prove the ultraviolet finiteness of Chern-Simons theory in three space-time dimensions [120]. In the latter, it was shown to preserve the local vector-supersymmetry characteristic of topological field theories [121–124]. This vector-supersymmetry, first identified in the Landau gauge, is crucial to the proof of the UV finiteness of Chern-Simons theory [125], as it imposes an extra constraint to the theory, and it is also present in algebraic non-covariant gauges [120,126–128].

We now make use of this interpolating gauge to study the algebraic renormalization of the Yang-Mills-Chern-Simons theory and prove its ultraviolet finiteness. The tree-level gauge-fixed YMCS action is given by the invariant action plus the interpolating gauge-fixing term [129]

$$S = S_{inv} + S_{gf}, \tag{4.28}$$

where the invariant action is given by

$$S_{inv} = \int d^3x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} m \epsilon^{\mu\rho\nu} \left(A^a_\mu \partial_\rho A^a_\nu + \frac{2g}{3} f^{abc} A^a_\mu A^b_\rho A^c_\nu \right) \right], \tag{4.29}$$

while the gauge-fixing action is given by eq. (4.22), which we write explicitly

$$\begin{aligned} S_{gf} = \int d^3x \, \big[& K^a (\zeta (n^{*\mu} \partial_\mu) D^a - \partial^\mu A^a_\mu) - K^a_- (\zeta (n^{*\mu} \partial_\mu) D^a_+ + \partial^\mu D^{ab}_\mu c^b) \\ & + b^a (D^a + n^\mu A^a_\mu) - \bar{c}^a (D^a_+ - n^\mu D^{ab}_\mu c^b + \frac{\alpha}{2} K^a K^a) \big] . \end{aligned} \tag{4.30}$$

As in the case of pure Yang-Mills, the gluon propagator makes clear the interpolation of the gauge-fixing for the limiting values of ζ between the light-cone ($\zeta \to \infty$) and the linear covariant ($\zeta \to 0$) gauges. It reads

$$\langle TA^a_\mu A^b_\nu \rangle = \frac{i\delta^{ab}}{\partial^2 + m^2}\left[\eta_{\mu\nu} - \frac{\partial^2\partial_\mu\partial_\nu}{\partial^2_\zeta} - \frac{m}{\partial^2}\epsilon_{\mu\rho\nu}\partial^\rho - \frac{\zeta n^{*\lambda}\partial_\lambda}{\partial^2_\zeta}(n_\mu\partial_\nu + n_\nu\partial_\mu) \right. \\ \left. + \frac{\alpha}{(\partial^2_\zeta)^2}\frac{\partial^2(\partial^2+m^2) + m^2\zeta(n^{*\lambda}\partial_\lambda)(n^\lambda\partial_\lambda)}{\partial^2}\partial_\mu\partial_\nu\right], \tag{4.31}$$

where we defined $\partial^2_\zeta = \partial^2 + \zeta(n^{*\lambda}\partial_\lambda)(n^\lambda\partial_\lambda)$. From (4.31), one can recover the covariant propagator

$$\langle TA^a_\mu A^b_\nu \rangle|_{\zeta=0} = \frac{i\delta^{ab}}{\partial^2 + m^2}\left[\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2} - \frac{m}{\partial^2}\epsilon_{\mu\rho\nu}\partial^\rho\right] + i\delta^{ab}\frac{\alpha}{\partial^2}\frac{\partial_\mu\partial_\nu}{\partial^2}, \tag{4.32}$$

and the light-cone propagator

$$\langle TA^a_\mu A^b_\nu \rangle|_{\zeta\to\infty} = \frac{i\delta^{ab}}{\partial^2 + m^2}\left[\eta_{\mu\nu} - \frac{m}{\partial^2}\epsilon_{\mu\rho\nu}\partial^\rho - \frac{n_\mu\partial_\nu + n_\nu\partial_\mu}{n^\lambda\partial_\lambda}\right]. \tag{4.33}$$

For finite values of ζ, the propagator is compatible with the requirements of the power-counting theorem and we can use the quantum action principle to study the renormalization properties of the YMCS action to all orders in perturbation theory. To make use of the algebraic renormalization procedure [43,117], we must introduce external sources coupled to the non-linear BRST-variations of the gauge (A^a_μ) and Faddeev-Popov ghost (c^a) fields to control their renormalization

$$S_{ext} = \int d^3x[\rho^{a\mu} sA^a_\mu + \sigma^a sc^a], \tag{4.34}$$

so that the complete tree-level action is given by

$$\Gamma^{(0)} = S_{inv} + S_{gf} + S_{ext}. \tag{4.35}$$

To ensure that the whole tree-level action is BRST-invariant, we require the sources to transform trivially under the BRST operator, that is, $s\rho^{a\mu} = s\sigma = 0$. The BRST-invariance of the tree-level action $\Gamma^{(0)}$ is expressed functionally through the Slavnov-Taylor identity

$$\mathcal{S}(\Gamma^{(0)}) = \int d^3x \left[\frac{\delta\Gamma^{(0)}}{\delta\rho^{a\mu}}\frac{\delta\Gamma^{(0)}}{\delta A^a_\mu} + \frac{\delta\Gamma^{(0)}}{\delta\sigma^a}\frac{\delta\Gamma^{(0)}}{\delta c^a} + b^a\frac{\delta\Gamma^{(0)}}{\delta\bar{c}^a} + K^a\frac{\delta\Gamma^{(0)}}{\delta K^a_-} + D^a_+\frac{\delta\Gamma^{(0)}}{\delta D^a}\right] = 0. \tag{4.36}$$

The linearized Slavnov-Taylor operator, which will be used later in the analysis of the BRST cohomology, is defined as

$$\mathcal{S}_{\Gamma^{(0)}} = \int d^3x \left[\frac{\delta\Gamma^{(0)}}{\delta\rho^{a\mu}}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\Gamma^{(0)}}{\delta A^a_\mu}\frac{\delta}{\delta\rho^{a\mu}} + \frac{\delta\Gamma^{(0)}}{\delta\sigma^a}\frac{\delta}{\delta c^a} + \frac{\delta\Gamma^{(0)}}{\delta c^a}\frac{\delta}{\delta\sigma^a} + b^a\frac{\delta}{\delta\bar{c}^a} + K^a\frac{\delta}{\delta K^a_-} + D^a_+\frac{\delta}{\delta D^a}\right]. \tag{4.37}$$

Beyond that, the tree-level action $\Gamma^{(0)}$ obeys a series of other functional identities, which will be useful in the determination of the counterterm. Those are linearly broken identities on the quantum fields, given by the equations of motion of some of the fields of the theory, whose extension for the quantum theory is guaranteed by the Quantum Action Principle [43]. They are four gauge conditions for the auxiliary Lagrange multiplier fields

$$\frac{\delta\Gamma^{(0)}}{\delta b^a} = D^a + n^\mu A^a_\mu, \tag{4.38}$$

$$\frac{\delta\Gamma^{(0)}}{\delta K^a} = \zeta(n^{*\mu}\partial_\mu)D^a - \partial_\mu A^a_\mu + \alpha K^a, \tag{4.39}$$

$$\frac{\delta\Gamma^{(0)}}{\delta D^a} = -\zeta(n^{*\mu}\partial_\mu)K^a + b^a, \tag{4.40}$$

$$\frac{\delta\Gamma^{(0)}}{\delta D^a_+} = -\zeta(n^{*\mu}\partial_\mu)K^a_- + \bar{c}^a, \tag{4.41}$$

two ghost equations

$$\frac{\delta\Gamma^{(0)}}{\delta\bar{c}^a} + n^\mu\frac{\delta\Gamma^{(0)}}{\delta\rho^{a\mu}} = -D^a_+, \tag{4.42}$$

$$\frac{\delta\Gamma^{(0)}}{\delta K^a_-} - \partial^\mu\frac{\delta\Gamma^{(0)}}{\delta\rho^{a\mu}} = -\zeta(n^{*\mu}\partial_\mu)D^a_+, \tag{4.43}$$

Field	A_μ	c	$\bar{c}$	b	K	K_-	D	D_+	ρ^μ	σ	g
d	1/2	-1/2	5/2	5/2	3/2	3/2	1/2	1/2	5/2	7/2	1/2
$\Phi\Pi$	0	1	-1	0	0	-1	0	1	-1	-2	0

Table 4.1: Dimension and ghost number of the field content and coupling constant of the YMCS action with the auxiliary fields of the interpolating gauge.

as well as an integrated anti-ghost equation

$$\mathcal{G}^a\Gamma^{(0)} = \Delta^a_{cl}, \tag{4.44}$$

where we define

$$\mathcal{G}^a = \int d^3x \left[\frac{\delta}{\delta c^a} + gf^{abc}K^a_-\frac{\delta}{\delta K^a} + gf^{abc}D^a\frac{\delta}{\delta D^a_+} + gf^{abc}\bar{c}^a\frac{\delta}{\delta b^a}\right], \tag{4.45}$$

and

$$\Delta^a_{cl} = \int d^3x\ gf^{abc}[c^b\sigma^c - \rho^{b\mu}A^c_\mu + \alpha K^b_- K^c]. \tag{4.46}$$

Before starting the discussion on renormalization, let us recap a little about the power counting for Yang-Mills-Chern-Simons. As stated in the previous chapter, the YMCS theory is superrenormalizable due to the dimensionful nature of its coupling constant g, of mass dimension $d_g = 1/2$ [91]. This reduces the effective dimension of the counterterms to 2, since any radiative correction appearing in the full vertex functional must be at least of order g^2 in the coupling constant. Since we have introduced new auxiliary fields, we collect the dimensions and ghost numbers of the full field content of the theory on Table 4.1. Notice how the change in the gauge-fixing alters the dimensions of the fields other than A^a_μ, which is fully defined by the kinetic term in the invariant action S_{inv} (4.29).

We also know that the gauge conditions (4.38) - (4.41) and ghost equations (4.42), (4.43) can be readily extended from the classical to the quantum theory, since they are at most linearly broken in the quantum fields [43]. However, that is not the case for the anti-ghost equation (4.44), since its classical breaking exhibits a non-linear term on the quantum fields, namely the $\alpha K^b_- K^c$ term. But due to its explicit dependence on the gauge parameter, this term can be eliminated by a suitable choice of α. We can choose $\alpha = 0$, corresponding to the Landau gauge, so that the breaking is now linear in the quantum fields and the antighost equation can be extended to the quantum theory. The use of the Landau gauge when quantizing a theory comes with some advantages: first, the nonrenormalization of the antighost equation controls the dependence of the counterterms on the ghost fields, as well as its nonrenormalization [87]; second, it provides an extra symmetry to the action, which is also valid for the vertex functional, given by the rigid gauge transformations

$$\mathcal{W}^a \Gamma^{(0)} = \int d^3x \sum_\phi f^{abc} \phi^b \frac{\delta \Gamma^{(0)}}{\delta \phi^c} = 0, \tag{4.47}$$

where ϕ stands for any field of the tree-level action (4.35). We assume the Landau gauge from now on.

We now discuss the extension of the Slavnov-Taylor identity for the vertex functional, which is defined by the cohomology of the linearized Slavnov-Taylor operator $\mathcal{S}_{\Gamma^{(0)}}$ in the sector of ghost number 1. As before, we already know that there is no anomaly in three space-time dimensions, that is, the cohomology sector of ghost number 1 is empty [92–96]. This is not altered by the introduction of the auxiliary fields, since they all enter as BRST-doublets and, therefore, can only appear in the trivial sector of the cohomology, that is, terms which are of the form $\mathcal{S}_{\Gamma^{(0)}}\Delta$. These are non-invariant counterterms, which appear through the regularization procedure employed in the evaluation of Feynman diagrams, and can be reabsorbed by the vertex functional at each order, since

$$\mathcal{S}_{\Gamma^{(0)}}\Gamma = \hbar^n \mathcal{S}_{\Gamma^{(0)}}\Delta + O(\hbar^{n+1}) \rightarrow \mathcal{S}_{\Gamma^{(0)}}(\Gamma - \hbar^n \Delta) = 0 + O(\hbar^{n+1}) \tag{4.48}$$

Moving over to the cohomology sector of ghost number 0, its solution represents the renormalization of physical parameters and redefinition of field amplitudes, which are given by nontrivial cocycles and coboundaries invariant under the Slavnov-Taylor operator. This corresponds to the stability of the tree-level action subject to a perturbation $\Gamma^{(0)} + \varepsilon\Gamma^{ct}$, where the invariant counterterm action Γ^{ct} is an integrated local polynomial in the fields and obeys the following constraints:

$$\frac{\delta\Gamma^{ct}}{\delta b^a} = \frac{\delta\Gamma^{ct}}{\delta K^a} = \frac{\delta\Gamma^{ct}}{\delta D^a} = \frac{\delta\Gamma^{ct}}{\delta D^a_+} = \int d^3x \frac{\delta\Gamma^{ct}}{\delta c^a} = 0, \tag{4.49}$$

which implies that Γ^{ct} does not depend on the fields b^a, K^a, D^a, D^a_+ and that the Faddeev-Popov ghost field can only enter through a derivative $\partial_\mu c^a$, due to the integrated antighost equation. Moreover, the ghost equations (4.42,4.43) intertwines the dependence on the fields $\bar{c}^a$ and K^a_- with the source $\rho^{a\mu}$, so that they can only appear as

$$\bar{\rho}^{a\mu} = \rho^{a\mu} - n^\mu \bar{c}^a + \partial^\mu K^a_-. \tag{4.50}$$

All these restrictions, together with dimension 2 bound on the counterterm and the requirement of invariance under $\mathcal{S}_{\Gamma^{(0)}}$, enforce that the only possible counterterm is proportional to the Chern-Simons action, which we can write as

$$\Gamma^{ct} = z_m m \frac{\partial}{\partial m}\Gamma^{(0)}, \tag{4.51}$$

where z_m is an arbitrary parameter.

Therefore, we recover the results discussed in the previous chapter, that the counterterm is invariant up to a total derivative, but not locally invariant. This allows us to conclude that no parameter of the theory in the interpolating gauge fixing receives any radiative correction [85], that is, all β-functions and anomalous dimensions are identically zero. We conclude that the Yang-Mills-Chern-Simons theory is ultraviolet finite to all orders in perturbation theory when quantized in the interpolating gauge [129].

Chapter 5

Yang-Mills-Chern-Simons and the Gribov Horizon

The algebraic renormalization of gauge theories relies on the invariance of the gauge-fixed action under BRST transformations. The extension of such symmetry to the quantum analogue of the classical action, the vertex functional, is the goal of renormalization, which can be done in a fully algebraic manner through the use of the Quantum Action Principle, as we have seen in the previous chapters. Therefore, the Faddeev-Popov gauge-fixing procedure [37] is crucial to the quantization of gauge theories, not only for the elimination of spurious degrees of freedom of the gauge fields, but as the foundation of the general framework we are pursuing here.

Gauge-fixing consists on choosing a representative for each inequivalent gauge field configuration, that is, those which are not related by a gauge transformation

$$A'^a_\mu = A^a_\mu - D^{ab}_\mu \omega^b. \tag{5.1}$$

The fields related by a gauge transformation compose a set called the gauge orbit. Therefore, to avoid overcounting of the gauge fields in the functional integration, we must integrate over the space of gauge orbits rather than the whole configuration space. We introduce this constraint in the Green functional by means of the Faddeev-Popov trick, which expresses the gauge condition in terms of the insertion of a unity.

Let us assume a gauge condition given by the equation $F[A]=0$, where F is some local, linear function of the gauge field. It is assumed to be ideal, i.e., to select only one representative per gauge orbit, such that $F[A'] \neq 0$, and therefore it has only one root. Recalling the composition property of the Dirac delta function $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ and generalizing it to the space of gauge functions, we get that

$$\int \mathcal{D}\omega \; \delta(F[A']) \left| \det\left(\frac{\delta F[A']}{\delta\omega}\right)\right| = 1 \tag{5.2}$$

where we integrate over all possible infinitesimal group transformations, with $\mathcal{D}\omega$ being the functional measure of the group. Since we assumed a linear gauge condition, the functional derivative $\frac{\delta F[A']}{\delta\omega}$ is independent of ω. The gauge field measure $\mathcal{D}A$ is invariant under gauge transformations, as well as the starting action. Therefore, we can insert (5.2) into the path integral, so that

$$\int \mathcal{D}A \; e^{iS} = \int \mathcal{D}\omega\mathcal{D}A \; \delta(F[A']) \left|\det\left(\frac{\delta F[A']}{\delta\omega}\right)\right| e^{iS}. \tag{5.3}$$

Using the invariance of the action and the measure $\mathcal{D}A$, we can change the integration variables $A \to A'$, so that the integrand is independent of ω and the integral over the gauge group can be factored out. Moreover, the determinant can be lifted to the exponential, giving rise to the Faddeev-Popov ghosts term

$$\int \mathcal{D}c\mathcal{D}\bar{c} \; e^{i\int dx \bar{c}^a \left(\frac{\delta F[A']}{\delta\omega}\right)^{ab} c^b}. \tag{5.4}$$

The delta function can also be lifted to the action through a gaussian integration, which will result in the introduction of the Lagrange multiplier Nakanishi-Lautrup fields [130, 131]. These are an important component of the BRST formulation, since they form a doublet structure with the antighost $\bar{c}$, allowing for the off-shell closure of the BRST algebra.

The construction of the ghost path-integral (5.4) relied on two assumptions: that the gauge condition $F[A]$ is ideal, i.e., it successfully selects only one representative of the gauge field per orbit; and that the Faddeev-Popov determinant, computed at the root of the gauge condition, is positive, which identifies it with its absolute value and guarantees that it does not develop zero-modes. However, as first shown by Gribov in the case of the Landau gauge [132], the gauge condition fails to properly fix the gauge in non-abelian theories, so that representatives of the same gauge orbit are always present. This issue, called the Gribov ambiguity, were later shown to be a general feature of non-Abelian Lie groups [133], not a peculiarity of the Landau gauge.

This is not a particular issue on the high-energy limit, where we expand perturbatively around the trivial vacuum configuration $A = 0$. Indeed, the agreement of perturbative calculations with experimental data, e.g. as seen in the LHC, leads us to the conclusion that Gribov copies are absent or at least that their role is negligible in this energy regime. However, as we go to the non-perturbative infrared region, they need to be properly addressed. The solution proposed by Gribov were to restrict the functional integration to a region free of copies, called the Gribov region.

