

# Stable maps from surfaces to the plane with prescribed branching data

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## Abstract

We consider the problem of constructing stable maps from surfaces to the plane with branch set a given set of curves immersed (except possibly with cusps) in the plane. Various constructions are used (1) piecing together regions immersed in the plane (2) modifying an existing stable map by a sequence of codimension one transitions (swallowtails etc) or by surgeries. In (1) the way the regions are pieced together is described by a bipartite graph (an edge  $C^*$  corresponds to a branch curve  $C$  with the vertices of  $C^*$  corresponding to the two regions containing  $C$ ). We show that any bipartite graph may be realized by a stable map and we consider the question of realizing graphs by fold maps (i.e. maps without cusps). For example, using Arnol'd's classification of immersed curves, we list all branch sets with at most two branch curves and four double points realizable by planar fold maps of the torus.

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## 1. Introduction

By a well-known theorem of Whitney, the singular set of any stable smooth map from a closed surface to the plane consists of curves of double points, possibly containing isolated cusp points. The branch set (i.e. the image of the singular set) consists of a number of immersed curves in the plane (possibly with cusps) whose self-intersections are all transverse and disjoint from the cusps (if any). Stable maps possess various topological invariants, the most important being the graph (as defined in [8]), which describes the position of the singular set in the surface, and the branching data, i.e. the restriction of the map to the singular set. In this work we address the question of characterizing branching data realizable by stable maps. In [8,9] necessary and sufficient conditions were found for the realizability of certain graphs by stable maps, with particular emphasis on the simplest case where the surface is a sphere, the graph in this case being (necessarily) a tree. We consider the question of realizability of more general graphs and the related question of the calculation of invariants of the branch set, in particular the local invariants which are characterized [14]

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by their behaviour at codimension one transitions, as classified by J. Rieger in [15] (for the multilocal classification see [5]).

Stable maps may be constructed in various ways. One way is to immerse the surface in 3-space and then project it into the plane. Or one may modify a given stable map by a (generic) homotopy consisting of a sequence of codimension one transitions: tangencies, swallowtails, beaks etc. Alternatively, one may perform surgery on a given map to obtain a new map (see Section 3). In Section 4, surgeries are used to construct stable maps with prescribed graphs. We show (Theorem 4.1) that any bipartite graph (with arbitrarily weighted vertices) is the graph of a stable map from an orientable closed surface to the plane. Surgeries are useful in calculating local invariants (rather than via homotopy). The relevant formulas are given in Section 5.

In Section 6 we consider fold maps (i.e. stable maps without cusps). Such maps and their higher dimensional analogues have been studied by various authors [1,2,6,16,17]. Fold maps of surfaces are closely related to immersions of surfaces (necessarily with non empty boundary) into the plane. These are classified, up to image isotopy, in [10,18]. In [9] graphs of fold maps of the sphere were characterized. In Section 6 this characterization is extended to planar fold maps of closed orientable surfaces. Such maps can be obtained by suitable assembly of immersed planar regions in the plane. This enables us to list all possible planar fold maps of the torus with  $\mu \leq 2$  and  $D \leq 4$  (Theorem 6.3).

## 2. Stable maps and their invariants

We first recall some definitions and basic results. Two smooth maps  $f$  and  $g$  from an orientable surface  $M$  to the plane are said to be *topologically equivalent* if there are orientation-preserving diffeomorphisms,  $l$  and  $k$ , such that  $g \circ l = k \circ f$ . The maps  $f$  and  $g$  are *isotopic* if both the above diffeomorphisms are isotopic to the identity. A map  $f$  is said to be *stable* if all maps sufficiently close to  $f$  (in the Whitney  $C^\infty$ -topology) are isotopic to  $f$ .

A point of the surface is *non singular* if the map is injective in a neighborhood of that point and is *singular* otherwise. Denote by  $\Sigma f$  the singular set of a map  $f$ . Its image  $Bf = f(\Sigma f)$  is the *branch set* of  $f$ . By Whitney's theorem, the singularities of any stable map consist of curves of fold points possibly containing (isolated) cusp points (see [7]). Except at the cusp points (if any), the singular set is immersed into the plane with at most finitely many transverse intersection points, none of which are cuspidal. The branch set is oriented as follows: as we traverse a branch curve following the orientation, nearby points on our left have two more inverse images than those on our right. The non-singular set (which is immersed into the plane by the map) consists of finitely many *regions*. Given orientations of the surface and the plane, a region is *positive* if the map preserves orientation and *negative* otherwise. The singular set is the frontier of each half of the surface (positive or negative) i.e. any singular curve lies in the frontier of a positive and a negative region.