For the rest of this chapter, we will work in Euclidean space-time signature $\eta_{\mu\nu} = \delta_{\mu\nu} = (1, 1, 1)$, so as to better compare the results with the literature involving the Gribov problem. Before proceeding, we must emphasize that the following exposition covers general aspects of the restriction of gauge theories to the Gribov horizon, but shall not go deep into details, as this is not the focus of this thesis. For a complete account of the results presented next, we refer to [134, 135] and references therein.

5.1 Solving the Gribov Problem

Let us see how Gribov copies appear in the Landau gauge, which is a gauge choice where is easy to explicitly verify that it fails the condition to be ideal [132]. Take the Landau gauge condition

$$\partial_\mu A^a_\mu = 0, \tag{5.5}$$

and an infinitesimal transformation of the gauge field

$$A'^a_\mu = A^a_\mu - D^{ab}_\mu \omega^b. \tag{5.6}$$

If the gauge-fixing were ideal, we would get $\partial_\mu A'^a_\mu \neq 0$, as only one representative per orbit would fulfill the condition (5.5). In terms of the gauge transformation, this implies

$$\partial_\mu (A^a_\mu - D^{ab}_\mu \omega^b) = -\partial_\mu D^{ab}_\mu \omega^b \neq 0. \tag{5.7}$$

The equation above expresses that the Faddeev-Popov operator $-\partial_\mu D^{ab}_\mu$ must attain non-zero values for any gauge fields other than the representative chosen by the gauge-fixing. But since we know that no gauge condition is ideal in non-abelian theories [133], we must have

$$\partial_\mu D^{ab}_\mu \omega^b = 0, \tag{5.8}$$

for certain values of ω^a. By studying the eigenvalue equation of the Faddeev-Popov operator, we can obtain information on its spectrum. Assuming an eigenvector Ψ of this operator, we have

$$-\partial_\mu D^{ab}_\mu \Psi^b = \varepsilon(A)\Psi^a \implies -(\delta^{ab}\partial^2 - g f^{abc} A^c_\mu \partial_\mu)\Psi^b = \varepsilon(A)\Psi^a, \tag{5.9}$$

where we used the transversality condition of the Landau gauge (5.5). This condition also implies that the Faddev-Popov operator is self-adjoint in the space of Lie-algebra valued fields in the Landau gauge, which can be seen through the inner product

$$\begin{aligned}
\int dx\ (-\partial_\mu D^{ab}_\mu \psi^b)^\dagger \phi^a &= \int dx\ (-\partial_\mu \partial_\mu \psi^a + g f^{abc} A^b_\mu \partial_\mu \psi^c)^\dagger \phi^a \\
&= \int dx\ (-\psi^{a\dagger} \partial_\mu \partial_\mu \phi^a - g f^{abc} \psi^{c\dagger} A^b_\mu \partial_\mu \phi^a) \\
&= \int dx\ (-\psi^{a\dagger} \partial_\mu \partial_\mu \phi^a - g f^{cba} \psi^{a\dagger} A^b_\mu \partial_\mu \phi^c) \qquad (5.10) \\
&= \int dx\ (-\psi^{a\dagger} \partial_\mu \partial_\mu \phi^a + g f^{abc} \psi^{a\dagger} A^b_\mu \partial_\mu \phi^c \\
&= \int dx\ \psi^{a\dagger} (-\partial_\mu D^{ab}_\mu \phi^b).
\end{aligned}$$

Therefore, its spectrum consists of real eigenvalues $\varepsilon(A)$. The presence of

Gribov copies is related to the existence of null eigenvalues of the Faddeev-Popov operator, that is, to the existence of zero modes of the operator. That this is always the case can be seen through the following remarks: by neglecting the term proportional to A^a_μ (ignoring the non-abelian nature of the group), we get an eigenvalue equation of the d'Alembertian operator

$$-\partial^2\Psi^a = \varepsilon\Psi^a, \tag{5.11}$$

which only has positive solutions $\varepsilon = p^2 > 0$, apart from the trivial one. This explains why no Gribov ambiguity appears in abelian theories and why the results from perturbation theory, where we expand around the trivial vacuum $A = 0$, are valid. In the perturbative regime, we can ensure that the two assumptions of the Faddeev-Popov procedure are met, i.e., that the gauge condition is ideal and that the Faddeev-Popov determinant is positive. As we move far from the perturbative regime, the contribution from A^a_μ can no longer be neglected, and may lead to zero and negative eigenvalues. Thus, we can see that the existence of Gribov copies is deeply intertwined with the presence of zero-modes of the Faddeev-Popov operator. For a detailed and pedagogical exposition of the subject, we refer to [136, 137].

To solve this problem, we must restrict the space of gauge configurations to a region Ω, called the Gribov region [132], such that

$$\Omega = \{A^a_\mu, \partial_\mu A^a_\mu = 0 | \mathcal{M}^{ab} > 0\}, \tag{5.12}$$

where $\mathcal{M}^{ab} = -\partial_\mu D^{ab}_\mu$ is the Faddeev-Popov operator. Thus, the assumptions in the gauge-fixing procedure are true for all $A^a_\mu \in \Omega$. The boundary $\delta\Omega$ of the Gribov region is the hypersurface where the first zero-mode of the Faddeev-Popov operator appears and is called the first Gribov horizon. Subsequent horizons are defined by the hypersurfaces where the second, third, etc, zero-modes appear. Since we will proceed always within the Gribov region, we only need to concern about the first horizon, hereafter simply called as the Gribov horizon.

The Gribov region presents a set of properties, beyond the ones presented so far [138–140]:

- It is intersected by every gauge orbit, ensuring that all different gauge configuration are accounted for in the functional integration;
- It contains the trivial configuration $A = 0$, which ensures that the perturbative vacuum is incorporated by the Gribov region;
- It is bounded in every direction. This implies that for every configuration $A^a_\mu \neq 0$ inside the Gribov region Ω, there is a configuration λA^a_μ, with $\lambda > 0$, which is outside Ω;
- It is convex. This means that any "line segment" connecting two distinct gauge configurations lies entirely within the Gribov region.

We should point out, however, that this construction does not rid the space of field configurations from every Gribov copy. Indeed, only infinitesimal copies are ruled out by the restriction to the Gribov region, while copies generated by finite gauge transformations are still present [141]. A conjectured complete removal of Gribov copies would be achieved in the Fundamental Modular Region Λ, which shares the properties of the Gribov region, being a proper subset of the latter ($\Lambda \subset \Omega$), as we shall see. It can be constructed as the set of absolute minima of the functional

$$||A||^2 = \frac{1}{2}\int dx\ A^a_\mu A^a_\mu, \tag{5.13}$$

which corresponds to the squared norm of the gauge fields A^a_μ. The variation of the norm gives us the stationary points

$$\begin{aligned}\delta||A||^2 &= \int dx\ \delta A^a_\mu A^a_\mu = -\int dx\ D^{ab}_\mu \omega^b A^a_\mu \\ &= -\int dx\ \partial_\mu \omega^a A^a_\mu = \int dx\ \omega^a \partial_\mu A^a_\mu = 0,\end{aligned} \tag{5.14}$$

for any ω^a belonging to a semi-simple Lie group, so that $\partial_\mu A^a_\mu = 0$, recovering the Landau gauge condition. The second variation ensures the positivity of the Faddeev-Popov operator by requiring the stationary points to be local minima of the functional

$$\delta^2||A||^2 = -\int dx\ \partial_\mu\omega^a\delta A^a_\mu = \int dx\omega^a(-\partial_\mu D^{ab}_\mu)\omega^b > 0. \tag{5.15}$$

This already defines the Gribov region, as discussed above. The further step to construct Λ is to select from each orbit the gauge configuration closest to the trivial vacuum $A = 0$, that is, we select the absolute minima of the functional $||A||^2$. Nevertheless, there is no known practical way of defining Λ and restricting the functional integration to it.

5.1.1 The no-pole condition

The restriction of the path integral to the Gribov region Ω was first proposed by Gribov [132] through the introduction of a factor $V(\Omega)$, so that

$$\int_\Omega \mathcal{D}A\ e^{-S_{inv}} = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\ \delta(\partial_\mu A^a_\mu)e^{-(S_{inv}+\int dx\bar{c}^a\partial_\mu D^{ab}_\mu c^b)}V(\Omega), \tag{5.16}$$

where S_{inv} stands for a gauge-invariant initial action. As can be seen through the delta-function, we are still working in the Landau gauge. The factor $V(\Omega)$ must act like a characteristic function of the Gribov region. To properly characterize it, we will analyze the exact ghost two-point function, given by

$$\langle T\bar{c}^a(x)c^b(y)\rangle = \int \mathcal{D}A\ \delta(\partial_\mu A^a_\mu)\det(\mathcal{M})[\mathcal{M}^{-1}]^{ab}(x,y)e^{-S_{inv}}V(\Omega), \tag{5.17}$$

which is directly related to the inverse of the Faddeev-Popov operator $\mathcal{M}^{-1}$, as can be seen from the expression above. Since we are inside the Gribov region, this operator is positive-definite, and the inverse is well-defined. For gauge configurations in the Gribov horizon, it can develop zero-modes, which translates to a pole in the ghost propagator. Since at tree-level the propagator diverges only at $k^2 \to 0$, the restriction to the Gribov region can be implemented by demanding that the full propagator does not develop new poles other then the trivial one, therefore known as the no-pole condition.

However, if we take the 1-loop approximation to the ghost propagator [33,34]

$$\langle T\bar{c}^a c^b\rangle(k) = \frac{\delta^{ab}}{k^2}\frac{1}{\left(1-\frac{11g^2N}{48\pi^2}\ln\frac{\Lambda^2}{k^2}\right)^{\frac{9}{44}}} \tag{5.18}$$

where Λ is an ultraviolet cut-off and N is the adjoint Casimir element of $SU(N)$, we see that it shows two distinct singularities, one at $k^2 = 0$ and other at $k^2 = \Lambda^2\exp\left(-\frac{48\pi^2}{11g^2N}\right)$. But the second pole indicates that the propagator is no longer positive for $k^2 < \Lambda^2\exp\left(-\frac{48\pi^2}{11g^2N}\right)$, which violates the positivity of the Faddeev-Popov operator, signaling that we are out of the Gribov region. Therefore, the only possible singularity is the one at $k^2 = 0$ and the factor $V(\Omega)$ must enforce this through the restriction to the Gribov region.

We can then use the ghost propagator to describe the no-pole condition, and with it define the explicit form of $V(\Omega)$ as a step function. Take the color-singlet, connected ghost two-point function

$$\begin{aligned}\frac{\langle T\bar{c}^a(x)c^a(y)\rangle}{N^2-1} &= \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\;\delta(\partial_\mu A^a_\mu)\frac{\bar{c}^a c^a}{N^2-1}e^{-(S_{inv}+\int dx\bar{c}^a\partial_\mu D^{ab}_\mu c^b)}\\ &= \int \mathcal{D}A\;\delta(\partial_\mu A^a_\mu)\mathcal{G}(x,y;A)e^{-(S_{inv})},\end{aligned} \tag{5.19}$$

where $\mathcal{G}(x,y;A)$ represents the ghost two-point function with the gluon A^a_μ treated as an external classical field. By taking its Fourier transform and expanding it to second order in perturbation theory (see [136,137] for details), we get the following expression

$$\begin{aligned}\mathcal{G}(k^2;A) &= \frac{1}{k^2} + \frac{1}{k^4}\frac{1}{V}\frac{Ng^2}{N^2-1}\int\frac{dq}{(2\pi)^d}A^a_\mu(-q)A^a_\mu(q)\\ &= \frac{1}{k^2}(1+\sigma(k^2,A)),\end{aligned} \tag{5.20}$$

where V is the space-time volume and

$$\sigma(k,A) = \frac{1}{k^2}\frac{1}{V}\frac{Ng^2}{N^2-1}\int\frac{dq}{(2\pi)^d}A^a_\mu(-q)A^a_\mu(q). \tag{5.21}$$

Since this is the second order approximation for an infinite number of diagrams, it can be rewritten as

$$\mathcal{G}(k^2;A) \approx \frac{1}{k^2}\frac{1}{1-\sigma(k,A)}, \tag{5.22}$$

and the no-pole condition then resumes to imposing that

$$\sigma(k,A) < 1. \tag{5.23}$$

However, as shown in [137], the function $\sigma(k,A)$ is decreasing for increasing k^2, so it suffices to demand that

$$\sigma(0,A) < 0. \tag{5.24}$$

The limit $k^2 \to 0$ yields

$$\begin{aligned}\sigma(0,A) &= \frac{1}{V}\frac{1}{d-1}\frac{Ng^2}{N^2-1}\lim_{k^2\to 0}\frac{k_\mu k_\nu}{k^2}\frac{d-1}{d}\int\frac{dq}{(2\pi)^d}A^a_\mu(-q)A^a_\mu(q)\frac{1}{q^2}\\ &= \frac{1}{V}\frac{1}{d}\frac{Ng^2}{N^2-1}\int\frac{dq}{(2\pi)^d}A^a_\mu(-q)A^a_\mu(q)\frac{1}{q^2},\end{aligned} \tag{5.25}$$

which renders a closed expression depending on the gauge field A^a_μ. The restriction to the Gribov region can then be implemented by making

$$V(\Omega) = \theta(1-\sigma(0,A)), \tag{5.26}$$

which can be inserted in the path integral through the integral representation of the step function

$$\theta(1-\sigma(0,A)) = \int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon}\frac{d\beta}{2\pi i\beta}e^{\beta(1-\sigma(0,A))}, \tag{5.27}$$

that leads to the final result

$$Z = N\int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon}\frac{d\beta}{2\pi i\beta}\int\mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\; e^{\beta(1-\sigma(0,A))}\delta(\partial_\mu A^a_\mu)e^{-(S_{inv}+\int dx\bar{c}^a\partial_\mu D^{ab}_\mu c^b)} \tag{5.28}$$

To obtain the gluon propagator, we can ignore the ghost fields and the interaction terms, retaining only the quadratic part in the gauge fields of the gauge-fixed action. We exchange the delta function enforcing the Landau gauge condition by the gauge-fixing term $\frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2$, where we must take the limit $\alpha \to 0$ at the end of the calculation. Writing in terms of the Fourier transforms of the gauge fields, we get

$$Z_{quad} = N \int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon} \frac{d\beta}{2\pi i\beta} \int \mathcal{D}A \; e^{\beta(1-\sigma(0,A))} e^{-\frac{1}{2}\int \frac{dk}{(2\pi)^d} A_\mu^a(k) K_{\mu\nu}^{ab}(k) A_\nu^b(-k)}, \quad (5.29)$$

where the wave operator $K_{\mu\nu}^{ab}$ depend on the invariant action. For simplicity, let us assume for now a Yang-Mills action, so that working out the free part of the theory and recalling the definition of the function $\sigma(0, A)$ (5.25), we get the full wave operator

$$Q_{\mu\nu}^{ab}(k) = \delta^{ab} \left(\frac{\beta}{V} \frac{2}{d} \frac{Ng^2}{N^2-1} \frac{\delta_{\mu\nu}}{k^2} + k^2 \delta_{\mu\nu} + \left(\frac{1}{\alpha} - 1 \right) k_\mu k_\nu \right). \quad (5.30)$$

Integrating the quadratic path integral (5.29) over the gauge field, we get

$$Z_{quad} = N \int \frac{d\beta}{2\pi i\beta} e^\beta \left(\det Q_{\mu\nu}^{ab} \right)^{-\frac{1}{2}} = \int \frac{d\beta}{2\pi i} e^{f(\beta)}, \quad (5.31)$$

where we used the identity $\ln\det A = \mathrm{Tr}\ln A$ and defined the function

$$f(\beta) = \beta - \ln\beta - \frac{1}{2} \mathrm{Tr}\ln Q_{\mu\nu}^{ab}. \quad (5.32)$$

The evaluation of the last term of $f(\beta)$ is done explicitly in the Appendix of [137], and we refer to that for a complete derivation. Here we simply state the result, which is

$$f(\beta) = \beta - \ln\beta - \frac{d-1}{2} V(N^2-1) \int \frac{dq}{(2\pi)^d} \ln \left(q^2 + \frac{\beta Ng^2}{N^2-1} \frac{2}{dV} \frac{1}{q^2} \right). \quad (5.33)$$

Applying the method of steepest descent to evaluate the integral over β, we evaluate it at the saddle point

$$Z_{quad} \approx e^{f(\beta_0)}, \tag{5.34}$$

where the value of β_0 is determined by the minimum condition

$$f'(\beta_0) = 0, \tag{5.35}$$

which implies

$$1 - \frac{1}{\beta_0} - \frac{d-1}{d} N g^2 \int \frac{dq}{(2\pi)^d} \frac{1}{\left(q^4 + \frac{\beta_0 N g^2}{N^2-1} \frac{2}{dV}\right)} = 0. \tag{5.36}$$

To make sense of the expression above, we must have $\beta_0 \sim V$, since $V \to \infty$, so that we can fix the finite parameter

$$\gamma^4 = \frac{\beta_0 N g^2}{N^2 - 1} \frac{2}{dV}, \tag{5.37}$$

called the Gribov parameter or Gribov mass. This parameter is self-consistently fixed by the minimum equation (5.36)

$$1 = \frac{d-1}{d} N g^2 \int \frac{dq}{(2\pi)^d} \frac{1}{q^4 + \gamma^4}, \tag{5.38}$$

called the gap equation, where we neglected the term proportional to $1/\beta_0$. We can now obtain the propagator at the saddle point $\beta = \beta_0$, which is just the inverse of the wave operator $Q^{ab}_{\mu\nu}(k)$, and take the limit $\alpha \to 0$ to recover the Landau gauge, resulting in

$$\langle T A^a_\mu A^b_\nu \rangle(k) = \delta^{ab} \frac{k^2}{k^4 + \gamma^4} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \tag{5.39}$$

This propagator exhibits a series of properties that differs from the usual gluon propagator, which can be seen as the effect of the restriction to the Gribov region:

- It is suppressed in the infrared regime, due to the Gribov parameter γ;

- It vanishes in the infrared limit $k^2 \to 0$, contrary to the YM propagator which diverges;
- It recover the YM propagator in the ultraviolet, where $k^2 \gg \gamma^2$;
- It possesses two complex poles $k^2 = \pm i\gamma^2$. Therefore, the gluon cannot be part of the physical spectrum of the theory, which allows us to interpret it as a signal of confinement.

The restriction to the Gribov region also has consequences to the 1-loop ghost propagator, which is enhanced in the infrared, being proportional to $1/k^4$. This more singular behavior may lead to IR divergences.