The singular sets of two equivalent maps are equivalent in the sense that there is a diffeomorphism of the surface carrying one singular set onto the other and similarly for the branch sets. Thus any diffeomorphism invariant of singular sets or of branch sets will automatically be a topological invariant of the map. Clearly, the number of connected components of the singular set as well as the topological types of the regions are topological invariants. This information may be conveniently encoded in a weighted graph from which the pair  $M, \Sigma f$  may be reconstructed (up to diffeomorphism) [8]. The edges and vertices of the graph correspond (respectively) to the singular curves and the regions (i.e. the connected components of the non-singular set). An edge is incident to a vertex if and only if the singular curve corresponding to the edge lies in the frontier of the region corresponding to the vertex. Since each singular curve lies in the frontier of a positive and a negative region, the graph is *bipartite*. The *weight*  $g_v$  of a vertex  $v$  is defined to be the genus of the corresponding region i.e. the genus of the closed surface obtained by adding a disk to each boundary curve. Fig. 1 shows a stable map and its weighted graph.

## 3. Surgery of stable maps

One way of constructing a stable map is to glue together two stable maps. In particular, in a surgery, a pair of disjoint disks in the surface is removed and replaced by a tube, the map then being extended over the interior of the tube. There are two types of surgery: horizontal and vertical. Horizontal surgery is a generalization of Ohmoto's connected sum [14]. Given a stable map  $h$ , a *bridge* is an embedded arc  $\beta$  in the plane which meets the branch set  $Bh$  in its two end points (and nowhere else) compatibly with the orientation of the branch set as shown in Fig. 2(a) [13]. The stable map  $h_\beta$  is constructed as follows. The bridge meets  $h(M)$  in its end points,  $h(p)$  and  $h(q)$ , say. As

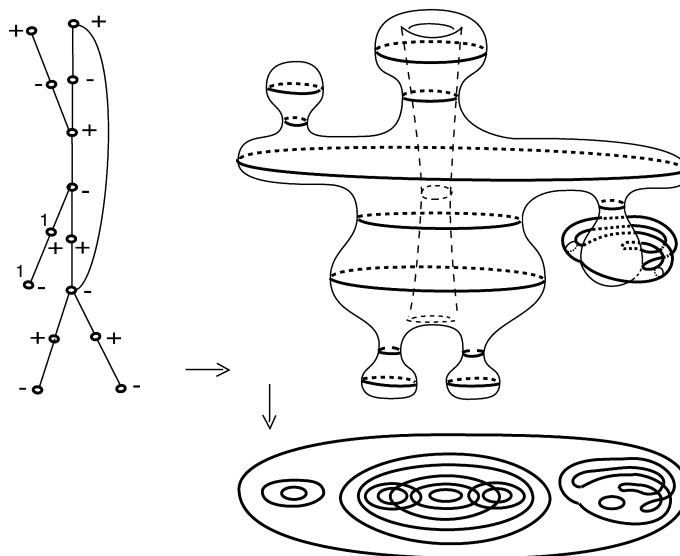


Fig. 1. Stable map and its weighted graph.

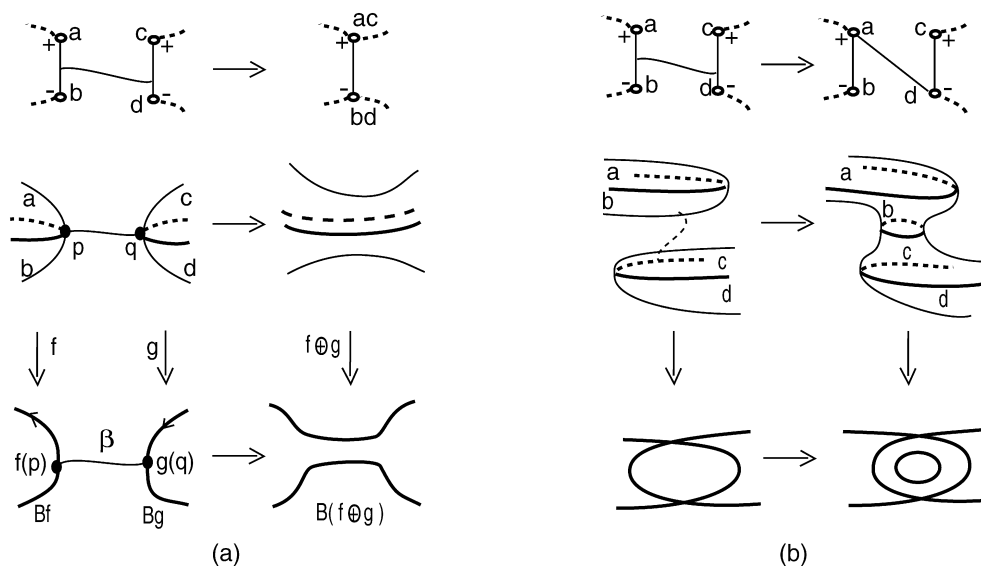


Fig. 2. Surgeries: (a) horizontal, (b) vertical.

in [14], choose small disks in  $M$  centered at  $p$  and  $q$  and replace their interiors by a tube (i.e. an annulus), respecting the orientation of  $M$ , so as to obtain an oriented surface. As illustrated in Fig. 2(a), the map  $h$  may then be extended over the tube to give the required stable map  $h_\beta$ . In particular, if  $M$  is the disjoint union of surfaces  $P$  and  $Q$  and  $f$  and  $g$  denote the restrictions of  $h$  to  $P$  and to  $Q$ , with  $p \in P$  and  $q \in Q$  then we obtain the *connected sum*  $f \oplus_\beta g$  (or simply  $f \oplus g$ ). In other words  $h = f \cup g$  and  $(f \cup g)_\beta = f \oplus g$ .