5.1.2 Zwanziger's Horizon Function and the Gribov-Zwanziger Action

Although it already presents new features in comparison with the standard perturbative Yang-Mills theory, and even agrees with it in the high-energy limit, Gribov's proposal to restrict the integral over the gauge fields is worked out only at leading order. The generalization to all orders was developed by Zwanziger in [142]. Instead of a self-consistent restriction based on the ghost propagator (Gribov's no-pole condition), he studied directly the spectrum of the Faddeev-Popov operator $\mathcal{M}$,

$$\mathcal{M}^{ab}\Psi^b = \varepsilon(A)\Psi^a, \tag{5.40}$$

in order to find its lowest positive eigenvalue $\lambda(A)$. By introducing the step function $\theta(\lambda(A))$, we ensure that the operator $\mathcal{M}$ is always positive and, thus, the integration is bounded to the Gribov region Ω. We do not delve into the computation here (see [137] for a complete account of it). The step function is written as

$$\theta(dV(N^2-1) - H(A)), \tag{5.41}$$

where $H(A)$, called the horizon function, is given by the expression

$$H(A) = g^2 \int dxdy\ f^{abc}A^b_\mu(x)[\mathcal{M}^{-1}]^{ad}(x,y)f^{dec}A^e_\mu(y). \tag{5.42}$$

At last, based on the result that the Gribov horizon $\delta\Omega$ is an ellipsoid in the A-space [139, 143], the theta function can be lifted into the Boltzmann factor of the path integral, so that we have

$$Z = N \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \; e^{-S_{GZ}^{nl}}, \tag{5.43}$$

where the Gribov-Zwanziger action S_{GZ}^{nl} is defined as

$$S_{GZ}^{nl} = S_{inv} + S_{GF} + \gamma^4 H(A) - \gamma^4 V d(N^2 - 1), \tag{5.44}$$

with the Gribov parameter γ now fixed through the horizon condition

$$\langle H(A) \rangle = V d(N^2 - 1), \tag{5.45}$$

taken with respect to the modified path integral (5.43). The Gribov-Zwanziger action (5.44) effectively restricts the domain of integration to the Gribov region, but does so at the cost of locality, since the horizon term (5.42) contains the inverse of the Faddeev-Popov operator in it. However, it can be made local through the introduction of appropriate auxiliary fields, much like we have shown in the previous chapter. As a closing remark, we point out that even though the two approaches followed different methods, extending Gribov's procedure to all orders in the perturbative expansion leads to Zwanziger's horizon function [144].

5.1.3 Localizing the GZ Action

In order to get a local action which restricts the integration to the Gribov region, we need to introduce some auxiliary fields. First, we deal we the inverse Faddeev-Popov operator appearing in the horizon term (5.42). Using the identity

$$\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\ \exp\left[-\int dxdy\ \bar{\phi}(x)A(x,y)\phi(y) + \int dx\ (\bar{\phi}(x)J_{\bar{\phi}}(x) + \phi(x)J_{\phi}(x))\right]$$
$$= C(\det A)^{-1}\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\left[-\int dxdy\ J_{\bar{\phi}}(x)A^{-1}(x,y)J_{\phi}(y)\right], \tag{5.46}$$

with C a constant, which can be absorbed by the normalization factor, A an invertible operator and J_ϕ, $J_{\bar{\phi}}$ are source terms, we see that the horizon term can be written as

$$e^{-\gamma^4 H(A)} = \exp\left[\int dxdy\ g\gamma^2 f^{abc}A^b_\mu(x)[-\mathcal{M}^{-1}]^{ad}(x,y)g\gamma^2 f^{dec}A^e_\mu(y)\right]$$
$$= [\det(-\mathcal{M})]^{d(N^2-1)}\int \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\exp\left[\int dxdy\ \bar{\varphi}^{ac}_\mu(x)\mathcal{M}^{ab}(x,y)\varphi^{bc}_\mu(y)\right.$$
$$\left. + \int dx\ \left(\bar{J}^{ab}_\mu\varphi^{ab}_\mu + \bar{\varphi}^{ab}_\mu J^{ab}_\mu\right)\right], \tag{5.47}$$

where we defined the sources $\bar{J}^{ac}_\mu = J^{ac}_\mu = g\gamma^2 f^{abc}A^b_\mu$ and introduced the complex conjugate bosonic fields $(\bar{\varphi}^{ab}_\mu, \varphi^{ab}_\mu)$. To lift the remaining determinant to the exponential, we follow the same trick used for the ghost action, introducing anti-commuting Grassmann variables $(\bar{\omega}^{ab}_\mu, \omega^{ab}_\mu)$, so that

$$[\det(-\mathcal{M})]^{d(N^2-1)} = \int \mathcal{D}\bar{\omega}\mathcal{D}\omega\exp\left[-\int dxdy\ \bar{\omega}^{ac}_\mu(x)\mathcal{M}^{ab}(x,y)\omega^{bc}_\mu(y)\right]. \tag{5.48}$$

Making these substitutions, the horizon term becomes

$$e^{-\gamma^4 H(A)} = \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}\bar{\omega}\mathcal{D}\omega\exp\left[\int dx\ \left(\bar{\varphi}^{ac}_\mu\mathcal{M}^{ab}\varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu\mathcal{M}^{ab}\omega^{bc}_\mu\right)\right.$$
$$\left. + \int dx\ g\gamma^2 f^{abc}A^b_\mu(x)(\bar{\varphi}+\varphi)^{ac}_\mu\right], \tag{5.49}$$

and we can finally define the local Gribov-Zwanziger action

$$
\begin{aligned}
S_{GZ} = S_{inv} + S_{GF} - \int dx \, \left(\bar{\varphi}^{ac}_\mu \mathcal{M}^{ab} \varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{M}^{ab} \omega^{bc}_\mu\right) \\
+ \int dx \; g\gamma^2 f^{abc} A^a_\mu(x)(\bar{\varphi} + \varphi)^{bc}_\mu - \gamma^4 V d(N^2-1),
\end{aligned}
\tag{5.50}
$$

which defines the partition function

$$
Z = N \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}\bar{\omega}\mathcal{D}\omega \; e^{S_{GZ}}. \tag{5.51}
$$

With a local expression to the Gribov-Zwanziger action, we can apply the usual techniques of perturbative quantum field theory. In fact, the S_{GZ} is not only local, but renormalizable to all orders [137, 142]. In this local form, the horizon condition reads

$$
\frac{\partial \Gamma}{\partial \gamma^2} = 0 \implies -\langle g f^{abc} A^a_\mu(x)(\bar{\varphi} + \varphi)^{bc}_\mu \rangle + 2\gamma^2 d(N^2-1) = 0, \tag{5.52}
$$

remembering that the vertex functional Γ is defined as

$$
e^{-\Gamma} = \int \mathcal{D}\Phi e^{-S_{GZ}}, \tag{5.53}
$$

where the measure $\mathcal{D}\Phi$ represents all fields of the Gribov-Zwanziger action, and we used the translation invariance of the horizon function to eliminate the volume term V. Moreover, the gluon and ghost propagators obtained from S_{GZ} have the same behavior as the ones obtained by imposing Gribov's no-pole condition [137].

5.2 The Refined Gribov-Zwanziger Action

For a time, results from lattice gauge theory simulations on the Landau gauge [145, 146] seemed to converge with the Gribov-Zwanziger action on the two-point functions behavior in the IR regime. They also showed a suppressed, vanishing gluon propagator and an enhanced ghost propagator. These results were also supported by other non-perturbative techniques, such as the Dyson-Schwinger equations [147–150] and the functional renormalization group [151] approaches.

However, more precise results coming from lattice data of larger volumes [152–157] contested this agreement, displaying a non-vanishing gluon propagator at null momentum and a non-enhanced ghost propagator for 3 and 4 dimensions, but maintaining the previous behavior for the two-dimensional case. These were later corroborated by further computations done via Dyson-Schwinger equations [158–160]. The question arose on how to reconcile these results to the Gribov-Zwanziger description of IR Yang-Mills theories.

The solution was to take into account further non-perturbative effects into the Gribov-Zwanziger action. A primary example of which are condensates (vacuum expectation values of local operators), which were already known to be relevant in the study of Quantum Chromodynamics [161, 162]. One such condensate, already extensively studied, is the $\langle A_\mu^2 \rangle$ (see [163–172]), as it was associated with a dynamical mass generation for the gluons [173–175]. The inclusion of a gluon condensate, however, did not accommodate a non-vanishing gluon propagator, and could not resolve the non-enhancement of the ghost propagator at one-loop order [172]. These are accounted for by the introduction of another condensate, involving the auxiliary fields $(\bar{\varphi}_\mu^{ab}, \varphi_\mu^{ab}, \bar{\omega}_\mu^{ab}, \omega_\mu^{ab})$, namely $\langle \bar{\varphi}_\mu^{ab} \varphi_\mu^{ab} - \bar{\omega}_\mu^{ab} \omega_\mu^{ab} \rangle$. One can justify the appearance of this condensate by two main reasons. First, since they are introduced to localize the horizon function, they translate the non-local dynamics of the GZ action to a local equivalent formulation, and as so should acquire their own quantum dynamics, inducing further non-perturbative effects beyond the restriction to the Gribov region. In fact, looking at the tree-level $A\varphi$-coupling, one can infer an influence of the auxiliary fields dynamics in the gluon propagator. Second, we already have a condensate involving the auxiliary fields in the Gribov parameter gap equation $\langle g f^{abc} A_\mu^a(x) (\bar{\varphi} + \varphi)_\mu^{bc} \rangle = 2\gamma^2 d(N^2 - 1)$ (5.52), so a possible $\bar{\varphi}\varphi$-condensation seems to be reasonable [176, 177].

The presence of such non-vanishing condensates can be attested already at one-loop order and are a direct consequence of the restriction to the Gribov region, which carries this non-perturbative information to a perturbative analysis [178, 179]. These condensates represent infrared instabilities of the Gribov-Zwanziger action, which should be taken into account from the starting action[1]. The inclusion of such terms into S_{GZ} (5.50)leads to the so-called Refined Gribov-Zwanziger action

$$S_{RGZ} = S_{GZ} + \frac{m^2}{2}\int dx A^a_\mu A^a_\mu - M^2 \int dx(\bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu), \tag{5.54}$$

where m and M are mass parameters associated with the vacuum expectation values of the condensates, which are dynamically fixed by their own gap equations

$$\frac{\partial\Gamma}{\partial m^2} = 0, \quad \frac{\partial\Gamma}{\partial M^2} = 0, \tag{5.55}$$

much as the Gribov mass parameter γ^2. This modifications lead to a tree-level gluon propagator in the Landau gauge given by

$$\langle A^a_\mu A^b_\nu\rangle(k) = \delta^{ab}\frac{k^2 + M^2}{(k^2 + m^2)(k^2 + M^2) + \gamma^4}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \tag{5.56}$$

It is easy to see that this propagator still presents a suppressed behavior in the IR, but now it attains a non-vanishing value at the $k \to 0$ limit, agreeing with the lattice results and other subsequent non-perturbative approaches. We must emphasize that, although the auxiliary field condensate is enough to obtain a qualitative agreement with the lattice data, the correction coming from the gluon condensate is crucial to a quantitative agreement [181]. As a final comment, we mention that the inclusion of such condensates does not spoil the renormalization of the Gribov-Zwanziger action, meaning that the Refined Gribov-Zwanziger theory in the Landau gauge is also proven to be renormalizable to all orders in perturbation theory [177].

[1]In $d = 2$ dimensions, it is not possible to introduce the condensates in the starting action, due to their singular behavior [180]

5.3 BRST Symmetry in the Gribov Region

An important feature of the Gribov-Zwanziger action (5.50)(and also of its refined version (5.54)) that must be taken into consideration is that it breaks the BRST symmetry, which can obfuscate its renormalizability, as well as the discussion on unitarity and the definition of the physical Hilbert space of the theory. However, this breaking is realized in a soft manner, with a breaking term of dimension lower than that of the action, due to its dependence on the Gribov parameter γ^2. Let us see how this appears.

Beyond the usual fields from the Faddeev-Popov action $(A, \bar{c}, c, b)$, the Gribov-Zwanziger action possesses two more sets of auxiliary fields $(\bar{\varphi}, \varphi, \bar{\omega}, \omega)$, one of which is bosonic, with ghost number 0, while the other is fermionic, with ghost numbers ± 1. Recalling that the BRST operator increases ghost number, we can define the following transformations [182]

$$\begin{aligned} s\varphi^{ab}_\mu &= \omega^{ab}_\mu, \quad s\omega^{ab}_\mu = 0, \\ s\bar{\omega}^{ab}_\mu &= \bar{\varphi}^{ab}_\mu, \quad s\bar{\varphi}^{ab}_\mu = 0, \end{aligned} \tag{5.57}$$

so that we still have $s^2 = 0$. Setting $\gamma = 0$, we can write the gauge-fixing and horizon terms in a manifestly BRST-invariant way

$$S^{\gamma=0}_{GZ} = S_{inv} + s \int dx (\bar{c}^a \partial_\mu A^a_\mu - \bar{\omega}^{ac}_\mu \mathcal{M}^{ab} \varphi^{bc}_\mu), \tag{5.58}$$

since the auxiliary fields enter as BRST-doublets and, therefore, can only enter in the trivial part of the BRST-cohomology. However, the explicit form of the action (5.58) differs from that of (5.50), with $\gamma = 0$, by the term

$$\int dx \; g f^{ade} (\partial_\mu \bar{\omega}^{ac}_\mu) \varphi^{ec} D^{df}_\mu c^f. \tag{5.59}$$

But this term can be eliminated by shifting the ω field as

$$\omega^{ab}_\mu \to \omega^{ab}_\mu + g f^{def} \int dy \left[\mathcal{M}^{-1}\right]^{ad} (x, y) \partial_\nu (\varphi^{fb} D^{eg}_\mu c^g), \tag{5.60}$$

which causes no further modification due to its trivial Jacobian. Therefore, as long as $\gamma = 0$, we can readily recover the Faddeev-Popov action physical content, since the auxiliary fields, being BRST-doublets, cannot enter in correlation functions of gauge invariant operators. This can also be seems by the fact that integrating over the auxiliary fields corresponds to an insertion of the unity in the partition function.

The breaking of the BRST symmetry comes then from the γ-dependent term in the action (5.50), denoting a characteristic of the restriction to the Gribov region. This breaking is given by

$$sS_{GZ} = g\gamma^2 f^{abc} \int dx \left(A^a_\mu \omega^{bc}_\mu - D^{ad}_\mu c^d (\bar{\varphi} + \varphi)^{bc}_\mu \right). \tag{5.61}$$

In the limit $\gamma \to 0$, away from the Gribov horizon and into the perturbative UV region, we recover BRST-invariance, with the vanishing of the breaking term. This characterizes it as a soft breaking term, which is explicitly so by its coefficient of positive mass dimension, and do not jeopardize the high-energy properties of the theory [183].

The origin of such breaking can be interpreted as the following: the restriction to the Gribov region eliminates gauge configurations connected by infinitesimal transformations. Thus, any gauge transformation will lead from a configuration within the horizon Ω to another one outside Ω. But since the BRST transformation of the gauge fields is formally equal to infinitesimal gauge transformations, with the gauge parameter exchanged by the ghost field c, it implies that the theory no longer enjoys this symmetry, resulting in the breaking term (5.61). It is, however, possible to modify our construction of the Gribov-Zwanziger action to enforce its invariance under BRST symmetry. It also comes from a more general construction, where we no longer depend on the chosen gauge condition.

We begin by introducing the field $A^{h,a}_\mu$, which can be constructed by minimizing the functional $||A||^2$ (5.13) for a given gauge orbit [184–186]. It can be written as a non-local formal series

$$A_\mu^{h,a} = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\left(A_\nu^a - ig\left[\frac{1}{\partial^2}\partial A^a, A_\nu^a\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A^a, \partial_\nu\frac{1}{\partial^2}\partial A^a\right] + O(A^3)\right), \tag{5.62}$$

which is manifestly transverse, due to the presence of the transverse projector, and BRST-invariant order by order, so that

$$\partial_\mu A_\mu^{h,a} = 0, \quad sA_\mu^{h,a} = 0. \tag{5.63}$$

We can then redefine the horizon function in terms of this new field

$$H(A) = H(A^h) - \int dxdy\ R^a(x,y)(\partial A^a)(y), \tag{5.64}$$

using the fact that all higher order terms of $A_\mu^{h,a}$ contain a divergence ∂A^a, where $R^a(x,y)$ is an infinite formal series in A_μ^a. By absorbing this second term into the Nakanishi-Lautrup b field

$$b^a \to b^a - \gamma^4 \int dy\ R^a(x,y), \tag{5.65}$$

which defines a shift with trivial Jacobian. The localization procedure for the horizon function proceeds exactly in the same manner, with the gauge field A_μ^a exchanged by the gauge-invariant non-local field $A_\mu^{h,a}$, resulting in a modified Gribov-Zwanziger action

$$S_{GZ} = S_{inv} + S_{GF} - \int dx \left(\bar\varphi\mathcal{M}(A^h)\varphi - \bar\omega\mathcal{M}(A^h)\omega + \gamma^2 A^h(\bar\varphi + \varphi)\right), \tag{5.66}$$

where we omitted color and Lorentz indices for simplicity.

We can see that the only terms of the action proportional to the field A^h are those coming from the restriction to the Gribov horizon. The determination of the region Ω can then be made in terms of the field A^h, ensuring that the determinant of the Faddeev-Popov operator is positive, which allows for the construction previously presented. Moreover, it allows us to define the theory for linear covariant gauges [187, 188], since now we do not depend on a construction based on the Landau gauge. However, we remain with a non-local action, the source of non-locality now coming from the invariant field A^h. Fortunately, it can be localized with the introduction of an auxiliary Stueckelberg-like field ξ^a, with the definition

$$h = e^{ig\xi^a T^a}, \tag{5.67}$$

so that the invariant gauge field A^h is written as

$$A^h_\mu = h^\dagger A_\mu h + \frac{i}{g} h^\dagger \partial_\mu h, \tag{5.68}$$

which is a local expression, although non-polynomial. By imposing the transversality constraint $\partial_\mu A^{h,a}_\mu = 0$ to this local version and solving iteratively for ξ^a, one recovers the non-local expression (5.62). The above construction also remains invariant under gauge transformations

$$A_\mu \to U^\dagger A_\mu U + \frac{i}{g} U^\dagger \partial_\mu U, \quad h \to U^\dagger h, \quad h^\dagger \to h^\dagger U, \tag{5.69}$$

so that

$$A^h_\mu \to A^h_\mu. \tag{5.70}$$

Therefore, we can write a BRST-invariant and local Gribov-Zwanziger action

$$\begin{aligned} S_{GZ} = S_{inv} + S_{GF} - \int dx \left(\bar{\varphi}^{ac}_\mu \mathcal{M}^{ab}(A^h) \varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{M}^{ab}(A^h) \omega^{bc}_\mu\right) \\ + \int dx\ g\gamma^2 f^{abc} A^{h,a}_\mu (\bar{\varphi} + \varphi)^{bc}_\mu + \int dx\ \tau^a \left(\partial_\mu A^{h,a}_\mu\right) - \int dx\ \bar{\eta}^a \mathcal{M}^{ab}(A^h) \eta^b, \end{aligned} \tag{5.71}$$

where τ^a is a Lagrange multiplier field introduced to enforce the transversality of $A^{h,a}_\mu$ and $(\bar{\eta}^a, \eta^a)$ are ghost fields coming from the equivalence of the local and non-local formulations upon integration of the Stueckelberg field [189]. The complete set of nilpotent BRST transformation which leave the action invariant is

$$
\begin{aligned}
sA^a_\mu &= -D^{ab}_\mu c^b, & sc^a &= \frac{g}{2} f^{abc} c^b c^c, \\
s\bar{c}^a &= b^a, & sb^a &= 0, \\
s\varphi^{ab}_\mu &= 0, & s\bar{\varphi}^{ab}_\mu &= 0, \\
s\omega^{ab}_\mu &= 0, & s\bar{\omega}^{ab}_\mu &= 0, \\
sA^{h,a}_\mu &= 0, & s\tau^a &= 0, \\
s\eta^a &= 0, & s\bar{\eta}^a &= 0, \\
sh^{ij} &= -igc^a (T^a)^{ik} h^{kj}, & s\xi^a &= g^{ab}(\xi) c^b,
\end{aligned}
\tag{5.72}
$$

with $g^{ab}(\xi)$ a power series in ξ, given by

$$
g^{ab}(\xi) = -\delta^{ab} + \frac{g}{2} f^{abc} \xi^c - \frac{g^2}{12} f^{amr} f^{mbq} \xi^r \xi^q + O(\xi^3). \tag{5.73}
$$

As before, we need to introduce the condensates to account for the IR instabilities of the action. By introducing the gluon condensate in terms of the invariant field A^h, we can see from the BRST transformations (5.72) that the RGZ action coming from (5.71) remains invariant. We can then use all of the theorems from the Quantum Action Principle to study the renormalizability of the RGZ action. In the following section, we will study the restriction of YMCS theories to the Gribov region using the algebraic renormalization framework [43].

5.4 Algebraic Renormalization of the Yang-Mills-Chern-Simons Theory in the Gribov Region

Before addressing the renormalizability analysis of the Yang-Mills-Chern-Simons theory restricted to the Gribov region, we must make a remark about the validity of the algebraic method [43] in such a case. This is necessary because the algebraic renormalization framework is established upon the Quantum Action Principle, which was proved in perturbation theory for general gauge theories. However, in the infrared energy scale, Yang-Mills theories become non-perturbative, due to the increase of the coupling constant coming from its negative β-function [65, 67]. Indeed, naive perturbation theory indicates a Landau pole for QCD on the infrared energy scale (10^2 MeV). Any results from a perturbative expansion on the coupling constant would fall short at this point.

However, some comments are necessary to avoid arising doubt and confusion. We talk about perturbative expansion in a formal sense in the algebraic renormalization program, which means that we look at the expansions order by order and are not concerned about the convergence of the sum [43]. We have also included in the starting action non-perturbative features of the theory, such as condensates and the restriction to the Gribov region Ω, so that perturbation theory will carry information from the non-perturbative regime. Moreover, there has been recent discussions about the absence of a QCD Landau pole, with a running coupling saturating scenario at low energies [190, 191], but for the pure Yang-Mills case, results from the lattice show that the coupling constant displays low values for the whole range of momenta [156], suggesting that perturbative approaches should be valid at all scales [192]. With these remarks pointed out, let us proceed with the renormalizability of YMCS theory in the Gribov region [193].

Since we are now working in Euclidean space-time, it is necessary to reintroduce the YMCS action, since it is slightly different from its Minkowski counterpart (3.2). For a general $SU(N)$ gauge group, it reads

$$S_{YMCS} = \frac{1}{4}\int d^3x\; F^a_{\mu\nu}F^a_{\mu\nu} - im\int d^3x\; \epsilon_{\mu\rho\nu}\left(\frac{1}{2}A^a_\mu\partial_\rho A^a_\nu + \frac{g}{3!}f^{abc}A^a_\mu A^b_\rho A^c_\nu\right). \tag{5.74}$$

The gauge fixing action for linear covariant gauges is given by a BRST-exact action

$$S_{GF} = s\int d^3x\; \bar{c}^a\left(\partial_\mu A^a_\mu - \frac{\alpha}{2}b^a\right) = \int d^3x\; \left(b^a\partial_\mu A^a_\mu - \frac{\alpha}{2}b^a b^a + \bar{c}^a\partial_\mu D^{ab}_\mu c^a\right), \tag{5.75}$$

so that the gauge-fixed action is given by

$$S = S_{YMCS} + S_{GF}. \tag{5.76}$$

With the restriction of such theory to the Gribov region as in (5.71) and the inclusion of condensates, we get the RGZ-YMCS action

$$\begin{aligned} S^{RGZ}_{YMCS} = S &- \int d^3x \ \left(\bar{\varphi}^{ac}_\mu \mathcal{M}^{ab}(A^h)\varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{M}^{ab}(A^h)\omega^{bc}_\mu\right) \\ &+ \int d^3x \ g\gamma^2 f^{abc} A^{h,a}_\mu (\bar{\varphi} + \varphi)^{bc}_\mu + \frac{M^2}{2}\int d^3x \ A^{h,a}_\mu A^{h,a}_\mu \\ &- \mu^2 \int d^3x \ \left(\bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu\right) + \int d^3x \left(\tau^a \partial_\mu A^{h,a}_\mu - \bar{\eta}^a \mathcal{M}^{ab}(A^h)\eta^b\right), \end{aligned} \tag{5.77}$$

which is invariant under the BRST transformations (5.72.) This action will be the starting point for our analysis.

5.4.1 Introducing External Sources

Following [189], we begin by introducing a set of external sources which take physical values at the end of the renormalization procedure, as well as the usual sources coupled to the non-linear BRST transformations. Starting from the latter, we introduce

$$S_{ext} = \int d^3x \ \left(-\Omega^a_\mu D^{ab}_\mu c^b + \frac{g}{2} L^a f^{abc} c^b c^c + K^a g^{ab}(\xi) c^b + \mathcal{J}^a_\mu A^{h,a}_\mu\right), \tag{5.78}$$

where we also introduced a source coupled to the invariant field $A^{h,a}_\mu$ due to its composite nature. All these sources transform trivially under BRST, i.e., $s\Omega^a_\mu = sL^a = sK^a = s\mathcal{J}^a_\mu = 0$, so that the action S_{ext} is BRST-invariant ($sS_{ext} = 0$).

The presence of the condensates $\langle A^{h,a}_\mu A^{h,a}_\mu \rangle$ and $\langle \bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu \rangle$ leads to the introduction of sources coupled to the corresponding local composite operators, as described in [173, 194],

$$S_{cond} = \int d^3x \, \left[J \left(A^{h,a}_\mu A^{h,a}_\mu \right) - \rho \left(\bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu \right) \right] . \tag{5.79}$$

This action will replace the condensate terms in (5.77). Since the composite operators are BRST-invariant, the associated sources also need to be, i.e., $sJ = s\rho = 0$. In the end, these sources should attain the physical values

$$J|_{phys} = \frac{M^2}{2}, \quad \rho|_{phys} = \mu^2. \tag{5.80}$$

In four dimensions, one needs to introduce a quadratic terms in the source J, in order to control divergencies coming from $\langle (A^{h,a}_\mu)^2(x)(A^{h,a}_\mu)^2(y)\rangle_{x\to y}$. However, such a term is not allowed in three dimensions due to power counting (the dimensions and charges of all fields and sources are collected in Tables 5.1, 5.2 and 5.3). As for the auxiliary fields condensate, there is no need for such a quadratic term due to the absence of UV divergencies coming from a $\langle \left(\bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu \right)(x) \left(\bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu \right)(y)\rangle_{x\to y}$ [177].

Next, we modify the Gribov mass term in the action (5.77) by introducing a set of new sources. This was first employed to study the renormalization of the softly broken RGZ theory in the Landau gauge [142, 177]. There, the introduction of such sources was motivated by the restoration of BRST symmetry, allowing for the use of cohomological arguments in the study of renormalization. These sources needed to obtain physical values in the end of the analysis, restoring the softly broken theory [195]. Here, even though our starting action is already BRST-invariant, the introduction of these sources remains useful, as it will restore the doublet structure of the auxiliary fields as in (5.57). We then make the substitution from

$$S_{\gamma^2} \equiv \int d^3x \; g\gamma^2 f^{abc} A^{h,a}_\mu \left(\bar{\varphi} + \varphi \right)^{bc}_\mu , \tag{5.81}$$

to

$$S_{\gamma^2} \equiv \int d^3x \ \left[M_\mu^{ai} D_\mu^{ab}(A^h)\varphi^{bi} + V_\mu^{ai} D_\mu^{ab}(A^h)\bar{\varphi}^{bi} + N_\mu^{ai} D_\mu^{ab}(A^h)\omega^{bi} \right. \\ \left. + U_\mu^{ai} D_\mu^{ab}(A^h)\bar{\omega}^{bi} - M_\mu^{ai} V_\mu^{ai} + N_\mu^{ai} U_\mu^{ai} \right], \tag{5.82}$$

where we used the multi-index notation $i = (a, \mu)$, concerning color and spacetime indices. The introduction of the sources $(M, N, V, U)_{\mu\nu}^{ab}$ renders us with additional Ward identities, further constraining the possible counterterms allowed, which will be proven useful when studying the renormalizability of the action. Once again, they must attune to physical values to recover the original action S_{γ^2}, given by

$$M_{\mu\nu}^{ab}|_{phys} = V_{\mu\nu}^{ab}|_{phys} = \gamma^2 \delta_{\mu\nu}\delta^{ab}, \\ N_{\mu\nu}^{ab}|_{phys} = U_{\mu\nu}^{ab}|_{phys} = 0. \tag{5.83}$$

Also, to preserve BRST-invariance of the total action, the sources must transform as singlets, i.e., $sM_{\mu\nu}^{ab} = sN_{\mu\nu}^{ab} = sV_{\mu\nu}^{ab} = sU_{\mu\nu}^{ab} = 0$. As a final remark, we emphasize that the introduction of these sources is done in the same fashion as in [142, 177] and that the quadratic source terms are allowed by power-counting.

Finally, we introduce an extra set of external sources coupled to the non-linear terms, as in [189],

$$S_{extra} = \int d^3x \ \left[-\Xi_\mu^a D_\mu^{ab}(A^h)\eta^a + X^i \eta^a \bar{\omega}^{ai} \right. \\ \left. + Y^i \eta^a \bar{\varphi}^{ai} + \bar{X}^{abi}\eta^a \omega^{bi} + \bar{Y}^{abi}\eta^a \varphi^{bi} \right]. \tag{5.84}$$

These new sources are also BRST singlets, so that

$$s\Xi_\mu^a = sX^i = sY^i = s\bar{X}^{abi} = s\bar{Y}^{abi} = 0. \tag{5.85}$$

With the introduction of all these external sources, we are left with an enlarged action which will be the focus of the algebraic renormalization analysis. It reads

$$\Sigma = S - \int d^3x \ \left(\bar{\varphi}^{ac}_{\mu}\mathcal{M}^{ab}(A^h)\varphi^{bc}_{\mu} - \bar{\omega}^{ac}_{\mu}\mathcal{M}^{ab}(A^h)\omega^{bc}_{\mu}\right) + \int d^3x \left(\tau^a\partial_\mu A^{h,a}_{\mu} - \bar{\eta}^a\mathcal{M}^{ab}(A^h)\eta^b\right) + S_{\gamma^2} + S_{cond} + S_{ext} + S_{extra}, \tag{5.86}$$

which is invariant under BRST, i.e.

$$s\Sigma = 0. \tag{5.87}$$

We also see that the original RGZ-YMCS action (5.77) is recovered in the limit where all the introduced sources attain the prescribed physical values (5.80) and (5.83), while the remaining sources are all set to zero

$$\begin{aligned} &\Omega^a_\mu|_{phys} = L^a|_{phys} = K^a|_{phys} = \mathcal{J}^a_\mu|_{phys} = 0, \\ &\Xi^a_\mu|_{phys} = X^i|_{phys} = Y^i|_{phys} = \bar{X}^{abi}|_{phys} = \bar{Y}^{abi}|_{phys} = 0, \end{aligned} \tag{5.88}$$

so that we have

$$\Sigma|_{phys} = S^{RGZ}_{YMCS}. \tag{5.89}$$

Before we start the characterization of all the enlarged action Σ symmetries, it will be useful to define an extended BRST symmetry Q, which will later simplify the characterization of the invariant counterterm.

5.4.2 Extended BRST symmetry

Following the strategy developed in [196] and already explored in the context of RGZ theories [189, 197], we introduce a BRST transformation for the gauge parameter α, making it a BRST doublet

$$s\alpha = \chi, \quad s\chi = 0, \tag{5.90}$$

where χ is a constant Grassmann variable of ghost number 1, which we can take to zero at the end of the analysis, recovering the original theory. In this way, we can control the dependence of correlation functions on the gauge parameter α through functional identities. Moreover, being now in a BRST doublet, the gauge parameter can only enter the trivial sector in the cohomology of the BRST operator, so that it does not affect gauge-invariant physical quantities.

Beyond that, due to the introduction of sources, the action Σ is also left invariant by another symmetry

$$\delta\Sigma = 0, \tag{5.91}$$

with δ being a Grassmann operator which acts on the fields and sources as

$$\begin{aligned}
&\delta\varphi^{ab} = \omega^{ab}_{\mu}, && \delta\omega^{ab}_{\mu} = 0,\\
&\delta\bar{\omega}^{ab}_{\mu} = \bar{\varphi}^{ab}_{\mu}, && \delta\bar{\varphi}^{ab}_{\mu} = 0,\\
&\delta N^{ai}_{\mu} = M^{ai}_{\mu}, && \delta M^{ai}_{\mu} = 0,\\
&\delta V^{ai}_{\mu} = U^{ai}_{\mu}, && \delta U^{ai}_{\mu} = 0,\\
&\delta Y^{i} = X^{i}, && \delta Y^{i} = 0,\\
&\delta\bar{X}^{abi} = \bar{Y}^{abi}, && \delta\bar{Y}^{abi} = 0,
\end{aligned} \tag{5.92}$$

with all other fields and sources transforming trivially under δ. From the transformations (5.92), we can see that this new operator is nilpotent, i.e., $\delta^2 = 0$. Given that the operators s and δ anticommute, i.e., $\{s, \delta\} = 0$, we can construct an extended nilpotent operator Q as

$$Q = s + \delta, \quad Q^2 = 0. \tag{5.93}$$

The transformations of fields and sources under Q are given by

$$
\begin{array}{ll}
QA^a_\mu = -D^{ab}_\mu c^b\,, & Qc^a = \frac{g}{2} f^{abc} c^b c^c\,, \\
Q\bar{c}^a = b^a\,, & Qb^a = 0\,, \\
Q\varphi^{ai} = \omega^{ai}\,, & Q\omega^{ai} = 0\,, \\
Q\bar{\omega}^{ai} = \bar{\varphi}^{ai}\,, & Q\bar{\varphi}^{ai} = 0\,, \\
QA^{h,a}_\mu = 0\,, & Q\tau^a = 0\,, \\
Q\bar{\eta}^a = 0\,, & Q\eta^a = 0\,, \\
Q\xi^a = g^{ab}(\xi) c^b\,, & Q\alpha = \chi\,, \\
Q\chi = 0\,, & QN^{ai}_\mu = M^{ai}_\mu\,, \\
QM^{ai}_\mu = 0\,, & QV^{ai}_\mu = U^{ai}_\mu\,, \\
QU^{ai}_\mu = 0\,, & QY^i = X^i\,, \\
QX^i = 0\,, & Q\bar{X}^{abi} = -\bar{Y}^{abi}\,, \\
Q\bar{Y}^{abi} = 0\,, & Q\Omega^a_\mu = 0\,, \\
QL^a = 0\,, & Q\mathcal{J}^a_\mu = 0\,, \\
QK^a = 0\,, & QJ = 0\,, \\
Q\Xi^a_\mu = 0\,. &
\end{array}
\tag{5.94}
$$

The introduction of the extended symmetry Q restores the doublet nature of the localizing auxiliary fields $(\bar{\varphi}, \varphi)^{ab}_\mu$ and $(\bar{\omega}, \omega)^{ab}_\mu$, which ensures that those fields can only appear in the trivial sector of the cohomology of the operator Q. Thus, we can also define the source ρ coupled to the refining auxiliary fields condensate as a Q-doublet, by introducing the following transformations

$$
Q\rho = \sigma, \quad Q\sigma = 0, \tag{5.95}
$$

where we introduced an extra source σ.