Fig. 2(b) illustrates a vertical surgery. Here there are two non-singular disks, one positive and one negative (as in the figure) whose images (in the plane) coincide. The disks are replaced by a tube which is mapped into the plane, with a singular curve running around the middle of the tube. Thus the surgery adds a disjoint embedded curve to the branch set. It is possible also to perform vertical surgery using a bridge, but this will not be needed here. Observe that both types of surgery increase or decrease the value of  $\mu$  by one.

#### 4. Realizing graphs with arbitrary weights

We next turn to the question of which weighted graphs can be realized by stable maps. A basic building block is a certain stable map of the torus to the plane with just one singular curve. This singular curve necessarily contains cusps (in fact two of them) and separates the torus into two regions, one of which is (must be) a disk. The map in question may be constructed by means of the map of the annulus to the plane given by

$$(r, (\cos r) \sin 2\theta + (\sin r) \cos \theta),$$

where  $\pi/2 \leq r \leq 3\pi/2$ ,  $r$  being the radial coordinate of the annulus and  $\theta$  the circular coordinate. Alternatively, as in Fig. 3, a pair of swallowtails may be introduced into the usual projection of the torus (to the plane) and then two of the resulting four cusps may be cancelled by a beaks transition. Or one may use the stable map of the sphere (as defined in [11, p. 154]) which has two singular curves, each containing one cusp point, see Fig. 4.

Fig. 5 illustrates the images of two curves (on either side of the singular curve), one bounding an immersed punctured torus, the other an immersed disk.

**Theorem 4.1.** *Any bipartite connected graph (with arbitrarily weighted vertices) is the graph of a stable map.*

**Proof.** As observed above, the graph of a stable map is necessarily bipartite since, along every singular curve, a positive region meets a negative region. Conversely, in [8] it was shown that any tree is realizable by a stable map of the sphere (so that all the regions are planar i.e. have weight zero). The stable maps in question were constructed by a sequence of lips transitions. Consider next a (connected) bipartite graph (with all weights zero). Choose a tree containing all the vertices of the graph. Realize the tree by a stable map (with all weights zero). The construction is completed by showing that for any stable map there exists a stable map whose graph is the graph of the original map plus an edge joining a pair of vertices of opposite signs. Thus, given a stable map and two regions of opposite signs,

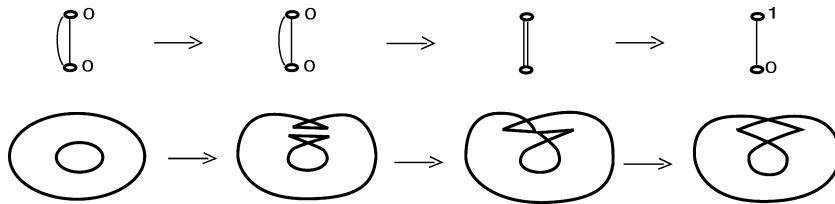


Fig. 3. Two-cusp transition in the torus.

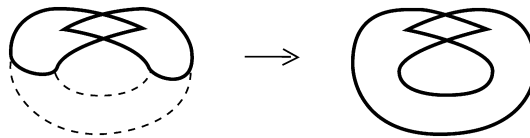


Fig. 4. Branch sets of stable maps with two cusps.

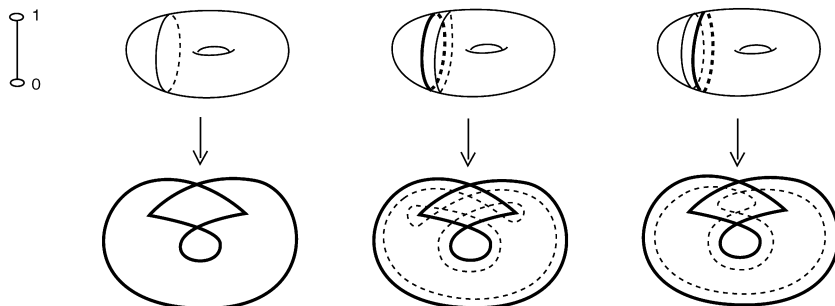


Fig. 5. Stable map of the torus with  $\mu = 1$ .