The complete action Σ, which is invariant under the extended BRST symmetry Q defined by the transformations (5.94) and (5.95), is given by

$$\begin{aligned}\Sigma = S_{YMCS} &+ \int d^3x \left(b^a \partial_\mu A^a_\mu - \frac{\alpha}{2} b^a b^a - \frac{\chi}{2} \bar{c}^a b^a + \bar{c}^a \partial_\mu D^{ab}_\mu c^a \right) \\ &- \int d^3x \left(\bar{\varphi}^{ai} \mathcal{M}^{ab}(A^h) \varphi^{bi} - \bar{\omega}^{ai} \mathcal{M}^{ab}(A^h) \omega^{bi} \right) \\ &+ \int d^3x \left[J(A^{h,a}_\mu A^{h,a}_\mu) - \rho(\bar{\varphi}^{ai}\varphi^{ai} - \bar{\omega}^{ai}\omega^{ai}) - \sigma \bar{\omega}^{ai}\varphi^{ai} \right] \\ &+ \int d^3x \left(\tau^a \partial_\mu A^{h,a}_\mu - \bar{\eta}^a \mathcal{M}^{ab}(A^h) \eta^b \right) + S_{ext} + S_{\gamma^2} + S_{extra},\end{aligned} \tag{5.96}$$

so that

$$Q\Sigma = 0. \tag{5.97}$$

As before, the action (5.96) reduces to the original one (5.77) when the sources assume physical values and the Grassmann parameter χ is set to zero. As already pointed out for the auxiliary fields, the extension of the BRST symmetry to the transformations Q (5.94) introduces a large set of the sources as Q-doublets, due to the δ-component of Q, limiting all of these sources to the trivial sector of the Q-cohomology. This restriction will control the counterterm candidates which can be constructed. Beyond that, the use of the multi-index $i = (a, \mu)$ entails another global symmetry of the action (5.96), to which we attribute another quantum number dubbed $U(f)$ charge [189] and show explicitly in the next section. Moreover, the localizing ghost-fields $(\bar{\eta}^a, \eta^a)$, introduced in the localization of $A^{h,a}_\mu$, carry a ghost number that is independent of the Faddeev-Popov ghosts. Therefore, we need to assign different names to each type of ghost number, namely $\Phi\Pi$ for the Faddeev-Popov ghost and η for the η-ghost, in order to avoid any confusion. We collect the dimensions and quantum numbers of all fields, sources and parameters in Tables 5.1, 5.2 and 5.3. In the following, we characterize all the functional identities obeyed by (5.96), which will allow us to investigate algebraically the most general counterterm which can be added to the initial action in the renormalization procedure.

Fields	A	b	c	$\bar{c}$	ξ	$\bar{\varphi}$	φ	$\bar{\omega}$	ω	α	χ
d	1/2	3/2	-1/2	3/2	-1/2	1/2	1/2	1/2	1/2	0	0
ΦΠ	0	0	1	-1	0	0	0	-1	1	0	1
η	0	0	0	0	0	0	0	0	0	0	0
$U(f)$	0	0	0	0	0	-1	1	-1	1	0	0
Nature	B	B	F	F	B	B	B	F	F	B	F

Table 5.1: The quantum numbers of fields and their nature, i.e., bosonic (B) meaning that they are commuting and fermionic (F) that implies an anti-commuting field.

Sources	Ω	L	K	$\mathcal{J}$	M	N	U	V	J	ρ	σ
d	5/2	7/2	7/2	5/2	3/2	3/2	3/2	3/2	2	2	2
ΦΠ	-1	-2	-1	0	0	-1	1	0	0	0	1
η	0	0	0	0	0	0	0	0	0	0	0
$U(f)$	0	0	0	0	-1	-1	1	1	0	0	0
Nature	F	B	F	B	B	F	F	B	B	B	F

Table 5.2: The quantum numbers of sources and their nature, i.e., bosonic (B) meaning that they are commuting and fermionic (F) that implies an anti-commuting source.

Fields/Sources	τ	η	$\bar{\eta}$	Ξ	X	Y	$\bar{X}$	$\bar{Y}$
d	3/2	1/2	1/2	3/2	2	2	2	2
ΦΠ	0	0	0	0	1	0	-1	0
η	0	1	-1	-1	-1	-1	-1	-1
$U(f)$	0	1	-1	0	1	1	-1	-1
Nature	B	F	F	F	B	F	B	F

Table 5.3: The quantum numbers of fields and sources and their nature, i.e., bosonic (B) meaning that they are commuting and fermionic (F) that they are anti-commuting

5.4.3 Slavnov-Taylor and other functional identities

The Q-invariance of the action Σ (5.96) can be functionally expressed by the Slavnov-Taylor identity, or rather, the extended Slavnov-Taylor identity (since Q is an extension of the BRST operator s). It reads

$$\mathcal{S}_Q(\Sigma) = 0, \tag{5.98}$$

where

$$\mathcal{S}_Q(\Sigma) = \int d^3x \left[\frac{\delta\Sigma}{\delta\Omega^a_\mu}\frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a} + \frac{\delta\Sigma}{\delta K^a}\frac{\delta\Sigma}{\delta\xi^a} + b^a\frac{\delta\Sigma}{\delta\bar{c}^a} + \omega^{ai}\frac{\delta\Sigma}{\delta\varphi^{ai}} + \bar{\varphi}^{ai}\frac{\delta\Sigma}{\delta\bar{\omega}^{ai}} \right.$$
$$\left. + M^{ai}_\mu\frac{\delta\Sigma}{\delta N^{ai}_\mu} + U^{ai}_\mu\frac{\delta\Sigma}{\delta V^{ai}_\mu} + \sigma\frac{\delta\Sigma}{\delta\rho} + X^i\frac{\delta\Sigma}{\delta Y^i} - \bar{Y}^{abi}\frac{\delta\Sigma}{\delta\bar{X}^{abi}} \right] + \chi\frac{\partial\Sigma}{\partial\alpha}. \tag{5.99}$$

Other useful identities can be readily extended to the quantum theory due to the quantum action principle [43], being linearly broken in the quantum fields. They are listed in the following:

The gauge-fixing condition

$$\frac{\delta\Sigma}{\delta b^a} = \partial_\mu A^a_\mu - \alpha b^a - \frac{1}{2}\chi\bar{c}^a; \tag{5.100}$$

The anti-ghost equation

$$\frac{\delta\Sigma}{\delta\bar{c}^a} - \partial_\mu\frac{\delta\Sigma}{\delta\Omega^a_\mu} = \frac{1}{2}\chi b^a; \tag{5.101}$$

The equations of motion of the lagrange multiplier τ^a

$$\frac{\delta\Sigma}{\delta\tau^a} - \partial_\mu\frac{\delta\Sigma}{\delta\mathcal{J}^a_\mu} = 0; \tag{5.102}$$

The global $U(f)$ symmetry, which can be seen from the particular contraction of fields and sources carrying the multi-index $i = (a, \mu)$, is functionally expressed as

$$U_{ij}\Sigma = 0, \tag{5.103}$$

with

$$\begin{aligned} U_{ij} = \int d^3x \Bigg[&\varphi^{ai}\frac{\delta}{\delta\varphi^{aj}} - \bar{\varphi}^{aj}\frac{\delta}{\delta\bar{\varphi}^{ai}} + \omega^{ai}\frac{\delta}{\delta\omega^{aj}} - \bar{\omega}^{aj}\frac{\delta}{\delta\bar{\omega}^{ai}} \\ &- M_\mu^{aj}\frac{\delta}{\delta M_\mu^{ai}} + V_\mu^{ai}\frac{\delta}{\delta V_\mu^{aj}} - N_\mu^{aj}\frac{\delta}{\delta N_\mu^{ai}} + U_\mu^{ai}\frac{\delta}{\delta U_\mu^{aj}} \\ &+ X^i\frac{\delta}{\delta X^j} + Y^i\frac{\delta}{\delta Y^j} - \bar{X}^{abj}\frac{\delta}{\delta\bar{X}^{abi}} - \bar{Y}^{abj}\frac{\delta}{\delta\bar{Y}^{abi}}\Bigg]; \end{aligned} \tag{5.104}$$

We also have a set of linearly broken identities mixing the auxiliary fields and sources:

$$\frac{\delta\Sigma}{\delta\bar{\varphi}^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta M_\mu^{ai}} + gf^{abc}V_\mu^{bi}\frac{\delta\Sigma}{\delta\mathcal{J}_\mu^c} = -\rho\varphi^{ai} + Y^i\eta^a; \tag{5.105}$$

$$\frac{\delta\Sigma}{\delta\varphi^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta V_\mu^{ai}} - gf^{abc}\bar{\varphi}^{bi}\frac{\delta\Sigma}{\delta\tau^c} + gf^{abc}M_\mu^{bi}\frac{\delta\Sigma}{\delta\mathcal{J}_\mu^c} = -\rho\bar{\varphi}^{ai} - \sigma\bar{\omega}^{ai} + \bar{Y}^{bai}\eta^b; \tag{5.106}$$

$$\frac{\delta\Sigma}{\delta\bar{\omega}^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta N_\mu^{ai}} - gf^{abc}U_\mu^{bi}\frac{\delta\Sigma}{\delta\mathcal{J}_\mu^c} = \rho\omega^{ai} + \sigma\varphi^{ai} - X^i\eta^a; \tag{5.107}$$

$$\frac{\delta\Sigma}{\delta\omega^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta U_\mu^{ai}} - gf^{abc}\bar{\omega}^{bi}\frac{\delta\Sigma}{\delta\tau^c} + gf^{abc}N_\mu^{bi}\frac{\delta\Sigma}{\delta\mathcal{J}_\mu^c} = -\rho\bar{\omega}^{ai} - \bar{X}^{bai}\eta^b; \tag{5.108}$$

The Faddeev-Popov and η-ghosts present two distinct global ghost number identities, given by

$$\begin{aligned} \int d^3x \Bigg[&c^a\frac{\delta\Sigma}{\delta c^a} - \bar{c}^a\frac{\delta\Sigma}{\delta\bar{c}^a} + \omega^{ai}\frac{\delta\Sigma}{\delta\omega^{ai}} - \bar{\omega}^{ai}\frac{\delta\Sigma}{\delta\bar{\omega}^{ai}} - \Omega_\mu^a\frac{\delta\Sigma}{\delta\Omega_\mu^a} - 2L^a\frac{\delta\Sigma}{\delta L^a} - K^a\frac{\delta\Sigma}{\delta K^a} \\ &+ U_\mu^{ai}\frac{\delta\Sigma}{\delta U_\mu^{ai}} - N_\mu^{ai}\frac{\delta\Sigma}{\delta N_\mu^{ai}} + X^i\frac{\delta\Sigma}{\delta X^i} - \bar{X}^{abi}\frac{\delta\Sigma}{\delta\bar{X}^{abi}}\Bigg] + \chi\frac{\partial\Sigma}{\partial\chi} = 0; \end{aligned} \tag{5.109}$$

$$\begin{aligned} \int d^3x \Bigg[&\eta^a\frac{\delta\Sigma}{\delta\eta^a} - \bar{\eta}^a\frac{\delta\Sigma}{\delta\bar{\eta}^a} - \Xi_\mu^a\frac{\delta\Sigma}{\delta\Xi_\mu^a} \\ &- X^i\frac{\delta\Sigma}{\delta X^i} - Y^i\frac{\delta\Sigma}{\delta Y^i} - \bar{X}^{abi}\frac{\delta\Sigma}{\delta\bar{X}^{abi}} - \bar{Y}^{abi}\frac{\delta\Sigma}{\delta\bar{Y}^{abi}}\Bigg] = 0; \end{aligned} \tag{5.110}$$

Beyond the global $U(f)$ symmetry, the auxiliary fields and sources satisfy another symmetry corresponding to an exchange between them with appropriate signs. This is named in the literature as a $\mathcal{R}_{ij}$ symmetry [189] and is given by the functional identity

$$\mathcal{R}_{ij}\Sigma = 0, \tag{5.111}$$

with

$$\begin{aligned}\mathcal{R}_{ij} = \int d^3x \bigg[&\varphi^{ai}\frac{\delta\Sigma}{\delta\omega^{aj}} - \bar{\omega}^{aj}\frac{\delta\Sigma}{\delta\bar{\varphi}^{ai}} + V^{ai}_{\mu}\frac{\delta\Sigma}{\delta U^{aj}_{\mu}} \\ &- N^{aj}_{\mu}\frac{\delta\Sigma}{\delta M^{ai}_{\mu}} + \bar{X}^{abj}\frac{\delta\Sigma}{\delta\bar{Y}^{abi}} + Y^{i}\frac{\delta\Sigma}{\delta X^{j}}\bigg] ;\end{aligned} \tag{5.112}$$

The auxiliary ghost fields $(\bar{\eta}^a, \eta^a)$, introduced in the localization of the invariant field A^h, are presented in a similar fashion to the Faddeev-Popov ghosts in the Landau gauge. In fact, they also lead to a η-ghost equation

$$\frac{\delta\Sigma}{\delta\bar{\eta}^a} - \partial_{\mu}\frac{\delta\Sigma}{\delta\Xi^a_{\mu}} = 0, \tag{5.113}$$

and an anti-ghost equation

$$\begin{aligned}\int d^3x &\left[\frac{\delta\Sigma}{\delta\eta^a} + gf^{abc}\bar{\eta}^b\frac{\delta\Sigma}{\delta\tau^c} - gf^{abc}\Xi^b_{\mu}\frac{\delta\Sigma}{\delta\mathcal{J}^c_{\mu}}\right] \\ &= \int d^3x \left[X^i\bar{\omega}^{ai} - Y^i\bar{\varphi}^{ai} + \bar{X}^{abi}\omega^{bi} - \bar{Y}^{abi}\varphi^{bi}\right] ;\end{aligned} \tag{5.114}$$

Finally, we have a series of identities that mix the auxiliary fields $(\bar{\varphi}, \varphi, \bar{\omega}, \omega)$ with the ghosts $(\bar{\eta}, \eta)$:

$$W^i_{(1)}\Sigma = \int d^3x \left(\bar{\omega}^{ai}\frac{\delta\Sigma}{\delta\bar{\eta}^a} + \eta^a\frac{\delta\Sigma}{\delta\omega^{ai}} + N^{ai}_{\mu}\frac{\delta\Sigma}{\delta\Xi^a_{\mu}} + \rho\frac{\delta\Sigma}{\delta X^i}\right) = 0; \tag{5.115}$$

$$W^i_{(2)}\Sigma = \int d^3x \left(\bar{\varphi}^{ai}\frac{\delta\Sigma}{\delta\bar{\eta}^a} - \eta^a\frac{\delta\Sigma}{\delta\varphi^{ai}} + M^{ai}_{\mu}\frac{\delta\Sigma}{\delta\Xi^a_{\mu}} - \rho\frac{\delta\Sigma}{\delta Y^i} + \sigma\frac{\delta\Sigma}{\delta X^i}\right) = 0; \tag{5.116}$$

$$W^i_{(3)}\Sigma = \int d^3x \left(\varphi^{ai}\frac{\delta\Sigma}{\delta\bar{\eta}^a} - \eta^a\frac{\delta\Sigma}{\delta\bar{\varphi}^{ai}} - gf^{abc}\frac{\delta\Sigma}{\delta\bar{Y}^{abi}}\frac{\delta\Sigma}{\delta\tau^c} - V^{ai}_\mu\frac{\delta\Sigma}{\delta\Xi^a_\mu} + \rho\frac{\delta\Sigma}{\delta\bar{Y}^{aai}}\right) = 0; \tag{5.117}$$

$$W^i_{(4)}\Sigma = \int d^3x \left(\omega^{ai}\frac{\delta\Sigma}{\delta\bar{\eta}^a} - \eta^a\frac{\delta\Sigma}{\delta\bar{\omega}^{ai}} + gf^{abc}\frac{\delta\Sigma}{\delta\bar{X}^{abi}}\frac{\delta\Sigma}{\delta\tau^c} + U^{ai}_\mu\frac{\delta\Sigma}{\delta\Xi^a_\mu} + \rho\frac{\delta\Sigma}{\delta\bar{X}^{aai}} + \sigma\frac{\delta\Sigma}{\delta\bar{Y}^{aai}}\right) = 0. \tag{5.118}$$

These identities show that the RGZ-YMCS action (5.96) share the same symmetry content of the standard RGZ action [189]. This is to be expected, since the addition of the Chern-Simons term is already compatible with the BRST invariance of the Yang-Mills action and does not affect the additional structure responsible for elimination of infinitesimal Gribov copies. In the next section, we study the renormalizability of the RGZ-YMCS action, in which we will make use of all these identities to characterize the most general invariant counterterm that is allowed to be constructed.

5.4.4 Finiteness of the YMCS-RGZ theory

Let us discuss the renormalizability of the RGZ-YMCS theory. A known property of the RGZ formalism is that it does not introduce new UV divergencies to the underlying theory [198]. Therefore, since the YMCS theory is UV finite, i.e., all of the β-functions and anomalous dimensions of its parameters and fields are identically zero, the restriction to the Gribov region should maintain such finite character.

As stated before, the most general invariant counterterm Σ_{CT} is an integrated, local polynomial on the fields and sources, bounded by dimension 2 (due to the superrenormalizable character of the theory) and ghost number 0. Within the algebraic renormalization set up, it is interpreted as a perturbation of the classical action $(\Sigma + \epsilon\Sigma_{CT})$, which implies that it obeys the following constraints:

$$
\begin{aligned}
&\mathcal{B}_Q \Sigma_{\rm CT} = 0, \\
&U_{ij} \Sigma_{\rm CT} = 0, \\
&\mathcal{R}_{ij} \Sigma_{\rm CT} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta b^a} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \bar{c}^a} - \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \Omega^a_\mu} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \tau^a} - \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \mathcal{J}^a_\mu} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \bar{\varphi}^{ai}} + \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta M^{ai}_\mu} + g f^{abc} V^{bi}_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \mathcal{J}^c_\mu} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \varphi^{ai}} + \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta V^{ai}_\mu} - g f^{abc} \bar{\varphi}^{bi} \frac{\delta \Sigma_{\rm CT}}{\delta \tau^c} + g f^{abc} M^{bi}_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \mathcal{J}^c_\mu} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \bar{\omega}^{ai}} + \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta N^{ai}_\mu} - g f^{abc} U^{bi}_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \mathcal{J}^c_\mu} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \omega^{ai}} + \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta U^{ai}_\mu} - g f^{abc} \bar{\omega}^{bi} \frac{\delta \Sigma_{\rm CT}}{\delta \tau^c} + g f^{abc} N^{bi}_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \mathcal{J}^c_\mu} = 0, \\
&\frac{\delta \Sigma_{\rm CT}}{\delta \bar{\eta}^a} - \partial_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \Xi^a_\mu} = 0, \\
&W^i_{(1,2,3,4)} \Sigma_{\rm CT} = 0, \\
&\int d^3x \left(\frac{\delta \Sigma_{\rm CT}}{\delta \eta^a} + g f^{abc} \bar{\eta}^b \frac{\delta \Sigma_{\rm CT}}{\delta \tau^c} - g f^{abc} \Xi^b_\mu \frac{\delta \Sigma_{\rm CT}}{\delta \mathcal{J}^c_\mu} \right) = 0.
\end{aligned}
\tag{5.119}
$$

In the first identity of (5.119), we define the linearized extended Slavnov-Taylor operator $\mathcal{B}_Q$, which is written as

$$
\begin{aligned}
\mathcal{B}_Q = \int d^3x \bigg(&\frac{\delta \Sigma}{\delta \Omega^a_\mu} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \Omega^a_\mu} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} \\
&+ \frac{\delta \Sigma}{\delta K^a} \frac{\delta}{\delta \xi^a} + \frac{\delta \Sigma}{\delta \xi^a} \frac{\delta}{\delta K^a} + b^a \frac{\delta}{\delta \bar{c}^a} + \omega^{ai} \frac{\delta}{\delta \varphi^{ai}} + \bar{\varphi}^{ai} \frac{\delta}{\delta \bar{\omega}^{ai}} \\
&+ M^{ai}_\mu \frac{\delta}{\delta N^{ai}_\mu} + U^{ai}_\mu \frac{\delta}{\delta V^{ai}_\mu} + \sigma \frac{\delta}{\delta \rho} + X^i \frac{\delta}{\delta Y^i} - \bar{Y}^{abi} \frac{\delta}{\delta \bar{X}^{abi}} \bigg) + \chi \frac{\partial}{\partial \alpha},
\end{aligned}
\tag{5.120}
$$

and obeys the identity

$$\mathcal{B}_Q\mathcal{B}_Q = 0, \tag{5.121}$$

i.e., it is a nilpotent operator. This property restricts the general solution to be of the form

$$\Sigma_{CT} = \Delta + \mathcal{B}_Q\Delta^{(-1)}. \tag{5.122}$$

The solution is then divided into two sector: the functional Δ belonging to the cohomology of the operator (5.120), i.e. it is a non-trivial Q-cocycle; while the functional $\mathcal{B}_Q\Delta^{(-1)}$ is a trivial cocycle, i.e. a Q-coboundary. The auxiliary fields and sources must belong to the latter, since they were introduced as Q-doublets. Therefore, the functionals Δ and $\Delta^{(-1)}$ are local, integrated polynomials bounded by dimension 2 and with ghost number 0 and -1, respectively. These restrictions lead to the following expressions

$$\Delta = \int d^3x \Big[a_0\epsilon_{\mu\rho\nu}\left(\frac{1}{2}A^a_\mu\partial_\rho A^a_\nu + \frac{g}{3!}f^{abc}A^a_\mu A^b_\rho A^c_\nu\right) + a_1 J + \lambda^{abcd}A^{h,a}_\mu A^{h,b}_\mu A^{h,c}_\nu A^{h,d}_\nu\Big], \tag{5.123}$$

and

$$\Delta^{(-1)} = \int d^3x\ \epsilon_{\mu\rho\nu}\left[b_1\bar{\omega}^{ab}_\mu\partial_\rho\varphi^{ab}_\nu + b_2 g f^{abd}\bar{\omega}^{ac}A^{h,d}_\rho\varphi^{bc}_\nu + b_3 N^{ab}_{\mu\rho}\varphi^{ab}_\nu + b_4 V^{ab}_{\mu\rho}\bar{\omega}^{ab}_\nu\right], \tag{5.124}$$

where, upon the action of $\mathcal{B}_Q$, the trivial term becomes

$$\begin{aligned}\mathcal{B}_Q\Delta^{(-1)} = \int d^3x\ \epsilon_{\mu\rho\nu}\big[&b_1(\bar{\varphi}^{ab}_\mu\partial_\rho\varphi^{ab}_\nu - \bar{\omega}^{ab}_\mu\partial_\rho\omega^{ab}_\nu) + b_2 g f^{abd}(\bar{\varphi}^{ac}_\mu A^{h,d}_\rho\varphi^{bc}_\nu \\ &- \bar{\omega}^{ac}_\mu A^{h,d}_\rho\omega^{bc}_\nu) + b_3(M^{ab}_{\mu\rho}\omega^{ab}_\nu - N^{ab}_{\mu\rho}\omega^{ab}_\nu) + b_4(U^{ab}_{\mu\rho}\bar{\omega}^{ab}_\nu - V^{ab}_{\mu\rho}\bar{\varphi}^{ab}_\nu)\big].\end{aligned} \tag{5.125}$$

However, these expressions must also obey to the other conditions in (5.119), which severely restricts the coefficients a_0, a_1, λ^{abcd}, b_1, b_2, b_3 and b_4. By enforcing (5.119), we have that the trivial solution vanishes, i.e., $b_1 = b_2 = b_3 = b_4 = 0$. This vanishing solution also enforces the independence of the counterterm from the gauge parameter α, since by introducing it as a Q-doublet, it could only appear in the trivial cocycle. This also allows us to further restrict the non-trivial sector, since the coefficients can then be considered in a simplified manner by taking $\alpha = 0$, i.e., by assuming the Landau gauge. But in this situation, the invariant field $A^{h,a}_\mu$ reduces to the usual gauge field A^a_μ (see the appendix of [189]), thanks to the decoupling of the Stueckelberg-like field ξ^a [199]. Thus, to preserve BRST-invariance (more precisely, Q-invariance), we must demand that $\lambda^{abcd} = 0$, so that the most general counterterm is given by

$$\Delta = \int d^3x \left[a_0 \epsilon_{\mu\rho\nu} \left(\frac{1}{2} A^a_\mu \partial_\rho A^a_\nu + \frac{g}{3!} f^{abc} A^a_\mu A^b_\rho A^c_\nu \right) + a_1 J \right] . \qquad (5.126)$$

Two comments are in order: the term proportional to the source J must be taken at its physical value, and is, therefore, nothing more than an additive constant. As such, it does not affect any correlation functions of the theory and does not interfere with the renormalizability. As for the a_0 term, we have, once again, the Chern-Simons action. Being BRST-invariant only as an integrated term, it does not contribute to the non-trivial part of the cohomology [82–85], as discussed in chapters 3 and 4. Therefore, the counterterm action is trivial and the YMCS theory in linear covariant gauges remains finite when taking into account the elimination of Gribov copies and the dynamical formation of condensates.

Chapter 6

Final Remarks

As we have seen throughout this thesis, quantum field theories in 3 space-time dimensions have many different properties than its four-dimensional counterpart, with the possibility of a gauge invariant mass term for gauge fields and superrenormalizability among these properties. It serves as a good theoretical laboratory where one can encounter new phenomena and study features of four dimensional Yang-Mills in a simpler setting, while also being of key interest for the application in condensed matter physics. Here we pursued an investigation of the renormalization properties of Yang-Mills-Chern-Simons theories using the algebraic renormalization framework, analyzing it in a couple of different scenarios.

We began with a review of the Quantum Action Principle, upon which the algebraic method is built, and then introduced the Yang-Mills-Chern-Simons theory in the Landau gauge. Even though the theory is superrenormalizable and the counterterms can be explicitly computed with a little effort, since they appear only up to two loops order, the use of the algebraic framework allows us to come up with a new functional identity, the local Callan-Symanzik equation, which proves the finiteness of YMCS without the need of explicit calculations.

Building upon it, we extend the known finiteness of YMCS to more general settings. First, we introduce the theory for a wider class of gauge conditions, namely one which interpolates between the light-cone gauge and linear covariant gauges. This specific choice of gauge-fixing is made since it obeys the requirements of the QAP, a property that non-covariant gauges do not usually present, while also retaining information on the more physical light-cone gauge, which is recovered for a limiting value of the interpolating parameter. With it, we were once again able to prove the renormalizability of YMCS to all orders in perturbation theory and, even more, its finite character.

Finally, we have investigated the restriction of YMCS to the first Gribov horizon through the implementation of the Gribov-Zwanziger formalism. This is possible due to the geometrical nature of the problem and can provide insights on the dynamics of gauge fields in the infrared regime of three dimensions. We verified that the introduction of the auxiliary fields responsible of implementing such a restriction does not lead to new divergencies, thus retaining the finiteness of YMCS theory to all orders.

These investigations open up possible paths of continuity. A purely algebraic investigation of supersymmetric extensions of YMCS theory seems like a natural complement to the work presented here. Also, a careful study of the phase diagram of the gluon propagator and the possibility of a confining/deconfining transition coming from the interplay between the different mass parameters present in the RGZ-YMCS theory, which can be enriched even further with the introduction of a Higgs mass term. Beyond that, the computation of the Gribov parameter and the dynamical condensate masses self-consistently through their gap equation is still a little explored topic within the Gribov-Zwanziger formalism, in both three and four dimensions, and should also be addressed to allow for comparisons with the lattice data.

Bibliography

[1] Chen-Ning Yang and Robert L. Mills. Conservation of Isotopic Spin and Isotopic Gauge Invariance. *Phys. Rev.*, 96:191–195, 1954. doi:10.1103/PhysRev.96.191.

[2] Ryoyu Utiyama. Invariant theoretical interpretation of interaction. *Phys. Rev.*, 101:1597–1607, 1956. doi:10.1103/PhysRev.101.1597.

[3] H. A. Bethe. The Electromagnetic shift of energy levels. *Phys. Rev.*, 72:339–341, 1947. doi:10.1103/PhysRev.72.339.

[4] R. P. Feynman. A Relativistic cutoff for classical electrodynamics. *Phys. Rev.*, 74:939–946, 1948. doi:10.1103/PhysRev.74.939.

[5] R. P. Feynman. Relativistic cutoff for quantum electrodynamics. *Phys. Rev.*, 74:1430–1438, 1948. doi:10.1103/PhysRev.74.1430.

[6] Julian S. Schwinger. Quantum electrodynamics. I A covariant formulation. *Phys. Rev.*, 74:1439, 1948. doi:10.1103/PhysRev.74.1439.

[7] Julian S. Schwinger. Quantum electrodynamics. II. Vacuum polarization and selfenergy. *Phys. Rev.*, 75:651, 1948. doi:10.1103/PhysRev.75.651.

[8] Julian S. Schwinger. Quantum electrodynamics. III: The electromagnetic properties of the electron: Radiative corrections to scattering. *Phys. Rev.*, 76:790–817, 1949. doi:10.1103/PhysRev.76.790.

[9] S. Tomonaga. On a relativistically invariant formulation of the quantum theory of wave fields. *Prog. Theor. Phys.*, 1:27–42, 1946. doi:10.1143/PTP.1.27.

[10] Zirô Koba, Takao Tati, and Shinichirô Tomonaga. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. II: Case of Interacting Electromagnetic and Electron Fields. *Prog. Theor. Phys.*, 2(3):101–116, 1947. doi:10.1143/ptp/2.3.101.

[11] Zirô Koba, Takao Tati, and Sinichirô Tomonaga. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. III: Case of Interacting Electromagnetic and Electron Fields. *Prog. Theor. Phys.*, 2(4):198–208, 1947. arXiv:https://academic.oup.com/ptp/article-pdf/2/4/198/5276565/2-4-198.pdf, doi:10.1143/ptp/2.4.198.

[12] F. J. Dyson. The Radiation theories of Tomonaga, Schwinger, and Feynman. *Phys. Rev.*, 75:486–502, 1949. doi:10.1103/PhysRev.75.486.

[13] G. F. Chew and Steven C. Frautschi. Principle of Equivalence for All Strongly Interacting Particles Within the S Matrix Framework. *Phys. Rev. Lett.*, 7:394–397, 1961. doi:10.1103/PhysRevLett.7.394.

[14] Abdus Salam and John Clive Ward. On a gauge theory of elementary interactions. *Nuovo Cim.*, 19:165–170, 1961. doi:10.1007/BF02812723.

[15] F. Englert and R. Brout. Broken Symmetry and the Mass of Gauge Vector Mesons. *Phys. Rev. Lett.*, 13:321–323, 1964. doi:10.1103/PhysRevLett.13.321.

[16] Peter W. Higgs. Broken Symmetries and the Masses of Gauge Bosons. *Phys. Rev. Lett.*, 13:508–509, 1964. doi:10.1103/PhysRevLett.13.508.

[17] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. Global Conservation Laws and Massless Particles. *Phys. Rev. Lett.*, 13:585–587, 1964. doi:10.1103/PhysRevLett.13.585.

[18] Sheldon L. Glashow. The renormalizability of vector meson interactions. *Nucl. Phys.*, 10:107–117, 1959. doi:10.1016/0029-5582(59)90196-8.

[19] Abdus Salam and John Clive Ward. Weak and electromagnetic interactions. *Nuovo Cim.*, 11:568–577, 1959. doi:10.1007/BF02726525.

[20] Steven Weinberg. A Model of Leptons. *Phys. Rev. Lett.*, 19:1264–1266, 1967. doi:10.1103/PhysRevLett.19.1264.

[21] Murray Gell-Mann. Symmetries of baryons and mesons. *Phys. Rev.*, 125:1067–1084, 1962. doi:10.1103/PhysRev.125.1067.

[22] Yuval Ne'eman. Derivation of strong interactions from a gauge invariance. *Nucl. Phys.*, 26:222–229, 1961. doi:10.1016/0029-5582(61)90134-1.

[23] Murray Gell-Mann. A Schematic Model of Baryons and Mesons. *Phys. Lett.*, 8:214–215, 1964. doi:10.1016/S0031-9163(64)92001-3.

[24] G. Zweig. An SU(3) model for strong interaction symmetry and its breaking. Version 1. 1 1964. doi:10.17181/CERN-TH-401.

[25] G. Zweig. An SU(3) model for strong interaction symmetry and its breaking. Version 2. pages 22–101, 2 1964. doi:10.17181/CERN-TH-412.

[26] Gerard 't Hooft. Renormalization of Massless Yang-Mills Fields. *Nucl. Phys. B*, 33:173–199, 1971. doi:10.1016/0550-3213(71)90395-6.

[27] Gerard 't Hooft. Renormalizable Lagrangians for Massive Yang-Mills Fields. *Nucl. Phys. B*, 35:167–188, 1971. doi:10.1016/0550-3213(71)90139-8.

[28] David J. Gross and R. Jackiw. Effect of anomalies on quasirenormalizable theories. *Phys. Rev. D*, 6:477–493, 1972. doi:10.1103/PhysRevD.6.477.

[29] S. L. Glashow, J. Iliopoulos, and L. Maiani. Weak Interactions with Lepton-Hadron Symmetry. *Phys. Rev. D*, 2:1285–1292, 1970. doi:10.1103/PhysRevD.2.1285.

[30] J. J. Aubert et al. Experimental Observation of a Heavy Particle J. *Phys. Rev. Lett.*, 33:1404–1406, 1974. doi:10.1103/PhysRevLett.33.1404.

[31] J. E. Augustin et al. Discovery of a Narrow Resonance in e^+e^- Annihilation. *Phys. Rev. Lett.*, 33:1406–1408, 1974. doi:10.1103/PhysRevLett.33.1406.

[32] F. J. Hasert et al. Observation of Neutrino Like Interactions without Muon or Electron in the Gargamelle Neutrino Experiment. *Nucl. Phys. B*, 73:1–22, 1974. doi:10.1016/0550-3213(74)90038-8.

[33] David J. Gross and Frank Wilczek. Ultraviolet Behavior of Nonabelian Gauge Theories. *Phys. Rev. Lett.*, 30:1343–1346, 1973. doi:10.1103/PhysRevLett.30.1343.

[34] H. David Politzer. Reliable Perturbative Results for Strong Interactions? *Phys. Rev. Lett.*, 30:1346–1349, 1973. doi:10.1103/PhysRevLett.30.1346.

[35] Serguei Chatrchyan et al. Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC. *Phys. Lett. B*, 716:30–61, 2012. arXiv:1207.7235, doi:10.1016/j.physletb.2012.08.021.

[36] Georges Aad et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Phys. Lett. B*, 716:1–29, 2012. arXiv:1207.7214, doi:10.1016/j.physletb.2012.08.020.

[37] L. D. Faddeev and V. N. Popov. Feynman Diagrams for the Yang-Mills Field. *Phys. Lett. B*, 25:29–30, 1967. doi:10.1016/0370-2693(67)90067-6.

[38] C. Becchi, A. Rouet, and R. Stora. The Abelian Higgs-Kibble Model. Unitarity of the S Operator. *Phys. Lett. B*, 52:344–346, 1974. doi:10.1016/0370-2693(74)90058-6.

[39] C. Becchi, A. Rouet, and R. Stora. Renormalization of the Abelian Higgs-Kibble Model. *Commun. Math. Phys.*, 42:127–162, 1975. doi:10.1007/BF01614158.

[40] C. Becchi, A. Rouet, and R. Stora. Renormalization of Gauge Theories. *Annals Phys.*, 98:287–321, 1976. doi:10.1016/0003-4916(76)90156-1.

[41] I. V. Tyutin. Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism. 1975. arXiv:0812.0580.

[42] Taichiro Kugo and Izumi Ojima. Local Covariant Operator Formalism of Nonabelian Gauge Theories and Quark Confinement Problem. *Prog. Theor. Phys. Suppl.*, 66:1–130, 1979. doi:10.1143/PTPS.66.1.

[43] O. Piguet and S.P. Sorella. *Algebraic Renormalization: Perturbative Renormalization, Symmetries and Anomalies.* Lecture Notes in Physics Monographs. Springer Berlin Heidelberg, 1995.

[44] Alexander M. Polyakov. Quark Confinement and Topology of Gauge Groups. *Nucl. Phys. B*, 120:429–458, 1977. doi:10.1016/0550-3213(77)90086-4.

[45] Gerard 't Hooft. On the Phase Transition Towards Permanent Quark Confinement. *Nucl. Phys. B*, 138:1–25, 1978. doi:10.1016/0550-3213(78)90153-0.

[46] F. A. Schaposnik. Pseudoparticles and Confinement in the Two-Dimensional Abelian Higgs Model. *Phys. Rev. D*, 18:1183–1191, 1978. doi:10.1103/PhysRevD.18.1183.

[47] Jonathan F. Schonfeld. A Mass Term for Three-Dimensional Gauge Fields. *Nucl. Phys. B*, 185:157–171, 1981. doi:10.1016/0550-3213(81)90369-2.

[48] Stanley Deser, R. Jackiw, and S. Templeton. Topologically Massive Gauge Theories. *Annals Phys.*, 140:372–411, 1982. [Erratum: Annals Phys. 185, 406 (1988)]. doi:10.1016/0003-4916(82)90164-6.

[49] Stanley Deser, R. Jackiw, and S. Templeton. Three-Dimensional Massive Gauge Theories. *Phys. Rev. Lett.*, 48:975–978, 1982. doi:10.1103/PhysRevLett.48.975.

[50] Yutaka Hosotani. Compact QED in Three-Dimensions and the Josephson Effect. *Phys. Lett. B*, 69:499, 1977. doi:10.1016/0370-2693(77)90854-1.

[51] Petter Minnhagen. Kosterlitz- Thouless transition for a two-dimensional superconductor: Magnetic-field dependence from a Coulomb-gas analogy. *Phys. Rev. B*, 23:5745–5761, Jun 1981. URL: https://link.aps.org/doi/10.1103/PhysRevB.23.5745, doi:10.1103/PhysRevB.23.5745.