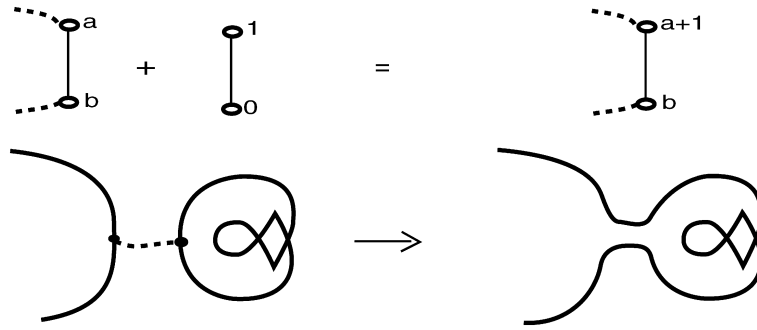


Fig. 6. Two-cusp surgery and its effect on the graph.

first perform a homotopy which does not alter the graph and after which the images (in the plane) of the two regions intersect. We may now choose a pair of disks, one in each region, whose images coincide. Using these disks, we perform a vertical surgery whose effect is to add the desired edge to the original graph. Repeating this procedure realizes the graph, with all weights zero since all the regions constructed are planar. We may now realize an arbitrary set of weights without altering the graph, since the genus of any given region may be increased by one (without altering the graph or the genus of the remaining regions) by means of connected sum with the stable map of the torus described above, Fig. 6. By a suitable sequence of such connected sums, any set of weights may be realized without altering the graph.  $\square$

## 5. Calculation of branch set invariants via surgery

There are three obvious topological invariants of the branch set:  $\mu$  (the number of singular curves),  $C$  (the number of cusps) and  $D$  (the number of double points). In [14] another invariant,  $BL(f)$ , was defined for stable maps  $f$  in terms of the self-linking of a certain Legendrian link associated to the branch set. The invariant  $BL$  is *local* in that it is characterized (up to an additive constant  $c_M$ ) by its behaviour at transitions. It is convenient to set

$$F(f) = BL(f) - BL(f_0),$$

where  $f_0$  is the standard projection of  $M$  (embedded in 3-space) into the plane. For a sum  $f \oplus g$ , where the images of  $f$  and  $g$  are separated by a line in the plane, one has (by [14])

$$BL(f \oplus g) = BL(f) + BL(g) - 2.$$

Clearly  $(f \oplus g)_0 = f_0 \oplus g_0$  so that  $F$  is additive, in other words

$$F(f \oplus g) = F(f) + F(g).$$

According to [14],  $\Delta F = \Delta D$  at a reverse tangency,  $\Delta F = -\Delta D$  at a direct tangency,  $\Delta F = \frac{1}{2}\Delta C$  for both beaks and lips transitions and  $\Delta F = 0$  for all other transitions. Recall that, since the space of maps into the plane is contractible, any two stable maps  $f$  and  $g$  may be joined by a homotopy which is generic, in the sense that it does not pass through any stratum of codimension two or more and meets the codimension one strata transversely (the intersections corresponding to the transitions). The transitions (see Fig. 7) were classified by Rieger [15] and Chínaro [5] (see also [14]).

$C$  and  $D$  are also local invariants whose jumps at transitions (i.e.  $\Delta C$  and  $\Delta D$ ) are easily calculated. On the other hand,  $\mu$  is not local since its jump at a transition depends not only on the transition in question but also on the rest of the map. Clearly, any surgery leaves  $C$  and  $D$  unaltered and its effect on  $\mu$  is easily calculated. Its effect on  $F$  is more subtle and depends on the position of the bridge.

**Lemma 5.1.** *If  $f \oplus g$  is a horizontal sum of  $f$  and  $g$  where the bridge (apart from its endpoints) lies in the unbounded component of the complement of the branch set of  $f \cup g$  then*