[52] R. B. Laughlin. Anomalous quantum Hall effect: An Incompressible quantum fluid with fractionallycharged excitations. *Phys. Rev. Lett.*, 50:1395, 1983. doi:10.1103/PhysRevLett.50.1395.

[53] P. W. Anderson. Remarks on the Laughlin theory of the fractionally quantized Hall effect. *Phys. Rev. B*, 28:2264–2265, 1983. URL: https://link.aps.org/doi/10.1103/PhysRevB.28.2264, doi:10.1103/PhysRevB.28.2264.

[54] F. D. M. Haldane. Fractional quantization of the Hall effect: A Hierarchy of incompressible quantum fluid states. *Phys. Rev. Lett.*, 51:605–608, 1983. doi:10.1103/PhysRevLett.51.605.

[55] J. H. Lowenstein. Differential vertex operations in Lagrangian field theory. *Commun. Math. Phys.*, 24:1–21, 1971. doi:10.1007/BF01907030.

[56] J. H. Lowenstein. Normal product quantization of currents in Lagrangian field theory. *Phys. Rev. D*, 4:2281–2290, 1971. doi:10.1103/PhysRevD.4.2281.

[57] Yuk-Ming P. Lam. Perturbation Lagrangian theory for scalar fields: Ward-Takahasi identity and current algebra. *Phys. Rev. D*, 6:2145–2161, 1972. doi:10.1103/PhysRevD.6.2145.

[58] Yuk-Ming P. Lam. Equivalence theorem on Bogolyubov-Parasiuk-Hepp-Zimmermann renormalized Lagrangian field theories. *Phys. Rev. D*, 7:2943–2949, 1973. doi:10.1103/PhysRevD.7.2943.

[59] Thomas E. Clark and John H. Lowenstein. Generalization of Zimmermann's Normal-Product Identity. *Nucl. Phys. B*, 113:109–134, 1976. doi:10.1016/0550-3213(76)90457-0.

[60] P. Breitenlohner and D. Maison. Dimensionally Renormalized Green's Functions for Theories with Massless Particles. 1. *Commun. Math. Phys.*, 52:39, 1977. doi:10.1007/BF01609070.

[61] P. Breitenlohner and D. Maison. Dimensionally Renormalized Green's Functions for Theories with Massless Particles. 2. *Commun. Math. Phys.*, 52:55, 1977. doi:10.1007/BF01609071.

[62] P. Breitenlohner and D. Maison. Dimensional Renormalization and the Action Principle. *Commun. Math. Phys.*, 52:11–38, 1977. doi:10.1007/BF01609069.

[63] Raymond F. Streater and Arthur S. Wightman. *PCT, Spin and Statistics, and All That.* Princeton University Press, 1989.

[64] H. Lehmann, K. Symanzik, and W. Zimmermann. On the formulation of quantized field theories. *Nuovo Cim.*, 1:205–225, 1955. doi:10.1007/BF02731765.

[65] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory.* Addison-Wesley, Reading, USA, 1995. doi:10.1201/9780429503559.

[66] Daniel H. T. Franco. *Elementos de análise funcional e aplicações na teoria quântica*. 2022.

[67] C. Itzykson and J.B. Zuber. *Quantum Field Theory*. Dover Books on Physics. Dover Publications, 2012.

[68] Warren Siegel. *Fields*. 2021. URL: `http://insti.physics.sunysb.edu/~siegel/plan.html`.

[69] P. Ramond. *Field Theory: A Modern Primer*. Frontiers in Physics : a lecture note and reprint series. Benjamin/Cummings Publishing Company, Advanced Book Program, 1981.

[70] Glenn Barnich, Friedemann Brandt, and Marc Henneaux. Local BRST cohomology in gauge theories. *Phys. Rept.*, 338:439–569, 2000. `arXiv:hep-th/0002245`, `doi:10.1016/S0370-1573(00)00049-1`.

[71] G. Barnich, M. Henneaux, and R. Tatar. Consistent interactions between gauge fields and the local BRST cohomology: The Example of Yang-Mills models. *Int. J. Mod. Phys. D*, 3:139–144, 1994. `arXiv:hep-th/9307155`, `doi:10.1142/S0218271894000149`.

[72] Daniel O. R. Azevedo, Oswaldo M. Del Cima, Thadeu D. S. Dias, Daniel H. T. Franco, Emílio D. Pereira, and Olivier Piguet. Spin One Matter Fields. 2024. `arXiv:2410.15407`.

[73] Steven Weinberg. High-energy behavior in quantum field theory. *Phys. Rev.*, 118:838–849, 1960. `doi:10.1103/PhysRev.118.838`.

[74] W. Zimmermann. The power counting theorem for Minkowski metric. *Commun. Math. Phys.*, 11:1–8, 1968. `doi:10.1007/BF01654298`.

[75] J. H. Lowenstein. Convergence Theorems for Renormalized Feynman Integrals with Zero-Mass Propagators. *Commun. Math. Phys.*, 47:53–68, 1976. `doi:10.1007/BF01609353`.

[76] J. H. Lowenstein and W. Zimmermann. The Power Counting Theorem for Feynman Integrals with Massless Propagators. *Commun. Math. Phys.*, 44:73–86, 1975. doi:10.1007/BF01609059.

[77] John H. Lowenstein and Eugene R. Speer. Distributional Limits of Renormalized Feynman Integrals with Zero-Mass Denominators. *Commun. Math. Phys.*, 47:43–51, 1976. doi:10.1007/BF01609352.

[78] C. Nash and S. Sen. *Topology and Geometry for Physicists*. Dover Books on Mathematics. Dover Publications, 2013. URL: https://books.google.com.br/books?id=TNvCAgAAQBAJ.

[79] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989. doi:10.1007/BF01217730.

[80] A. Blasi and R. Collina. Finiteness of the Chern-Simons Model in Perturbation Theory. *Nucl. Phys. B*, 345:472–492, 1990. doi:10.1016/0550-3213(90)90397-V.

[81] F. Delduc, C. Lucchesi, O. Piguet, and S. P. Sorella. Exact Scale Invariance of the Chern-Simons Theory in the Landau Gauge. *Nucl. Phys. B*, 346:313–328, 1990. doi:10.1016/0550-3213(90)90283-J.

[82] Oswaldo M. Del Cima, Daniel H. T. Franco, Jose A. Helaël-Neto, and Olivier Piguet. On the nonrenormalization properties of gauge theories with Chern-Simons terms. *JHEP*, 02:002, 1998. arXiv:hep-th/9711191, doi:10.1088/1126-6708/1998/02/002.

[83] Oswaldo M. Del Cima, Daniel H. T. Franco, Jose A. Helayël-Neto, and Olivier Piguet. Exact scale invariance of the BF Yang-Mills theory in three-dimensions. *JHEP*, 04:010, 1998. arXiv:hep-th/9803247, doi:10.1088/1126-6708/1998/04/010.

[84] O. M. Del Cima, D. H. T. Franco, J. A. Helayël-Neto, and O. Piguet. An Algebraic proof on the finiteness of Yang-Mills-Chern-Simons theory in D = 3. *Lett. Math. Phys.*, 47:265, 1999. arXiv:math-ph/9904030, doi:10.1023/A:1007595121742.

[85] Glenn Barnich. A General nonrenormalization theorem in the extended antifield formalism. *JHEP*, 12:003, 1998. arXiv:hep-th/9805030, doi:10.1088/1126-6708/1998/12/003.

[86] Wolfhart Zimmermann. Composite operators in the perturbation theory of renormalizable interactions. *Annals Phys.*, 77:536–569, 1973. doi:10.1016/0003-4916(73)90429-6.

[87] A. Blasi, O. Piguet, and S. P. Sorella. Landau gauge and finiteness. *Nucl. Phys. B*, 356:154–162, 1991. doi:10.1016/0550-3213(91)90144-M.

[88] G. Giavarini, C. P. Martin, and F. Ruiz Ruiz. Chern-Simons theory as the large mass limit of topologically massive Yang-Mills theory. *Nucl. Phys. B*, 381:222–280, 1992. arXiv:hep-th/9206007, doi:10.1016/0550-3213(92)90647-T.

[89] Robert D. Pisarski and Sumathi Rao. Topologically Massive Chromodynamics in the Perturbative Regime. *Phys. Rev. D*, 32:2081, 1985. doi:10.1103/PhysRevD.32.2081.

[90] C. P. Martin. Dimensional Regularization of Chern-Simons Field Theory. *Phys. Lett. B*, 241:513–521, 1990. doi:10.1016/0370-2693(90)91862-6.

[91] Nicola Maggiore, Olivier Piguet, and Mathieu Ribordy. Algebraic renormalization of N=2 supersymmetric Yang-Mills Chern-Simons theory in the Wess-Zumino gauge. *Helv. Phys. Acta*, 68:264–281, 1995. arXiv:hep-th/9504065.

[92] Friedemann Brandt, Norbert Dragon, and Maximilian Kreuzer. The Gravitational Anomalies. *Nucl. Phys. B*, 340:187–224, 1990. doi:10.1016/0550-3213(90)90161-6.

[93] Glenn Barnich, Friedemann Brandt, and Marc Henneaux. Local BRST cohomology in Einstein Yang-Mills theory. *Nucl. Phys. B*, 455:357–

408, 1995. arXiv:hep-th/9505173, doi:10.1016/0550-3213(95)00471-4.

[94] Glenn Barnich and Marc Henneaux. Renormalization of gauge invariant operators and anomalies in Yang-Mills theory. *Phys. Rev. Lett.*, 72:1588–1591, 1994. arXiv:hep-th/9312206, doi:10.1103/PhysRevLett.72.1588.

[95] G. Bandelloni, A. Blasi, C. Becchi, and R. Collina. Nonsemisimple gauge models: 1. classical theory and the properties of ghost states. *Ann. Inst. H. Poincare Phys. Theor.*, 28:225–254, 1978.

[96] G. Bandelloni, A. Blasi, C. Becchi, and R. Collina. Nonsemisimple gauge models: 2. renormalization. *Ann. Inst. H. Poincare Phys. Theor.*, 28:255–285, 1978.

[97] Alfredo Iorio, L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger. Weyl gauging and conformal invariance. *Nucl. Phys. B*, 495:433–450, 1997. arXiv:hep-th/9607110, doi:10.1016/S0550-3213(97)00190-9.

[98] Curtis G. Callan, Jr., Sidney R. Coleman, and Roman Jackiw. A New improved energy-momentum tensor. *Annals Phys.*, 59:42–73, 1970. doi:10.1016/0003-4916(70)90394-5.

[99] M. Srednicki. *Quantum field theory*. Cambridge University Press, 1 2007. doi:10.1017/CBO9780511813917.

[100] Suraj N Gupta. Theory of longitudinal photons in quantum electrodynamics. *Proceedings of the Physical Society. Section A*, 63(7):681, 1950. URL: https://dx.doi.org/10.1088/0370-1298/63/7/301, doi:10.1088/0370-1298/63/7/301.

[101] K. Bleuler. Eine neue methode zur behandlung der longitudinalen und skalaren photonen. *Helv. Phys. Acta*, 23:567, 1950. doi:10.5169/seals-112124.

[102] George Leibbrandt. *Noncovariant Gauges.* WORLD SCIENTIFIC, 1994. URL: https://www.worldscientific.com/doi/abs/10.1142/2014, arXiv:https://www.worldscientific.com/doi/pdf/10.1142/2014, doi:10.1142/2014.

[103] G. Passarino and M. J. G. Veltman. One Loop Corrections for e^+e^- Annihilation Into mu+ mu- in the Weinberg Model. *Nucl. Phys. B*, 160:151–207, 1979. doi:10.1016/0550-3213(79)90234-7.

[104] George Leibbrandt. Introduction to Noncovariant Gauges. *Rev. Mod. Phys.*, 59:1067, 1987. doi:10.1103/RevModPhys.59.1067.

[105] J. Frenkel. A class of ghost free nonabelian gauge theories. *Phys. Rev. D*, 13:2325–2334, 1976. doi:10.1103/PhysRevD.13.2325.

[106] Stanley Mandelstam. Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model. *Nucl. Phys. B*, 213:149–168, 1983. doi:10.1016/0550-3213(83)90179-7.

[107] Lars Brink, Olof Lindgren, and Bengt E. W. Nilsson. N=4 Yang-Mills Theory on the Light Cone. *Nucl. Phys. B*, 212:401–412, 1983. doi:10.1016/0550-3213(83)90678-8.

[108] Lars Brink, Olof Lindgren, and Bengt E. W. Nilsson. The Ultraviolet Finiteness of the N=4 Yang-Mills Theory. *Phys. Lett. B*, 123:323–328, 1983. doi:10.1016/0370-2693(83)91210-8.

[109] Michael B. Green and John H. Schwarz. Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory. *Phys. Lett. B*, 149:117–122, 1984. doi:10.1016/0370-2693(84)91565-X.

[110] A Bassetto, G Nardelli, and R Soldati. *Yang-Mills Theories in Algebraic Non-Covariant Gauges.* WORLD SCIENTIFIC, 1991. URL: https://www.worldscientific.com/doi/abs/10.1142/1341, arXiv:https://www.worldscientific.com/doi/pdf/10.1142/1341, doi:10.1142/1341.

[111] N. N. Bogolyubov and D. V. Shirkov. *INTRODUCTION TO THE THEORY OF QUANTIZED FIELDS*, volume 3. 1959.

[112] George Leibbrandt. The Light Cone Gauge in Yang-Mills Theory. *Phys. Rev. D*, 29:1699, 1984. doi:10.1103/PhysRevD.29.1699.

[113] A. Bassetto, M. Dalbosco, and R. Soldati. Renormalization of the Yang-Mills Theories in the Light Cone Gauge. *Phys. Rev. D*, 36:3138, 1987. doi:10.1103/PhysRevD.36.3138.

[114] A. Bassetto, G. Nardelli, and R. Soldati. Local and Nonlocal Counterterms in Algebraic Noncovariant Gauges. *Mod. Phys. Lett. A*, 3:1663, 1988. doi:10.1142/S0217732388001999.

[115] Su-Long Nyeo. Yang-Mills Selfenergy in a Class of Linear Gauges. *Phys. Rev. D*, 36:2512, 1987. doi:10.1103/PhysRevD.36.2512.

[116] O. Piguet, G. Pollak, and M. Schweda. The Cancellation of Nonlocal Divergences in Light Cone Theories. *Nucl. Phys. B*, 328:527–544, 1989. doi:10.1016/0550-3213(89)90341-6.

[117] A. Boresch, O. Moritsch, M. Schweda, T. Sommer, H. Zerrouki, and S. Emery. *Applications of noncovariant gauges in the algebraic renormalization procedure*. WORLD SCIENTIFIC, 1988.

[118] W. Zimmermann. Convergence of Bogolyubov's method of renormalization in momentum space. *Commun. Math. Phys.*, 15:208–234, 1969. doi:10.1007/BF01645676.

[119] H. Balasin, M. Schweda, M. Stierle, and O. Piguet. The Cohomology Problems of Rigid Lorentz Transformations in Axial Gauge Theories. *Phys. Lett. B*, 215:328–330, 1988. doi:10.1016/0370-2693(88)91442-6.

[120] K. Landsteiner, M. Langer, M. Schweda, and S. P. Sorella. Interpolating gauge fixing for Chern-Simons theory. *Phys. Lett. B*, 337:294–302, 1994. arXiv:hep-th/9309111, doi:10.1016/0370-2693(94)90978-4.

[121] Danny Birmingham, Mark Rakowski, and George Thompson. Renormalization of Topological Field Theory. *Nucl. Phys. B*, 329:83–97, 1990. doi:10.1016/0550-3213(90)90058-L.

[122] Danny Birmingham and Mark Rakowski. Vector supersymmetry in topological field theory. *Phys. Lett. B*, 269:103–108, 1991. doi:10.1016/0370-2693(91)91459-9.

[123] Claudio Lucchesi, Olivier Piguet, and Silvio Paolo Sorella. Renormalization and finiteness of topological BF theories. *Nucl. Phys. B*, 395:325–353, 1993. arXiv:hep-th/9208047, doi:10.1016/0550-3213(93)90219-F.

[124] F. Delduc, F. Gieres, and S. P. Sorella. Supersymmetry of the $d = 3$ Chern-Simons Action in the Landau Gauge. *Phys. Lett. B*, 225:367–370, 1989. doi:10.1016/0370-2693(89)90584-4.

[125] Claudio Lucchesi and Olivier Piguet. Local supersymmetry of the Chern-Simons theory and finiteness. *Nucl. Phys. B*, 381:281–300, 1992. doi:10.1016/0550-3213(92)90648-U.

[126] A. Brandhuber, M. Langer, M. Schweda, O. Piguet, and S. P. Sorella. A Short comment on the supersymmetric structure of Chern-Simons theory in the axial gauge. *Phys. Lett. B*, 300:92–95, 1993. arXiv:hep-th/9212112, doi:10.1016/0370-2693(93)90753-5.

[127] T. Gajdosik and W. Kummer. Finiteness of Chern-Simons theory for noncovariant gauges. *Phys. Rev. D*, 48:2897–2904, 1993. doi:10.1103/PhysRevD.48.2897.

[128] A. Brandhuber, M. Langer, M. Schweda, S. P. Sorella, S. Emery, and O. Piguet. Symmetries of the Chern-Simons theory in the axial gauge. *Helv. Phys. Acta*, 66:551–566, 1993. arXiv:hep-th/9305132.

[129] Daniel O. R. Azevedo, Oswaldo M. Del Cima, Thadeu S. Dias, and Emílio D. Pereira. Interpolating gauge-fixing for Yang–Mills–Chern–Simons theory in $D = 3$. *Eur. Phys.*

J. C, 84(9):917, 2024. arXiv:2406.09515, doi:10.1140/epjc/s10052-024-13279-3.

[130] Noboru Nakanishi. Covariant Quantization of the Electromagnetic Field in the Landau Gauge. *Prog. Theor. Phys.*, 35:1111–1116, 1966. doi:10.1143/PTP.35.1111.

[131] B. Lautrup. Canonical quantum electrodynamics in covariant gauges. 1967.

[132] V. N. Gribov. Quantization of Nonabelian Gauge Theories. *Nucl. Phys. B*, 139:1, 1978. doi:10.1016/0550-3213(78)90175-X.

[133] I. M. Singer. Some Remarks on the Gribov Ambiguity. *Commun. Math. Phys.*, 60:7–12, 1978. doi:10.1007/BF01609471.