$$F(f \oplus g) = F(f \cup g).$$

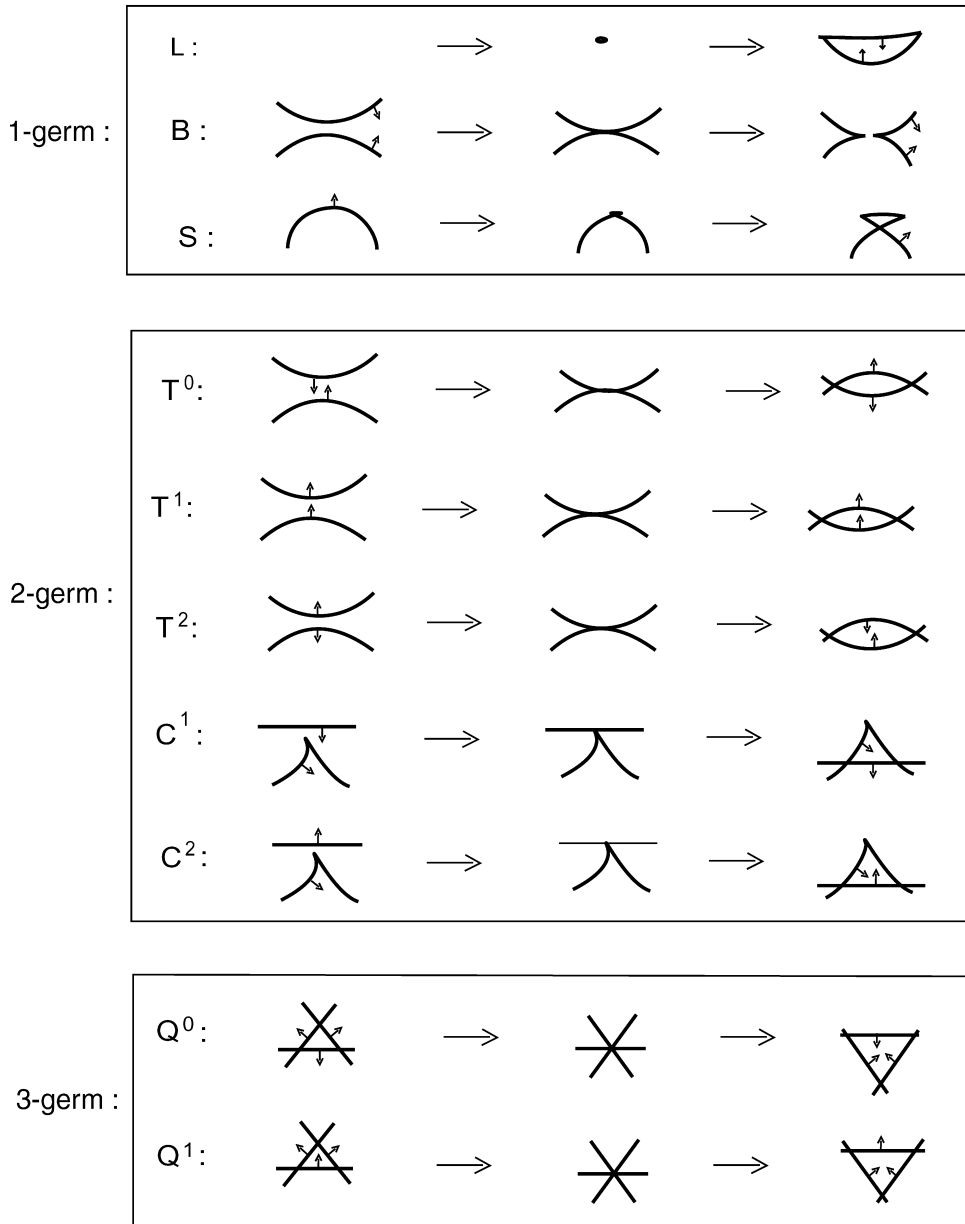


Fig. 7. Local transitions of codimension one.

**Proof.** By the hypothesis on the bridge, there is a (translation) homotopy  $g_t$  between  $g$  and  $h$  (say) where the images of  $h$  and  $f$  are separated by a line in the plane, together with a family of bridges  $\beta_t$  for  $f \cup g_t$ . Thus there is a homotopy between  $f \oplus g$  and  $f \oplus h$  where the transitions occur at points away from the connecting tube. Consequently

$$F(f \oplus h) - F(f \oplus g) = F(f \cup h) - F(f \cup g).$$

Since  $F(f \oplus h) = F(f \cup h)$ , the lemma follows.  $\square$

More generally, one has

**Theorem 5.2.** Let  $W(f \cup g)$  be the sum of the winding numbers of the branch curves of  $f \cup g$  (oriented by the usual convention) relative to any point in the interior of the bridge. Then

$$F(f \oplus g) = F(f \cup g) - 4W(f \cup g).$$

**Proof.** As in Fig. 8, the bridge  $\beta$  may be pulled into the unbounded component of  $B(f \cup g)$  by a homotopy  $f_t \cup g_t$ , thus obtaining a second bridge ( $\gamma$ ) between  $f_1$  and  $g_1$ . During this homotopy the only transitions are  $d$  direct and  $r$  reverse tangencies. Thus  $F(f_1 \cup g_1) - F(f \cup g) = 4(r - d)$ . From the orientation convention it follows that  $W(f \cup g) = d - r$ . Now  $F(f_1 \oplus g_1) - F(f \oplus g) = 0$  because there are as many direct as indirect tangencies during the homotopy between  $f \oplus g$  and  $f_1 \oplus g_1$ . By Lemma 5.1,  $F(f_1 \cup g_1) = F(f_1 \oplus g_1)$ . Thus  $F(f \oplus g) = F(f \cup g) - 4W(f \cup g)$ .  $\square$

Figs. 9 and 10 illustrate how to calculate the increments in the invariants via homotopy and surgery, in particular the increment of  $F$ .

The possible branch sets for stable maps of the sphere for which  $\mu \leq 3$ ,  $D \leq 4$  and  $C = 0$  where determined in [9]. Fig. 11 shows how to obtain all such maps with  $\mu = 1$  as horizontal sums of stable maps of the sphere without cusps. In the figure, dotted lines indicate bridges. The invariant  $F$  on each one of these maps is easily calculated using Theorem 5.2.

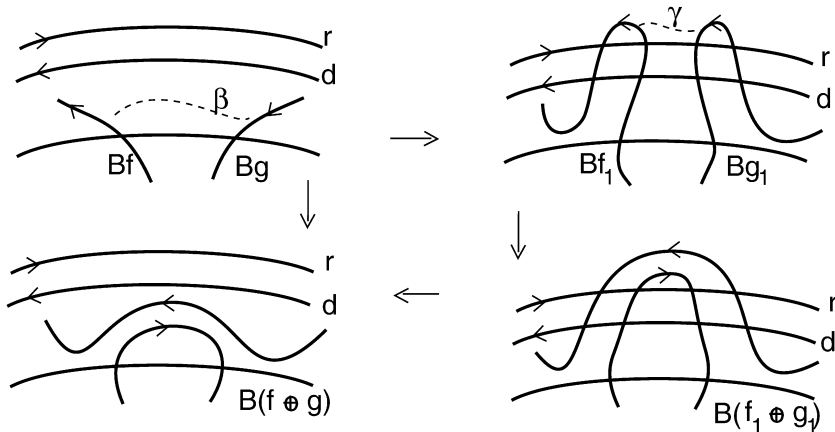


Fig. 8. Moving the bridge.

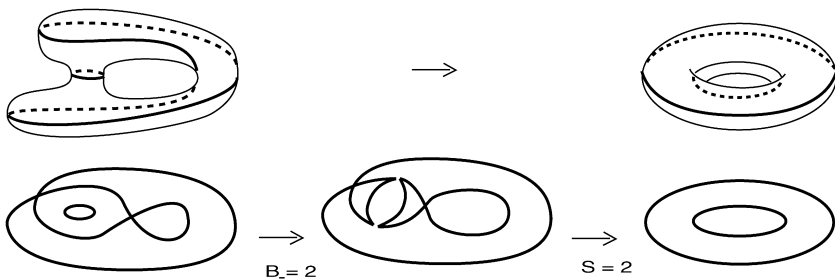


Fig. 9. Transition in the torus.

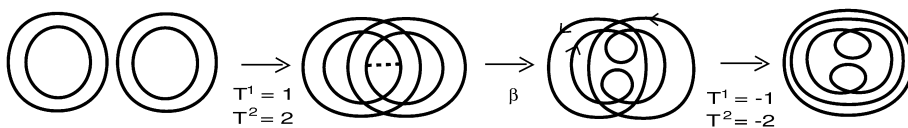
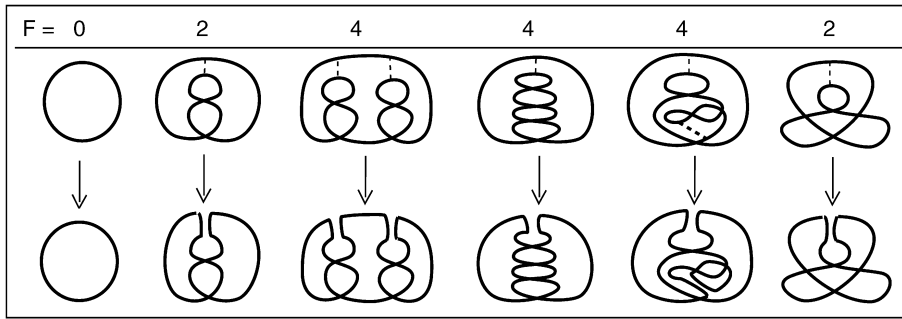


Fig. 10. Surgery on a map.

Fig. 11. Branch sets for stable maps on the sphere with  $\mu = 1$ .

## 6. Planar fold maps

More delicate is the question of characterizing those graphs which are graphs of fold maps (i.e. stable maps without cusps). In the spherical case there is a necessary and sufficient condition [9] *a weighted tree is the graph of a fold map of the sphere if and only if it is balanced and all of its weights are zero*. Recall that a bipartite graph is said to be *balanced* if it has the same number of positive and negative vertices.

We say that a stable map is *planar* if all regions in the complement of the singular set are planar; see Fig. 12. In other words, the vertices of the graph all have weight zero. As the figure shows, the graph of a planar map need not, of course, be planar.