[134] Nele Vandersickel. *A Study of the Gribov-Zwanziger action: from propagators to glueballs.* PhD thesis, 2011. arXiv:1104.1315.

[135] Antonio Duarte Pereira. *Exploring new horizons of the Gribov problem in Yang-Mills theories.* PhD thesis, Niteroi, Fluminense U., 2016. arXiv:1607.00365.

[136] R. F. Sobreiro and S. P. Sorella. Introduction to the Gribov ambiguities in Euclidean Yang-Mills theories. In *13th Jorge Andre Swieca Summer School on Particle and Fields*, 4 2005. arXiv:hep-th/0504095.

[137] N. Vandersickel and Daniel Zwanziger. The Gribov problem and QCD dynamics. *Phys. Rept.*, 520:175–251, 2012. arXiv:1202.1491, doi:10.1016/j.physrep.2012.07.003.

[138] Daniel Zwanziger. Nonperturbative Modification of the Faddeev-popov Formula and Banishment of the Naive Vacuum. *Nucl. Phys. B*, 209:336–348, 1982. doi:10.1016/0550-3213(82)90260-7.

[139] G. Dell'Antonio and D. Zwanziger. Ellipsoidal Bound on the Gribov Horizon Contradicts the Perturbative Renormalization Group. *Nucl. Phys. B*, 326:333–350, 1989. doi:10.1016/0550-3213(89)90135-1.

[140] G. Dell'Antonio and D. Zwanziger. Every gauge orbit passes inside the Gribov horizon. *Commun. Math. Phys.*, 138:291–299, 1991. `doi:10.1007/BF02099494`.

[141] Pierre van Baal. More (thoughts on) Gribov copies. *Nucl. Phys. B*, 369:259–275, 1992. `doi:10.1016/0550-3213(92)90386-P`.

[142] D. Zwanziger. Local and Renormalizable Action From the Gribov Horizon. *Nucl. Phys. B*, 323:513–544, 1989. `doi:10.1016/0550-3213(89)90122-3`.

[143] Daniel Zwanziger. Action From the Gribov Horizon. *Nucl. Phys. B*, 321:591–604, 1989. `doi:10.1016/0550-3213(89)90263-0`.

[144] M. A. L. Capri, D. Dudal, M. S. Guimaraes, L. F. Palhares, and S. P. Sorella. An all-order proof of the equivalence between Gribov's no-pole and Zwanziger's horizon conditions. *Phys. Lett. B*, 719:448–453, 2013. `arXiv:1212.2419`, `doi:10.1016/j.physletb.2013.01.039`.

[145] Attilio Cucchieri, Tereza Mendes, and Andre R. Taurines. Positivity violation for the lattice Landau gluon propagator. *Phys. Rev. D*, 71:051902, 2005. `arXiv:hep-lat/0406020`, `doi:10.1103/PhysRevD.71.051902`.

[146] A. Sternbeck, E. M. Ilgenfritz, M. Muller-Preussker, and A. Schiller. The Gluon and ghost propagator and the influence of Gribov copies. *Nucl. Phys. B Proc. Suppl.*, 140:653–655, 2005. `arXiv:hep-lat/0409125`, `doi:10.1016/j.nuclphysbps.2004.11.154`.

[147] Lorenz von Smekal, Reinhard Alkofer, and Andreas Hauck. The Infrared behavior of gluon and ghost propagators in Landau gauge QCD. *Phys. Rev. Lett.*, 79:3591–3594, 1997. `arXiv:hep-ph/9705242`, `doi:10.1103/PhysRevLett.79.3591`.

[148] Reinhard Alkofer and Lorenz von Smekal. The Infrared behavior of QCD Green's functions: Confinement dynamical symmetry breaking,

and hadrons as relativistic bound states. *Phys. Rept.*, 353:281, 2001. arXiv:hep-ph/0007355, doi:10.1016/S0370-1573(01)00010-2.

[149] Christoph Lerche and Lorenz von Smekal. On the infrared exponent for gluon and ghost propagation in Landau gauge QCD. *Phys. Rev. D*, 65:125006, 2002. arXiv:hep-ph/0202194, doi:10.1103/PhysRevD.65.125006.

[150] Reinhard Alkofer, W. Detmold, C. S. Fischer, and P. Maris. Analytic properties of the Landau gauge gluon and quark propagators. *Phys. Rev. D*, 70:014014, 2004. arXiv:hep-ph/0309077, doi:10.1103/PhysRevD.70.014014.

[151] Jan M. Pawlowski, Daniel F. Litim, Sergei Nedelko, and Lorenz von Smekal. Infrared behavior and fixed points in Landau gauge QCD. *Phys. Rev. Lett.*, 93:152002, 2004. arXiv:hep-th/0312324, doi:10.1103/PhysRevLett.93.152002.

[152] Attilio Cucchieri and Tereza Mendes. What's up with IR gluon and ghost propagators in Landau gauge? A puzzling answer from huge lattices. *PoS*, LATTICE2007:297, 2007. arXiv:0710.0412, doi:10.22323/1.042.0297.

[153] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck. The Landau gauge gluon and ghost propagators in 4D SU(3) gluodynamics in large lattice volumes. *PoS*, LATTICE2007:290, 2007. arXiv:0710.1968, doi:10.22323/1.042.0290.

[154] A. Cucchieri and T. Mendes. Constraints on the IR behavior of the gluon propagator in Yang-Mills theories. *Phys. Rev. Lett.*, 100:241601, 2008. arXiv:0712.3517, doi:10.1103/PhysRevLett.100.241601.

[155] Attilio Cucchieri and Tereza Mendes. Constraints on the IR behavior of the ghost propagator in Yang-Mills theories. *Phys. Rev. D*, 78:094503, 2008. arXiv:0804.2371, doi:10.1103/PhysRevD.78.094503.

[156] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck. Lattice gluodynamics computation of Landau gauge Green's functions in the deep infrared. *Phys. Lett. B*, 676:69–73, 2009. arXiv:0901.0736, doi:10.1016/j.physletb.2009.04.076.

[157] Attilio Cucchieri and Tereza Mendes. Landau-gauge propagators in Yang-Mills theories at $\beta = 0$: Massive solution versus conformal scaling. *Phys. Rev. D*, 81:016005, 2010. arXiv:0904.4033, doi:10.1103/PhysRevD.81.016005.

[158] A. C. Aguilar, D. Binosi, and J. Papavassiliou. Gluon and ghost propagators in the Landau gauge: Deriving lattice results from Schwinger-Dyson equations. *Phys. Rev. D*, 78:025010, 2008. arXiv:0802.1870, doi:10.1103/PhysRevD.78.025010.

[159] Philippe Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero. On the IR behaviour of the Landau-gauge ghost propagator. *JHEP*, 06:099, 2008. arXiv:0803.2161, doi:10.1088/1126-6708/2008/06/099.

[160] Christian S. Fischer, Axel Maas, and Jan M. Pawlowski. On the infrared behavior of Landau gauge Yang-Mills theory. *Annals Phys.*, 324:2408–2437, 2009. arXiv:0810.1987, doi:10.1016/j.aop.2009.07.009.

[161] Mikhail A. Shifman, A. I. Vainshtein, and Valentin I. Zakharov. QCD and Resonance Physics. Theoretical Foundations. *Nucl. Phys. B*, 147:385–447, 1979. doi:10.1016/0550-3213(79)90022-1.

[162] Mikhail A. Shifman, A. I. Vainshtein, and Valentin I. Zakharov. QCD and Resonance Physics: Applications. *Nucl. Phys. B*, 147:448–518, 1979. doi:10.1016/0550-3213(79)90023-3.

[163] G. Burgio, F. Di Renzo, G. Marchesini, and E. Onofri. Λ^2 contribution to the condensate in lattice gauge theory. *Phys. Lett. B*, 422:219–226, 1998. arXiv:hep-ph/9706209, doi:10.1016/S0370-2693(98)00057-4.

[164] Valentin I. Zakharov. Gluon condensate and beyond. *Int. J. Mod. Phys. A*, 14:4865–4880, 1999. arXiv:hep-ph/9906264, doi:10.1142/S0217751X9900230X.

[165] F. V. Gubarev, Leo Stodolsky, and Valentin I. Zakharov. On the significance of the vector potential squared. *Phys. Rev. Lett.*, 86:2220–2222, 2001. arXiv:hep-ph/0010057, doi:10.1103/PhysRevLett.86.2220.

[166] F. V. Gubarev and Valentin I. Zakharov. On the emerging phenomenology of $\langle {A_{min}}^2 \rangle$. *Phys. Lett. B*, 501:28–36, 2001. arXiv:hep-ph/0010096, doi:10.1016/S0370-2693(01)00085-5.

[167] Philippe Boucaud et al. Lattice calculation of $1/p^2$ corrections to α_s and of Λ_{QCD} in the $M\tilde{O}M$ scheme. *JHEP*, 04:006, 2000. arXiv:hep-ph/0003020, doi:10.1088/1126-6708/2000/04/006.

[168] KeiIchi Kondo. Vacuum condensate of mass dimension 2 as the origin of mass gap and quark confinement. *Phys. Lett. B*, 514:335–345, 2001. arXiv:hep-th/0105299, doi:10.1016/S0370-2693(01)00817-6.

[169] Philippe Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, F. De Soto, A. Donini, H. Moutarde, and J. Rodriguez-Quintero. Instantons and $\langle A^2 \rangle$ condensate. *Phys. Rev. D*, 66:034504, 2002. arXiv:hep-ph/0203119, doi:10.1103/PhysRevD.66.034504.

[170] D. Dudal, H. Verschelde, and S. P. Sorella. The Anomalous dimension of the composite operatorA^2 in the Landau gauge. *Phys. Lett. B*, 555:126–131, 2003. arXiv:hep-th/0212182, doi:10.1016/S0370-2693(03)00043-1.

[171] Xiangdong Li and C. M. Shakin. Description of gluon propagation in the presence of an A^2 condensate. *Phys. Rev. D*, 71:074007, 2005. arXiv:hep-ph/0410404, doi:10.1103/PhysRevD.71.074007.

[172] D. Dudal, R. F. Sobreiro, S. P. Sorella, and H. Verschelde. The Gribov parameter and the dimension two gluon condensate in Euclidean Yang-

Mills theories in the Landau gauge. *Phys. Rev. D*, 72:014016, 2005. `arXiv:hep-th/0502183`, `doi:10.1103/PhysRevD.72.014016`.

[173] H. Verschelde, K. Knecht, K. Van Acoleyen, and M. Vanderkelen. The Nonperturbative groundstate of QCD and the local composite operator ${A_\mu}^2$. *Phys. Lett. B*, 516:307–313, 2001. `arXiv:hep-th/0105018`, `doi:10.1016/S0370-2693(01)00929-7`.

[174] R. E. Browne and J. A. Gracey. Two loop effective potential for $\langle {A_\mu}^2 \rangle$ in the Landau gauge in quantum chromodynamics. *JHEP*, 11:029, 2003. `arXiv:hep-th/0306200`, `doi:10.1088/1126-6708/2003/11/029`.

[175] D. Dudal, H. Verschelde, J. A. Gracey, V. E. R. Lemes, M. S. Sarandy, R. F. Sobreiro, and S. P. Sorella. Dynamical gluon mass generation from $\langle {A_\mu}^2 \rangle$ in linear covariant gauges. *JHEP*, 01:044, 2004. `arXiv:hep-th/0311194`, `doi:10.1088/1126-6708/2004/01/044`.

[176] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde. New features of the gluon and ghost propagator in the infrared region from the Gribov-Zwanziger approach. *Phys. Rev. D*, 77:071501, 2008. `arXiv:0711.4496`, `doi:10.1103/PhysRevD.77.071501`.

[177] David Dudal, John A. Gracey, Silvio Paolo Sorella, Nele Vandersickel, and Henri Verschelde. A Refinement of the Gribov-Zwanziger approach in the Landau gauge: Infrared propagators in harmony with the lattice results. *Phys. Rev. D*, 78:065047, 2008. `arXiv:0806.4348`, `doi:10.1103/PhysRevD.78.065047`.

[178] D. Dudal, S. P. Sorella, and N. Vandersickel. The dynamical origin of the refinement of the Gribov-Zwanziger theory. *Phys. Rev. D*, 84:065039, 2011. `arXiv:1105.3371`, `doi:10.1103/PhysRevD.84.065039`.

[179] J. A. Gracey. Alternative refined Gribov-Zwanziger Lagrangian. *Phys. Rev. D*, 82:085032, 2010. `arXiv:1009.3889`, `doi:10.1103/PhysRevD.82.085032`.

[180] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde. The Effects of Gribov copies in 2D gauge theories. *Phys. Lett. B*, 680:377–383, 2009. arXiv:0808.3379, doi:10.1016/j.physletb.2009.08.055.

[181] D. Dudal, O. Oliveira, and N. Vandersickel. Indirect lattice evidence for the Refined Gribov-Zwanziger formalism and the gluon condensate $\langle A^2 \rangle$ in the Landau gauge. *Phys. Rev. D*, 81:074505, 2010. arXiv:1002.2374, doi:10.1103/PhysRevD.81.074505.

[182] Daniel Zwanziger. Renormalizability of the critical limit of lattice gauge theory by BRS invariance. *Nucl. Phys. B*, 399:477–513, 1993. doi:10.1016/0550-3213(93)90506-K.

[183] L. Baulieu and S. P. Sorella. Soft breaking of BRST invariance for introducing non-perturbative infrared effects in a local and renormalizable way. *Phys. Lett. B*, 671:481–485, 2009. arXiv:0808.1356, doi:10.1016/j.physletb.2008.11.036.

[184] Daniel Zwanziger. Quantization of Gauge Fields, Classical Gauge Invariance and Gluon Confinement. *Nucl. Phys. B*, 345:461–471, 1990. doi:10.1016/0550-3213(90)90396-U.

[185] Martin Lavelle and David McMullan. Constituent quarks from QCD. *Phys. Rept.*, 279:1–65, 1997. arXiv:hep-ph/9509344, doi:10.1016/S0370-1573(96)00019-1.

[186] M. A. L. Capri, D. Dudal, D. Fiorentini, M. S. Guimaraes, I. F. Justo, A. D. Pereira, B. W. Mintz, L. F. Palhares, R. F. Sobreiro, and S. P. Sorella. Exact nilpotent nonperturbative BRST symmetry for the Gribov-Zwanziger action in the linear covariant gauge. *Phys. Rev. D*, 92(4):045039, 2015. arXiv:1506.06995, doi:10.1103/PhysRevD.92.045039.

[187] M. A. L. Capri, D. Dudal, D. Fiorentini, M. S. Guimaraes, I. F. Justo, A. D. Pereira, B. W. Mintz, L. F. Palhares, R. F. Sobreiro, and S. P.

Sorella. Local and BRST-invariant Yang-Mills theory within the Gribov horizon. *Phys. Rev. D*, 94(2):025035, 2016. arXiv:1605.02610, doi:10.1103/PhysRevD.94.025035.

[188] M. A. L. Capri, D. Fiorentini, A. D. Pereira, R. F. Sobreiro, S. P. Sorella, and R. C. Terin. Aspects of the refined Gribov-Zwanziger action in linear covariant gauges. *Annals Phys.*, 376:40–62, 2017. arXiv:1607.07912, doi:10.1016/j.aop.2016.10.023.

[189] M. A. L. Capri, D. Fiorentini, A. D. Pereira, and S. P. Sorella. Renormalizability of the refined Gribov-Zwanziger action in linear covariant gauges. *Phys. Rev. D*, 96(5):054022, 2017. arXiv:1708.01543, doi:10.1103/PhysRevD.96.054022.

[190] Zhu-Fang Cui, Jin-Li Zhang, Daniele Binosi, Feliciano de Soto, Cédric Mezrag, Joannis Papavassiliou, Craig D. Roberts, Jose Rodríguez-Quintero, Jorge Segovia, and Savvas Zafeiropoulos. Effective charge from lattice QCD. *Chin. Phys. C*, 44(8):083102, 2020. arXiv:1912.08232, doi:10.1088/1674-1137/44/8/083102.

[191] Alexandre Deur, Stanley J. Brodsky, and Craig D. Roberts. QCD running couplings and effective charges. *Prog. Part. Nucl. Phys.*, 134:104081, 2024. arXiv:2303.00723, doi:10.1016/j.ppnp.2023.104081.

[192] Marcela Peláez, Urko Reinosa, Julien Serreau, Matthieu Tissier, and Nicolás Wschebor. A window on infrared QCD with small expansion parameters. *Rept. Prog. Phys.*, 84(12):124202, 2021. arXiv:2106.04526, doi:10.1088/1361-6633/ac36b8.

[193] Daniel O. R. Azevedo and Antonio D. Pereira. Finiteness of the Yang-Mills-Chern-Simons action in linear covariant gauges by taking into account gauge copies. 2025. arXiv:2502.03284.

[194] K. Knecht and H. Verschelde. A New start for local composite operators. *Phys. Rev. D*, 64:085006, 2001. arXiv:hep-th/0104007, doi:10.1103/PhysRevD.64.085006.

[195] K. Symanzik. Renormalizable models with simple symmetry breaking. 1. Symmetry breaking by a source term. *Commun. Math. Phys.*, 16:48–80, 1970. doi:10.1007/BF01645494.

[196] Olivier Piguet and Klaus Sibold. Gauge Independence in Ordinary Yang-Mills Theories. *Nucl. Phys. B*, 253:517–540, 1985. doi:10.1016/0550-3213(85)90545-0.

[197] M. A. L. Capri, D. Dudal, A. D. Pereira, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, and S. P. Sorella. Non-perturbative aspects of Euclidean Yang-Mills theories in linear covariant gauges: Nielsen identities and a BRST-invariant two-point correlation function. *Phys. Rev. D*, 95(4):045011, 2017. arXiv:1611.10077, doi:10.1103/PhysRevD.95.045011.

[198] M. A. L. Capri, M. S. Guimaraes, I. Justo, L. F. Palhares, and S. P. Sorella. On general ultraviolet properties of a class of confining propagators. *Eur. Phys. J. C*, 76(3):141, 2016. arXiv:1510.07886, doi:10.1140/epjc/s10052-016-3974-3.

[199] M. A. L. Capri, D. Dudal, M. S. Guimaraes, A. D. Pereira, B. W. Mintz, L. F. Palhares, and S. P. Sorella. The universal character of Zwanziger's horizon function in Euclidean Yang–Mills theories. *Phys. Lett. B*, 781:48–54, 2018. arXiv:1802.04582, doi:10.1016/j.physletb.2018.03.058.