The result above on weighted trees extends easily to more general graphs:

**Theorem 6.1.** *A weighted graph is the graph of a planar fold map if and only if it is balanced and all of its weights are zero.*

**Proof.** The Euler number of the positive half of the surface may be written as follows

$$\chi^+ = \Sigma \chi_R = \Sigma (2 - \mu_v) = 2V^+ - \mu,$$

where the sum is over positive vertices  $v$  and  $\mu_v$  denotes the number of edges containing  $v$ . Similarly for  $\chi^-$ . But, for any fold map,  $\chi^+ = \chi^-$  [9] from which it follows that  $V^+$  and  $V^-$ , the numbers of positive and negative vertices, are equal.

Conversely, from a balanced graph all of whose weights are zero one may obtain a balanced tree by removal of appropriate edges. This tree may be realized by a fold map of the sphere [9] and the remaining edges realized by

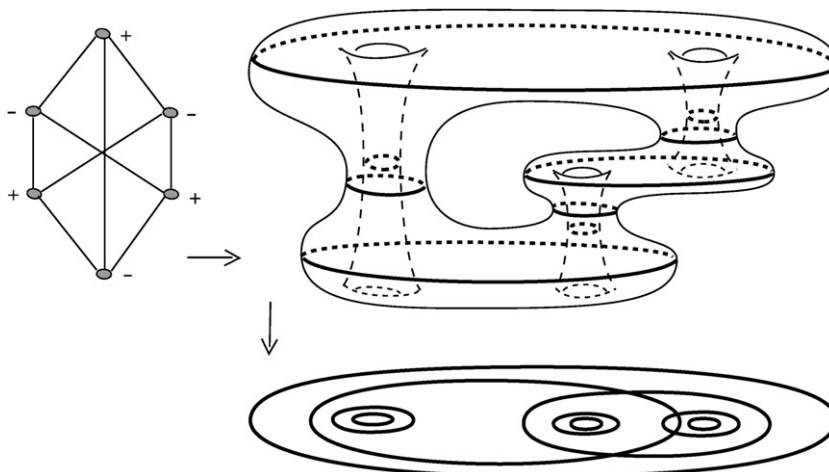


Fig. 12. Examples of planar fold maps.



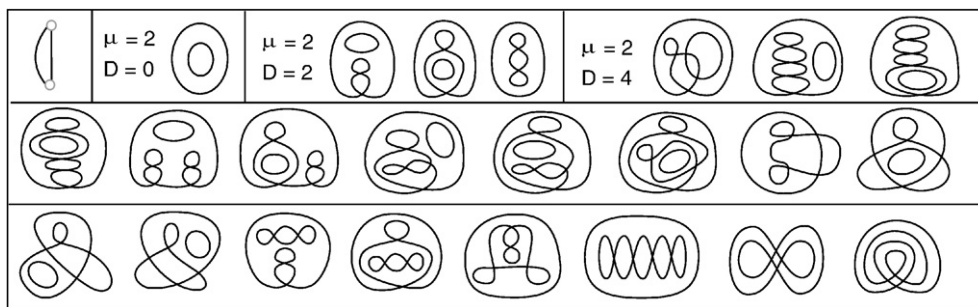


Fig. 13. Branch sets for planar fold maps of the torus.

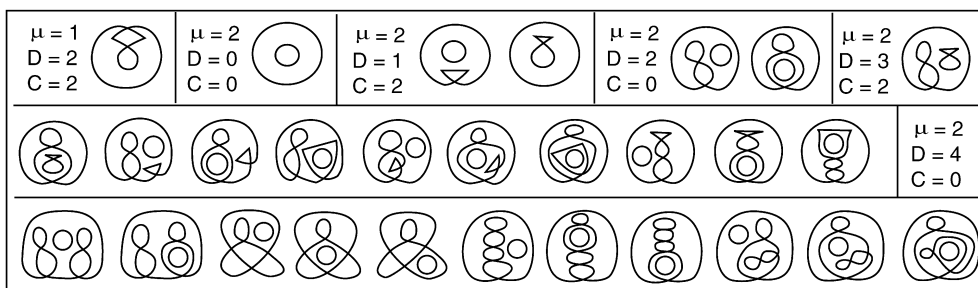
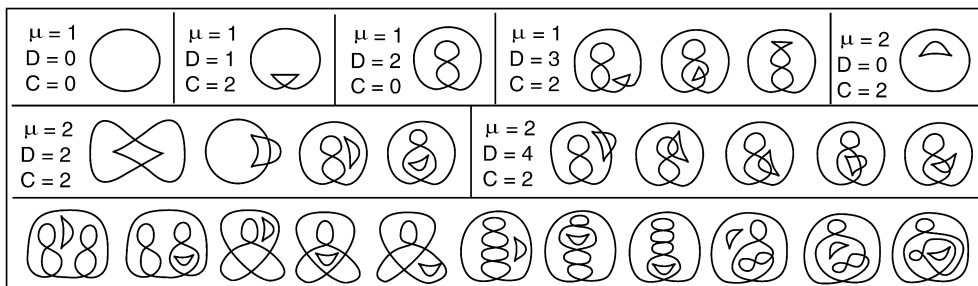


Fig. 14. Branch sets with cusps.

performing vertical surgeries on this map, as before. Since the vertical surgeries do not introduce any cusps, the resulting map continues to be a (planar) fold map.  $\square$

**Lemma 6.2.** *For any planar fold map of a connected, genus  $g$  surface,  $\mu + g$  is odd and  $D$  is even.*

**Proof.** First, observe that, for balanced graphs,  $V$  is even, since  $V = V^+ + V^- = 2V^+$ . For any planar map of a connected surface, a simple Euler characteristic calculation yields  $g = \mu - V + 1$ . Thus  $\mu + g$  is odd. As for  $D$ , one has, by [8], that, for any stable map of a connected surface,

$$\mu + \frac{C}{2} + D = (1 + g) \pmod{2}.$$

For fold maps,  $C = 0$  so that  $D$  is even.  $\square$

**Theorem 6.3.** *The table in Fig. 13 lists all possible pairs of graphs and branch sets of planar fold maps from the torus to the plane with  $\mu = 2$  and  $D \leq 4$ .*

**Proof.** For any planar fold map of the torus with  $\mu = 2$ , the two singular curves separate the torus into two annuli. One now lists all possible branch sets consisting of two curves (both chosen from Arnol'd's list, [3,4]) with  $D \leq 4$ .

The sum of the Whitney indices of the two branch curves is necessarily equal to the Euler characteristic (i.e. zero) and Troyer's condition for curves must be satisfied [12]. The possibilities are shown in Fig. 13. In fact, for each of these branch sets, the required immersed annuli are easily found by inspection.  $\square$

If cusps are allowed, then Fig. 14 shows (possibly all) branch sets realizable by planar stable maps of the sphere and the torus with  $\mu \leq 2$ ,  $C \leq 2$  and  $D \leq 4$ .

## References

- [1] Y. Ando, Existence theorems of fold maps, *Japan J. Math.* 30 (2004) 29–73.
- [2] Y. Ando, Fold-maps and the space of base point preserving maps of spheres, *J. Math. Kyoto Univ.* 41 (2002) 691–735.
- [3] V.I. Arnol'd, Plane curves, their invariants, perestroikas and classifications, in: *Advances in Soviet Math.*, vol. 21, AMS, Providence, RI, 1994, pp. 33–91.
- [4] V.I. Arnol'd, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series, vol. 5, AMS, Providence, RI, 1994.
- [5] E.A. Chínaro, Bifurcations of Whitney maps, Doctoral Thesis, Universidade Federal de Minas Gerais, Brasil, 1978.
- [6] Y. Eliashberg, On singularities of folding type, *Math. USSR-Izv.* 4 (1970) 1119–1134.
- [7] M. Golubitsky, V. Guillemin, *Stable Mappings and Their Singularities*, Springer, Berlin, 1976.
- [8] D. Hacon, C. Mendes de Jesus, M.C. Romero Fuster, Topological invariants of stable maps from a surface to the plane from a global viewpoint, in: *Real and Complex Singularities*, in: *Lecture Notes in Pure and Appl. Math.*, vol. 232, Dekker, New York, 2003, pp. 227–235.
- [9] D. Hacon, C. Mendes de Jesus, M.C. Romero Fuster, Fold maps from the sphere to the plane, *Experimental Math.* (2006), in press.
- [10] L. Kauffman, Planar surface immersions, *Illinois J. Math.* 23 (1979) 648–665.
- [11] H. Levine, Stable maps: an introduction with low dimensional examples, *Bol. Soc. Bras. Mat.* 7 (1976) 145–184.
- [12] I.P. Malta, N.C. Saldanha, C. Tomei, *Geometria e Análise Numérica de Funções do Plano no Plano*, Colóquio Brasileiro de Matemática, vol. 19, IMPA, 1993.
- [13] C. Mendes de Jesus, M.C. Romero Fuster, Bridges, channels and Arnol'd invariants for generic plane curves, *Topology and its Applications* 125 (2002) 505–524.
- [14] T. Ohmoto, F. Aicardi, First order local invariants of apparent contours, *Topology* 45 (2006) 27–45.
- [15] J. Rieger, Families of maps from the plane to the plane, *J. London Math. Soc.* (2) 36 (2) (1987) 351–369.
- [16] O. Saeki, Fold maps on 4-manifolds, *Comment. Math. Helv.* 78 (2003) 627–647.
- [17] K. Sakuma, On the topology of simple fold maps, *Tokyo J. Math.* 17 (1994) 21–31.
- [18] M. Yamamoto, Immersions of surfaces with boundary into the plane, *Pacific J. Math.* 212 (2003) 371–376.